Dynamical Systems
and Space Mission Design
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**Constructing Orbits of Prescribed Itineraries: Outline**

- **Outline of Lecture 4A:**
  - Using the proof of the **Main Theorem** as the guide, we develop a procedure to construct orbits with **prescribed itineraries**.
  - Example: An orbit with itinerary \((X, J, S, J, X)\).
  - Application: **Petit Grand Tour** of Jovian moons.
The Petit Grand Tour of the Jovian Moons
Global Orbit Structure: Review

- **Symbolic sequence** used to label itinerary of each orbit.

- **Main Theorem**: For any admissible itinerary, e.g., (... , X, J, S, J, X, ...) in the informal notation, there exists an orbit whose *whereabouts* matches this itinerary.

- Can even specify **number of revolutions** the comet (or spacecraft) makes around $L_1$ & $L_2$ as well as Sun & Jupiter.
Construction of Orbits with Prescribed Itineraries

Recall from Lecture 3B, that the Main Theorem concerning bi-infinite itineraries (e.g., \((\ldots, X, J, S, J, X, \ldots)\)) relied on the construction of the invariant set, a “cloud of points.”

To generate usable trajectories, truncate construction of invariant set at a finite number of iterations of the Poincaré map \(P\).
Construction of Orbits with Prescribed Itineraries

- The sets of orbits with different itineraries are easily visualizable on our chosen Poincaré section as areas in which all the orbits have the same finite itinerary.

- We will also no longer be limited to a small neighborhood of a homoclinic-heteroclinic chain, but can obtain more global results.

- Example: An orbit with itinerary \((J, X, J)\).
Construction of Orbits with Prescribed Itineraries

Example: An orbit with itinerary \((X, J, S)\), passing from the exterior region to the interior region via the Jupiter region.

Look at Poincaré section where \(L_2\) periodic orbit unstable manifold intersects \(L_1\) periodic orbit stable manifold.

Intersection region contains orbits with itinerary \((X, J, S)\).
The Flow near $L_1$ and $L_2$: Review

- Recall from Lecture 2B, for energy values just above that of $L_2$,
  - **Hill’s region** contains a "neck" about $L_1$ and $L_2$.

- [Conley] The flow in the **equilibrium region** $\mathcal{R}$ has **4 types** of orbits: **periodic**, **asymptotic**, **transit** and **non-transit** orbits.
McGehee Representation: Equilibrium Region $\mathcal{R}$

4 types of orbits in equilibrium region $\mathcal{R}$:

- **Black** circle $l$ is the unstable periodic orbit.
- 4 cylinders of asymptotic orbits form pieces of stable and unstable manifolds. They intersect the bounding spheres at asymptotic circles, separating **spherical caps**, which contain transit orbits, from spherical zones, which contain non-transit orbits.
Cylinders of asymptotic orbits form “tubes” in the energy surface.

- Periodic orbit **stable** and **unstable** manifold tubes.
- To transit from one region to another, must be inside the tubes.
- e.g., transit from Jupiter to Sun region via $L_1$ periodic orbit tubes.
Spherical Caps Containing Transit Orbits

- We will be focusing on the **spherical caps of transit orbits**. In particular, we will look at their images and pre-images on a suitable Poincaré section, since these spherical caps will be building blocks from which we construct orbits of prescribed itineraries.

  - The images and pre-images of the spherical caps form the tubes that partition the energy surface.
For instance, on a Poincaré section between $L_1$ and $L_2$,

- We look at the **image of the cap** on the left bounding sphere of the $L_2$ equilibrium region $\mathcal{R}_2$ containing orbits leaving $\mathcal{R}_2$.
- We also look at the **pre-image of the cap** on the right bounding sphere of $\mathcal{R}_1$ containing orbits entering $\mathcal{R}_1$. 

Poincaré Section $U_3$ ($x = 1 - \mu, y > 0$)
Poincaré Section: Images of Bounding Sphere Caps

Right bounding sphere \( n_{1,2} \) of \( L_1 \) equilibrium region \( (R_1) \)

Image of Spherical Cap \( d_{1,2}^+ \)

Pre-Image of Spherical Cap \( d_{1,2}^+ \)

\((J,S)\)

\((X;J)\)

Image of Spherical Cap \( d_{2,1}^- \)

Poincaré Section \( U_3 \)

\((x = 1 - \mu, y > 0)\)

Stable Manifold

Unstable Manifold

Jupiter region \( (J) \)

Left bounding sphere \( n_{2,1} \) of \( L_2 \) equilibrium region \( (R_2) \)
Construction of Orbits with Prescribed Itineraries

The Poincaré cut of the unstable manifold of the $L_2$ periodic orbit forms the boundary of the image of the cap containing transit orbits leaving $\mathcal{R}_2$.

- All of these orbits came from the exterior region and are now in the Jupiter region, so we label this region $(X; J)$. 

![Diagram of Poincaré section and orbits](image)
Construction of Orbits with Prescribed Itineraries

Similarly, the Poincaré cut of the \textbf{stable manifold} of the $L_1$ periodic orbit forms the boundary of the \textbf{pre-image of the cap} of transit orbits entering $\mathcal{R}_1$.

- All of these orbits are now in the Jupiter region and are headed for the interior (Sun) region, so we label this region $(; J, S)$.
Construction of Orbits with Prescribed Itineraries

- Note that the image of the $R_2$ cap and the pre-image of the $R_1$ cap intersect. We label this intersection region $\Delta J$.
  - This is the set of orbits which came from the exterior region, are now in the Jupiter region, and are headed for the interior (Sun) region, so we label this region $(X; J, S)$.
  - Integrating an initial condition in this set forwards and backwards would give us an orbit with the desired itinerary $(X, J, S)$. 

![Diagram showing the intersection region $\Delta J$]
We can partition the intersection region even further using this technique. Smaller regions of orbits with longer itineraries emerge.
Example Itinerary: \((X, J, S, J, X)\)

We wish to illustrate these techniques by constructing an orbit with an itinerary \((X, J, S, J, X)\).

- This is the itinerary for the comet \textit{Oterma} in the Sun-Jupiter system during the time interval of interest.
Example Itinerary: (X, J, S, J, X)

Before beginning, we review the notation regarding the location of the Poincaré sections used for the construction.

- $U_1$ in interior (Sun) region, $U_4$ in exterior region
- $U_2$ and $U_3$ in Jupiter region
Example Itinerary: \((X, J, S, J, X)\)

- Note also the locations of the **stable** and **unstable** manifolds of the \(L_1\) and \(L_2\) periodic orbits.
  - Note the mirror symmetry about the Sun-Jupiter line.
Example Itinerary: \((X, J, S, J, X)\)

We compute the first few transversal cuts of the \(L_1\) stable and \(L_2\) unstable periodic orbit manifolds on the \(U_3\) Poincaré section in the Jupiter region.

- Notice the intersection region \(\Delta \mathcal{J}\), in which all orbits have the central block itinerary \((X; J, S)\).
Example Itinerary: (X, J, S, J, X)

- All orbits entering the Sun region must pass through the first cut of the $L_1$ p.o. stable manifold.
  - Notice that the $L_2$ p.o. unstable manifold swings around Jupiter several times before intersecting the first cut of the $L_1$ p.o. stable manifold.
  - The intersection, containing $(X; J, S)$ orbits, is shown shaded.
Example Itinerary: \((X, J, S, J, X)\)

- Follow the shaded \(\Delta J = (X, J, S)\) region forward under the flow until it intersects the \(U_1\) Poincaré section in the Sun region.
  - The image \((X, J; S)\) is contained entirely inside the region bounded by the \(L_1\) p.o. unstable manifold, as expected.
  - Pieces that also intersect the \(L_1\) p.o. stable manifold will return to the Jupiter region and are labelled \(\Delta S = (X, J; S, J)\).
Example Itinerary: \((X, J, S, J, X)\)

- Follow the larger strip of \(\Delta^S = (X, J; S, J)\) forward until it returns to the Jupiter region, intersecting the \(U_2\) Poincaré section.
  - The **piece** contained inside the region bounded by the \(L_2\) p.o. stable manifold will return to the exterior region.
  - Region \(\Delta = (X, J, S; J, X)\) contains orbits with desired itinerary.
**Example Itinerary:** \((X, J, S, J, X)\)

- Pick any initial condition in the region \(\Delta = (X, J, S; J, X)\) and integrate forward and backward.
  - We obtain orbit with the desired prescribed itinerary \((X, J, S, J, X)\).
Application: Petit Grand Tour of Jovian Moons

Recently, there has been interest in sending a scientific spacecraft to orbit and study Europa.

- The Jovian moon may have subterranean oceans and life.
Application: Petit Grand Tour of Jovian Moons

The trajectory design involved in effecting an orbital capture by Europa presents formidable challenges to the traditional two-body “patched conic” approach.

- Regimes of motion involved depend heavily on three-body dynamics.

New three-body perspective is necessary to design a feasible trajectory from Earth to Europa using a very limited fuel budget.

- Fortunately, Jupiter’s vicinity and the Galilean moons provide enough interesting orbital dynamics that the construction of a relatively low fuel mission is possible.
Petit Grand Tour: Ganymede and Europa

For this initial exploration, we study the final leg of the trip, once the spacecraft is in a large orbit around Jupiter.

The goal is to perform a tour of two of the Jovian moons, Ganymede and Europa.

- Ending with a capture orbit around Europa.
Petit Grand Tour: Ganymede and Europa

Itinerary:

- Starting from outside Ganymede’s orbit, get captured for one orbit around Ganymede.
- Leaving Ganymede, transfer to Europa.
- Get captured by Europa for several orbits.
Starting Model: Coupled Three-Body Problem

Solution space of full 4-body problem is little known.

Look at coupled 3-body problem:

- Two nested 3-body systems, Jupiter-Ganymede-spacecraft and Jupiter-Europa-spacecraft, approximate the dynamics well.
Petit Grand Tour: Ganymede and Europa

The **transfer trajectory** from Ganymede to Europa is the most difficult part of the tour to construct. That is where we begin.

- We wish to go from the “capture region” of Ganymede to the “capture region” of Europa.
- Use the intersection of **unstable** and **stable** manifold tubes.
**Intersection of Tubes**

- Find intersection of **Ganymede’s $L_1$ periodic orbit unstable manifold** and **Europa’s $L_2$ periodic orbit stable manifold**.

- The two manifolds are at different energies, so a rocket burn ($\Delta V$) will be necessary.
Intersection of Tubes: Poincaré Section

On a suitably chosen Poincaré section

- Look for an intersection between the manifolds.
- Compute magnitude of rocket burn ($\Delta V$).
### Intersection of Tubes: Poincaré Section

- **Poincaré section**: Vary configuration of the moons until,
  - \((x, \dot{x})\)-plane: Intersection found!
  - \((x, \dot{y})\)-plane: Velocity discontinuity since energies are different.
  - A rocket burn \(\Delta V\) of this magnitude will make transfer trajectory "jump" from one tube to the other.

![Graphs showing intersection points and orbit manifolds for Ganymede's \(L_1\) and Europa's \(L_2\) periodic orbits.]

"Intersection point (and \(\Delta V\))"
**Intersection of Tubes: Transfer Trajectory**

- Using an initial condition in the intersection region, forward and backward integrate to generate a **transfer trajectory**.

- Integrate full 4-body problem: Jupiter-Ganymede-Europa-S/C.
Intersection of Tubes: Transfer Trajectory

- Rocket burn required for transfer trajectory: $\Delta V = 1200$ m/s
- Compare with standard Hohmann transfer: $\Delta V = 2800$ m/s
  - Coupled 3-body approach uses only 40% of the fuel of the conventional coupled 2-body approach. Significant savings!

- Trajectory shown below in both inertial and rotating frames.
  - Transfer trajectory takes only 25 days.
**Summary: Using Tubes to Construct Trajectories**

- The study of the planar circular restricted 3-body problem (PCR3BP) provides a **systematic method** for the numerical construction of a transfer trajectory for a space mission.

- **We work in the Case 3 energy regime** of the PCR3BP.

\[ \text{Case 3 : } E_2 < E < E_3 \text{ or } C_2 > C > C_3 \]
Summary: Using Tubes to Construct Trajectories

- $L_1$ and $L_2$ periodic orbit **stable** and **unstable** manifold tubes partition the energy surface into **transit** and **nontransit** orbits.

Transit orbits, contained inside the tubes, transit between the interior, capture, and exterior regions.
Summary: Using Tubes to Construct Trajectories

Construct “leap-frogging” transfer trajectories between moons:

- Jump from inside of **unstable manifold tube** of one 3-body system to the inside of **stable manifold tube** of an adjacent 3-body system.
- Use rocket burn ($\Delta V$) at tube intersection point.
Summary: Using Tubes to Construct Trajectories

- e.g., Petit Grand Tour of Jovian moons Ganymede and Europa.
- Uses less than half the fuel of standard 2-body approach.
Summary: Using Tubes to Construct Trajectories

Construction of many other low-fuel transfers is possible.

- Earth to Moon transfer: “Shoot the Moon” in an upcoming lecture.
Looking Ahead: Natural Transfers in the Solar System

Natural (zero-fuel) transfers may exist in the Solar System.

- Comet transitions between resonances of Jupiter (e.g., Oterma).
- Transfer of comets between Jupiter and Saturn systems.
- Transfer of material between Asteroid Belt and Kuiper Belt.