Statistics and Transport in the Restricted Three-Body Problem

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Some problems in dynamical astronomy suggest a three-body analysis

- e.g., Jupiter-family comets and scattered Kuiper Belt objects (under Neptune’s control)
- By applying dynamical systems methods to the planar, circular restricted three-body problem, several questions regarding these populations may be addressed
- Comparison with observational data is made
We want to answer several questions regarding the transport and origin of some kinds of solar system material:

- How do we characterize the motion of Jupiter-family comets (JFCs) and scattered Kuiper Belt objects (SKBOs)?
- How likely is a transition between the exterior and interior regions (e.g., Oterma)?
- How probable is a Shoemaker-Levy 9-type collision with Jupiter? Or an asteroid collision with Earth (e.g., KT impact)?

Harder questions:

- How does an SKBO become a JFC (and vice versa)?
- How does impact ejecta get from Mars to Earth?
Jupiter-family comets

- JFCs and lines of constant **Tisserand parameter**,

\[
T = \frac{1}{a} + 2\sqrt{a(1 - e^2)},
\]

an approximation of the Jacobi constant
Scattered Kuiper Belt Objects

- Current SKBO locations in black, with some Tisserand values w.r.t. Neptune in red ($T \approx 3$)
Scattered Kuiper Belt Objects

- Seen in inertial space

![Graph of Scattered Kuiper Belt Objects](image-url)
Motion of JFCs and SKBOs

- Theory, observation, and numerical experiment show motion along nearly constant Tisserand parameter (most of the time)
- We approximate the short-timescale motion of JFCs and SKBOs as occurring within an energy shell of the restricted three-body problem
- Several objects may be in nearly the same energy shell, i.e., all have \(|T - T^*| \leq \delta T|
- Can we analyze the structure of an energy shell to determine likely locations of JFCs and SKBOs?
Recall the planar, circular restricted three-body problem from Jerry Marsden’s talk.

For fixed $\mu$, an energy shell (or energy manifold) of energy $\varepsilon$ is

$$\mathcal{M}(\mu, \varepsilon) = \{(x, y, \dot{x}, \dot{y}) \mid E(x, y, \dot{x}, \dot{y}) = \varepsilon\}.$$  

The $\mathcal{M}(\mu, \varepsilon)$ are 3-dimensional surfaces foliating the 4-dimensional phase space.
Study Poincaré surface of section at fixed energy $\varepsilon$:

$$\Sigma_{(\mu, \varepsilon)} = \{(x, \dot{x}) | y = 0, \dot{y} = f(x, \dot{x}, \mu, \varepsilon) < 0\}$$

reducing the system to an area preserving map on the plane. Motion takes place on the cylinder, $S^1 \times \mathbb{R}$. 

Poincaré surface-of-section and map $P$
The energy shell has regular components (KAM tori) and irregular components. Large connected irregular component is the "chaotic sea."
The motion within the chaotic sea is understood as the movement of trajectories among resonance regions (see Meiss [1992] and Schroer and Ott [1997]).
Movement among resonances

- This is confirmed by numerical computation.
- Shaded region bounded by stable and unstable invariant manifolds of an unstable resonant (periodic) orbit.
The unstable and stable manifolds are understood as the backbone of the dynamics. This is the “homoclinic trellis” in the words of Poincaré.
There are several approaches to computing useful transport quantities.

- Markov model where the energy shell is partitioned into stochastic regions separated by partial barriers (Meiss et al.)
- Set oriented methods where a graph is created to model the underlying dynamical behavior (Dellnitz et al.)
- Lobe dynamics; following intersections of stable and unstable invariant manifolds of periodic orbits (Wiggins et al.)

These methods are preferred over the “brute force” asteroid dynamics calculations seen in the literature since they are based on first principles.
Transition Rates

Fluxes give rates and probabilities

- Jaffé, Ross, Lo, Marsden, Farrelly, and Uzer [2002] computed the rate of escape of asteroids temporarily captured by Mars.
- RRKM-like statistical approach
  - similar to chemical dynamics, see Truhlar [1996]
- Consider an asteroid (or other body) in orbit around Mars (perhaps impact ejecta) at a 3-body energy such that it can escape toward the Sun.
- Interested in rate of escape of such bodies at a fixed energy, i.e. $F_{M,S}(t)$
RRKM assumption: all asteroids in the Mars region at fixed energy are equally likely to escape. Then

\[ \text{Escape rate} = \frac{\text{flux across potential barrier}}{\text{Mars region phase space volume}} \]

Compare with Monte Carlo simulations of 107,000 particles

- randomly selected initial conditions at constant energy
Theory and numerical simulations agree well.

- Monte Carlo simulation (dashed) and theory (solid)
If the planar, circular restricted three-body problem is ergodic, then a statistical mechanics can be built (cf. ZhiGang [1999]).

Recent work suggests there may be regions of the energy shell for which the motion is ergodic, in particular the “chaotic sea” (Jaffé et al. [2002]).

This suggests we compute the steady state distribution of some observable for particles in the chaotic sea; a simple method for obtaining the likely locations of any particles within it.
Assuming ergodicity,

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t A(x, y, p_x, p_y) d\tau = \int A(x, y, p_x, p_y) \left( \frac{C}{|\frac{\partial H}{\partial p_y}|} \right) dp_x dx dy,
\]

where \( A(x, y, p_x, p_y) \) is any physical observable (e.g., semimajor axis), one finds that the density function, \( \rho(x, p_x) \), on the surface-of-section, \( \Sigma(\mu, \varepsilon) \), is constant.

We can determine the steady state distribution of semimajor axes; define \( N(a) da \) as the number of particles falling into \( a \to a + da \) on the surface-of-section, \( \Sigma(\mu, \varepsilon) \).
SKBOs should be in regions of high density.
Collisions with Jupiter

- **Shoemaker Levy-9**: similar energy to Oterma
  - Temporary capture and collision; came through L1 or L2

Possible *Shoemaker-Levy 9* orbit seen in rotating frame (Chodas, 2000)
Collision Probabilities

- Low velocity impact probabilities
- Assume object enters the planetary region with an energy slightly above L1 or L2
  - eg, Shoemaker-Levy 9 and Earth-impacting asteroids
Stable and unstable manifold tubes

- Control transport through the potential barrier.
Collision Probabilities

Collision probabilities
- Compute from tube intersection with planet on Poincaré section
- Planetary diameter is a parameter, in addition to $\mu$ and energy $E$
Collision Probabilities

Collision probabilities

Poincare Section: Tube Intersecting a Planet

Collision
Non-Collision

← Diameter of planet →
Collision Probabilities

Collision Probability for Jupiter Family Comets

- L1
- L2

Energy
Collision probability (%)

SL9
Collision Probability for Near-Earth Asteroids

Collision probability (%)

Energy (scaled)

Collision Probabilities
References


For papers, movies, etc., visit the website: http://www.cds.caltech.edu/~shane

The End