# Optimal flapping strokes for self-propulsion in a perfect fluid

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# Rowing and flapping in aquatic animals

escape response of daphnia or “water flea”

(from Walker, Univ. Southern Maine)

# Semi-aquatic animals: appendages for walking on land

swimming beetle

# At low speeds, rowing provides high maneuverability

high maneuverability of boxfish

# In this talk

- Design and control of vehicles with articulated bodies
  - Jointed four-link model of self-propulsion via large shape changes
  - Geometric structure of propulsive shape change strokes
  - Efficient strokes: mechanical structure preserving optimal control code

# Locomotion model

- Symmetrical four-link model propelling from rest
  - vortex shedding not solely responsible for locomotion, as noted by Saffman [1967]
  - applies methods used previously on three-link carangiform fish (Kanso et al. [2005])

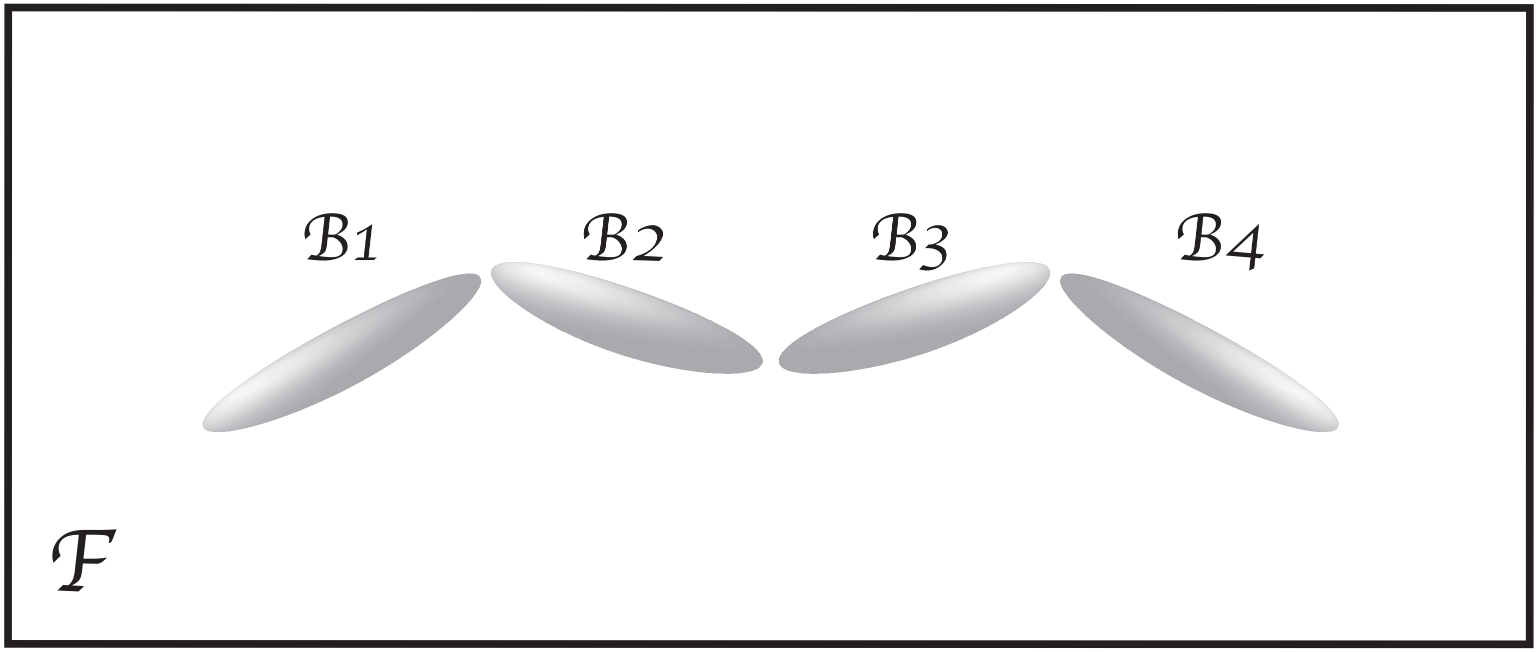
# Locomotion model

- example of "holonomy drive"
  - seen in, e.g., self-propulsion of microorganisms at low Reynolds number
  - **locomotion** based on sequence of shapes, not relative speed of shape change
  - but, **control effort** is based on relative speed of shape change

more efficient

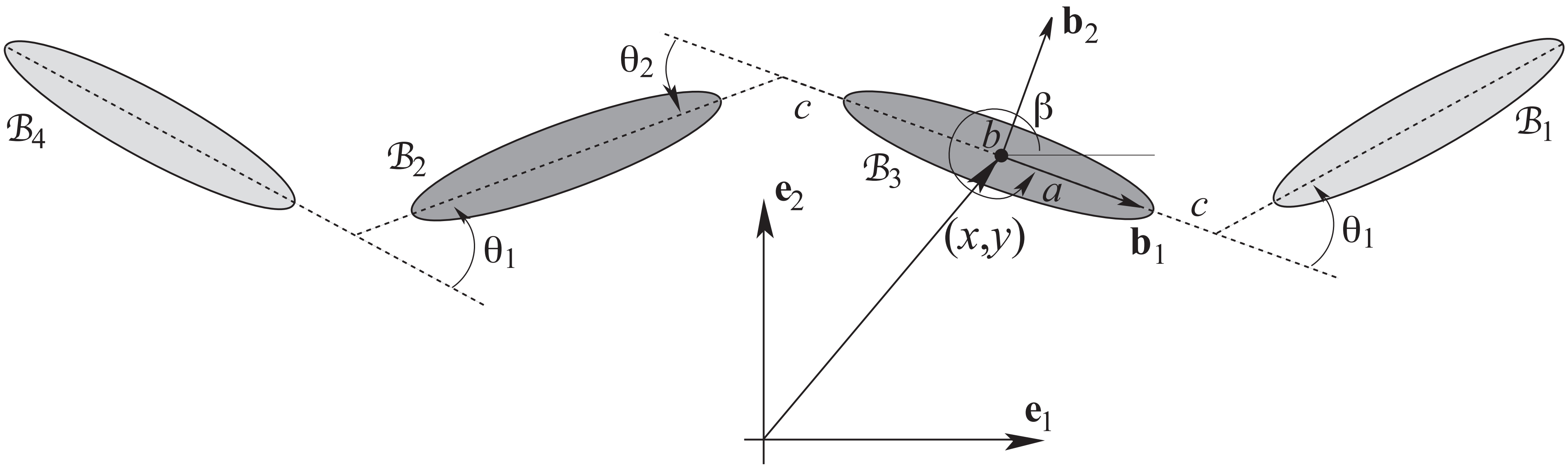
less efficient

# Four-link flapper model




- $\mathcal{F}$ is assumed to be inviscid, incompressible and irrotational for all time
- Potential flow ($\mathrm{u} = \nabla\phi, \nabla^2 u = 0$) with slip across solid boundaries
- Articulated body of four 4 rigid links $\mathcal{B}_i$ connected by hinge joints
- Bilaterally symmetric "flapping": four links is minimum necessary for locomotion in potential flow; allows for non-reciprocal shape changes

# Four-link flapper model




- Neutrally buoyant identical links
- Links: slender ellipsoidal geometry with axes $a, b$, where $b/a \ll 1$
- Joints: equipped with muscles which generate torques to achieve a desired stroke
- $g = (\beta, x, y)$, orientation, position of $\mathcal{B}_3$ w.r.t. $\{\mathrm{e}_1, \mathrm{e}_2\}$ – **net locomotion variables**
- $\theta = (\theta_1, \theta_2)$, orientation of $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_4$ relative to $\mathcal{B}_3$ – **shape space variables**

# Solid-fluid Lagrangian $L = T_s + T_f$

- Lagrangian = solid + fluid kinetic energy,

$$L = \sum_{i=1}^{4} T_{\mathcal{B}_i} + T_f = \frac{1}{2}\sum_{i=1}^{4} \xi_i^T \, \mathbb{I} \, \xi_i,$$

with $\xi_i = (\Omega_i, v_i)^T$ the velocity of the link $\mathcal{B}_i$ w.r.t. the $\mathcal{B}_i$-fixed frame and $\mathbb{I}_{ij} = \mathbb{I}$, including the added inertia, is the same for all links.

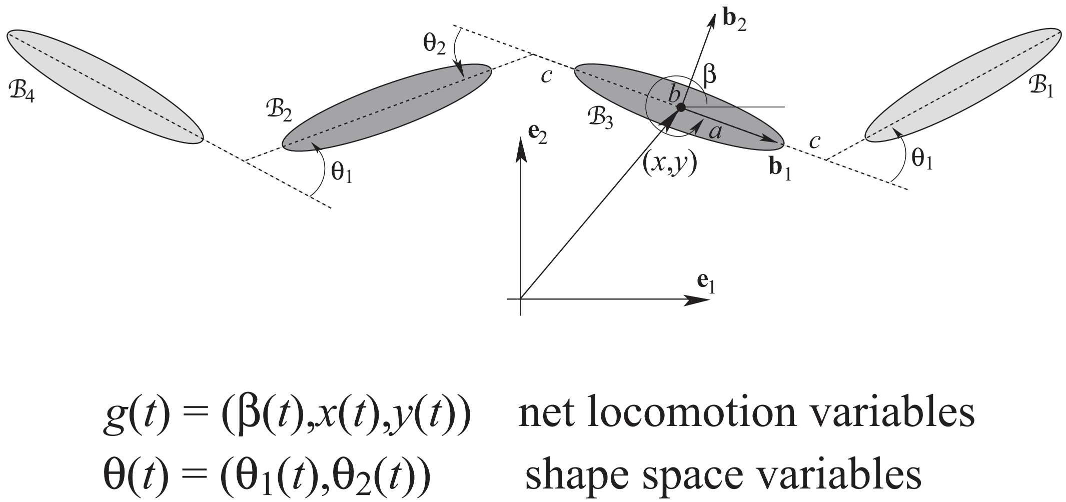
- The Lagrange-d'Alembert variational principle yields the forced Euler-Lagrange equations:

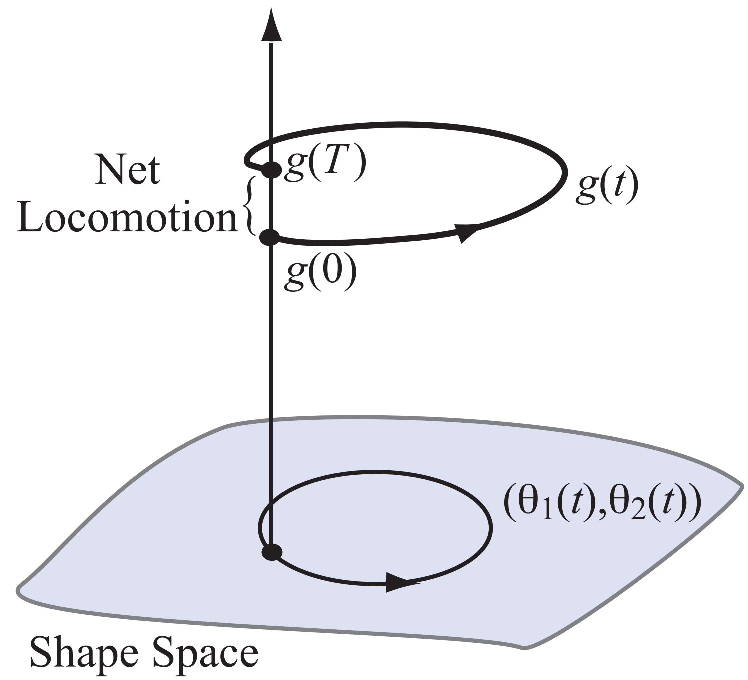
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{g}_i}\right) - \frac{\partial L}{\partial g_i} = 0, \qquad i = 1, 2, 3, \tag{1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = \tau_i, \qquad i = 1, 2, \tag{2}$$

where the internal torques $\tau(t)$ are exerted by actuators (or muscles) associated with the joints.

# Geometric mechanics description



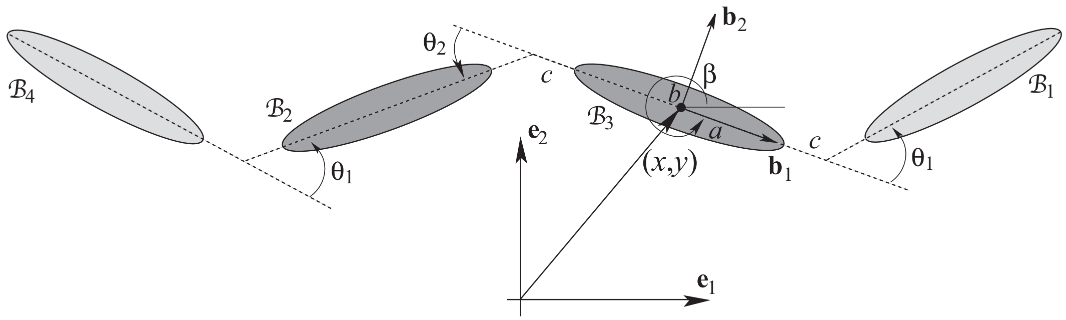

- When the motion starts from rest, (1) yields

$$g^{-1}\dot{g} = -\mathcal{A}(\theta)\dot{\theta}, \qquad (3)$$

where $g \in SE(2)$, the group of rotations and translations in $\mathbb{R}^2$.

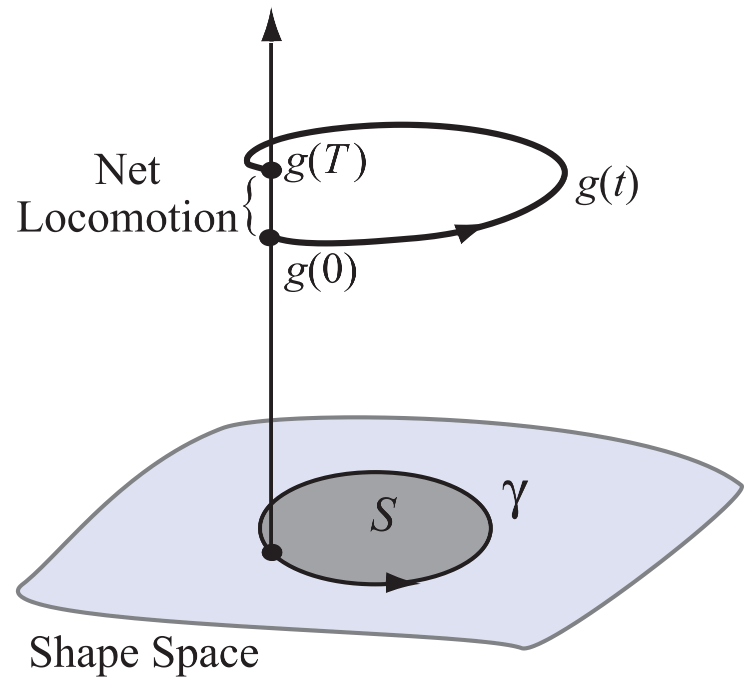
- Given a curve $\theta(t) = \big(\theta_1(t), \theta_2(t)\big), t \in [0, T]$, we solve (3) for $g(t) = \big(\beta(t), x(t), y(t)\big)$ and solve (2) for the torques $\tau(t)$

# Stroke: closed loop in shape space




$g(t) = (\beta(t), x(t), y(t))$ net locomotion variables

$\theta(t) = (\theta_1(t), \theta_2(t))$ shape space variables



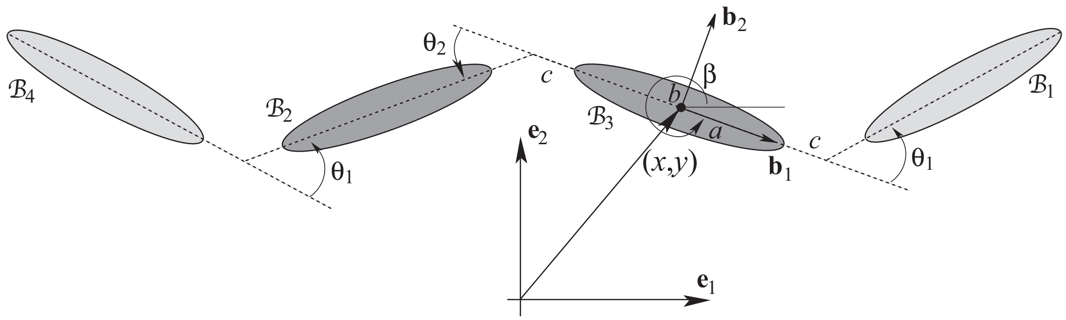

- A **stroke**: if $\theta(t)$ traces out a **closed loop** $\gamma$ in shape space $\Theta$ from time 0 to $T$,

$$g(T) = g(0) \exp\left( - \int_S d\mathcal{A}(\theta) \right).$$

  where $S$ is the region of $\Theta$ whose boundary is the loop $\gamma$

- Note: independent of time parametrization of curve $\gamma$

# Net locomotion from one stroke

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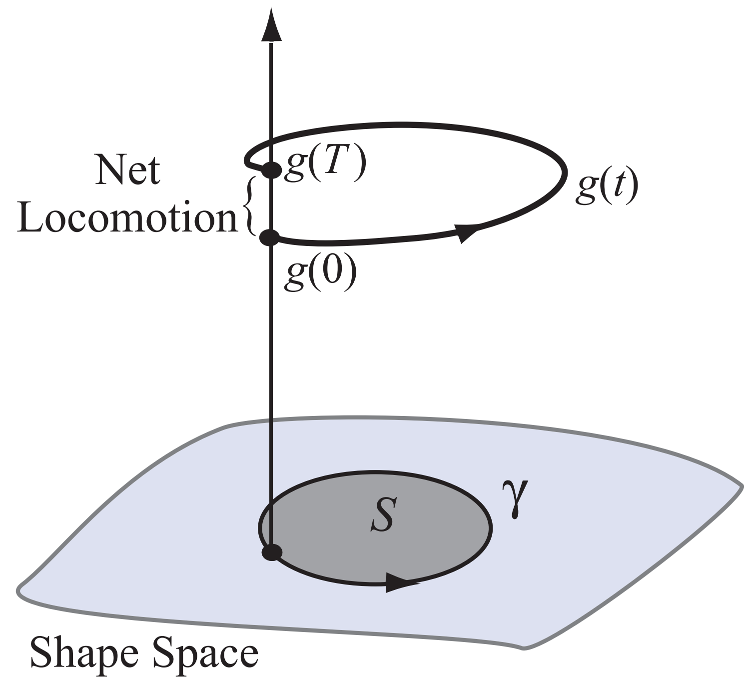
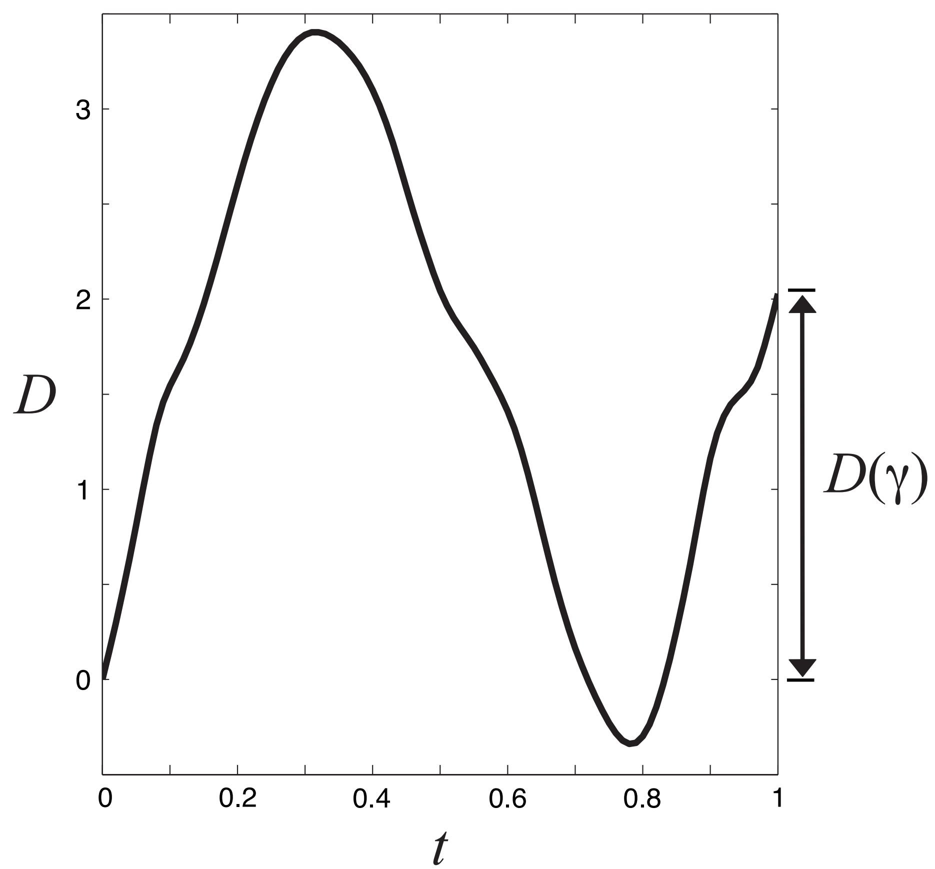


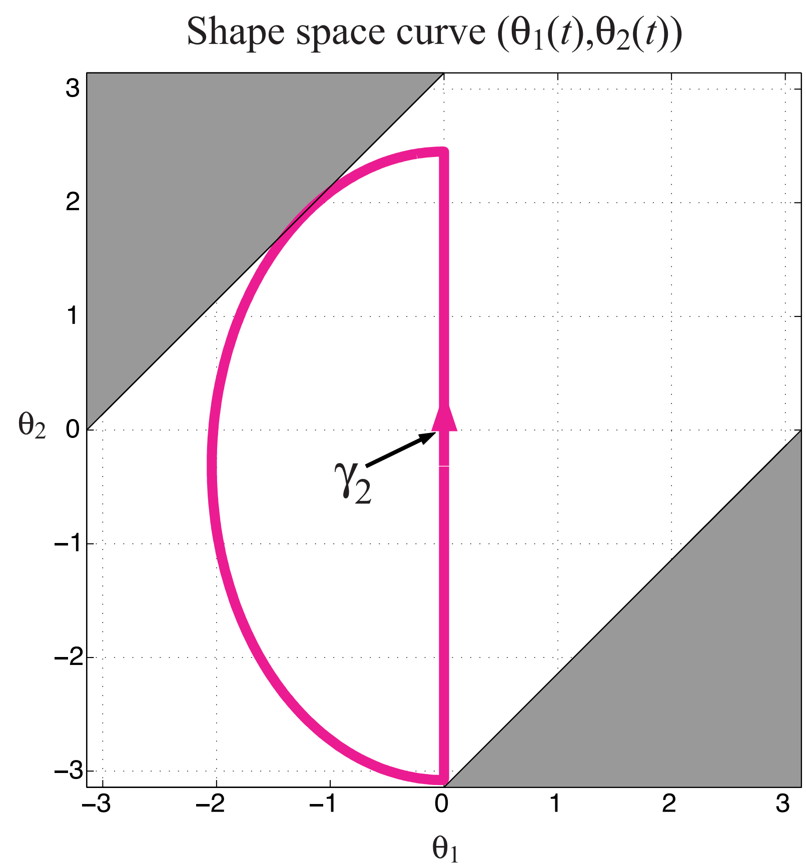

- i.e., **net locomotion** achieved, $g(T) - g(0)$, is a function of the loop geometry only (not on the instantaneous speeds along which the loop is traversed)

# Net locomotion from one stroke




- When a flapper has completed one stroke, it is back to its original shape, but has translated a distance $D(\gamma)$
- $D(\gamma) = D([\gamma])$ where $[\gamma]$ = equivalence class of strokes w.r.t. time reparametrization

# Example stroke loops $\gamma$




- simple expressions for closed curves $(\theta_1(t), \theta_2(t))$

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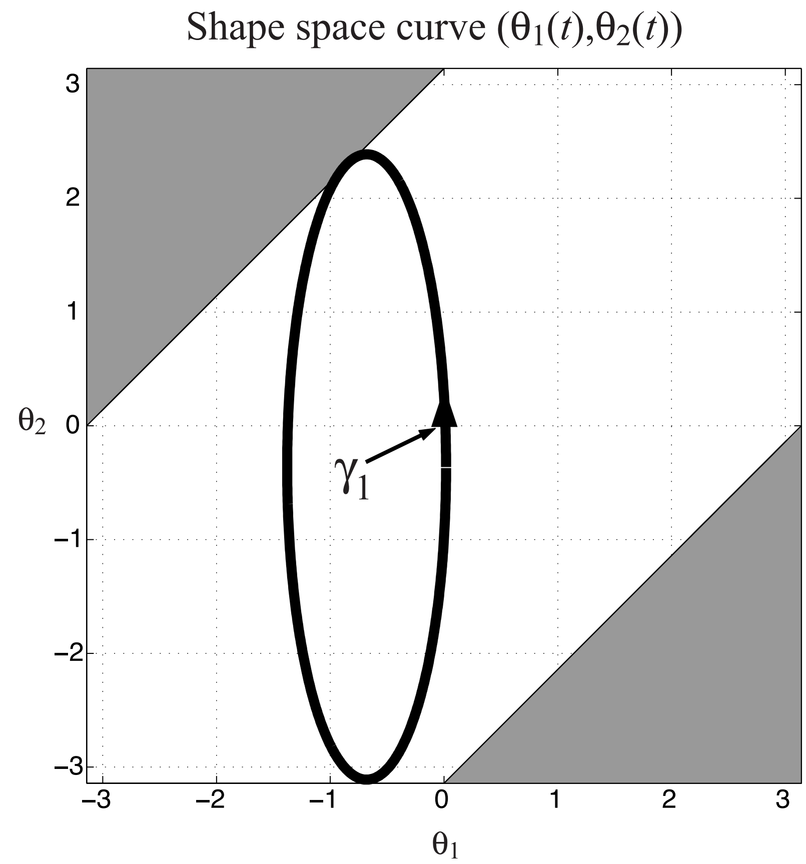


- simple expressions for closed curves $(\theta_1(t), \theta_2(t))$

# Optimal stroke loops

- Optimal strokes minimize the (torque) control effort per unit distance travelled

$$\delta(\gamma) = W(\gamma)/D([\gamma])$$

where

$$W(\gamma) \;=\; \int_0^T |\tau|^2 \, dt,$$

# Optimal stroke loops

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- Approximate $q(t) = (g(t), \theta(t))$ by a discrete path $q_n$ at $t_n = \{0, h, 2h, \ldots, Nh\}$

  And approximate control $\tau(t)$ by discrete torques $\tau_n$.

- Use DMOC (Discrete Mechanics & Optimal Control) algorithm of Junge, et al. [2005]
- Based on discretization of Lagrange D'Alembert variational principle
  $\Rightarrow$ discrete forced Euler-Lagrange equations
- Preserves mechanical structure; conserves momentum

# Optimization via DMOC

- Search over initial curves $\gamma_{\text{init}}$

- DMOC algorithm

$\gamma_{\text{init}} \Longrightarrow$

Minimize discrete cost function

$$\delta_d = \sum_n C_d(q_n, \tau_n, t_n) \;=\; \sum_n \Big(\sum_i \tau_{n\,i}^2\Big) h$$

subject to forced Euler-Lagrange equations
(as nonlinear equality constraints)

$$\begin{aligned} D_2 L_d(q_{n-1}, q_n) + D_1 L_d(q_n, q_{n+1}) + \tau_{n-1} + \tau_n &= 0 \\ p_0 + D_1 L_d(q_0, q_1) + \tau_0 &= 0 \\ -p_1 + D_2 L_d(q_{N-1}, q_N) + \tau_{N-1} &= 0 \end{aligned}$$

$\Longrightarrow \gamma_{\text{opt}}$

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- Implemented using SQP package of Matlab
- With $N = 100$, optimization usually takes a few minutes

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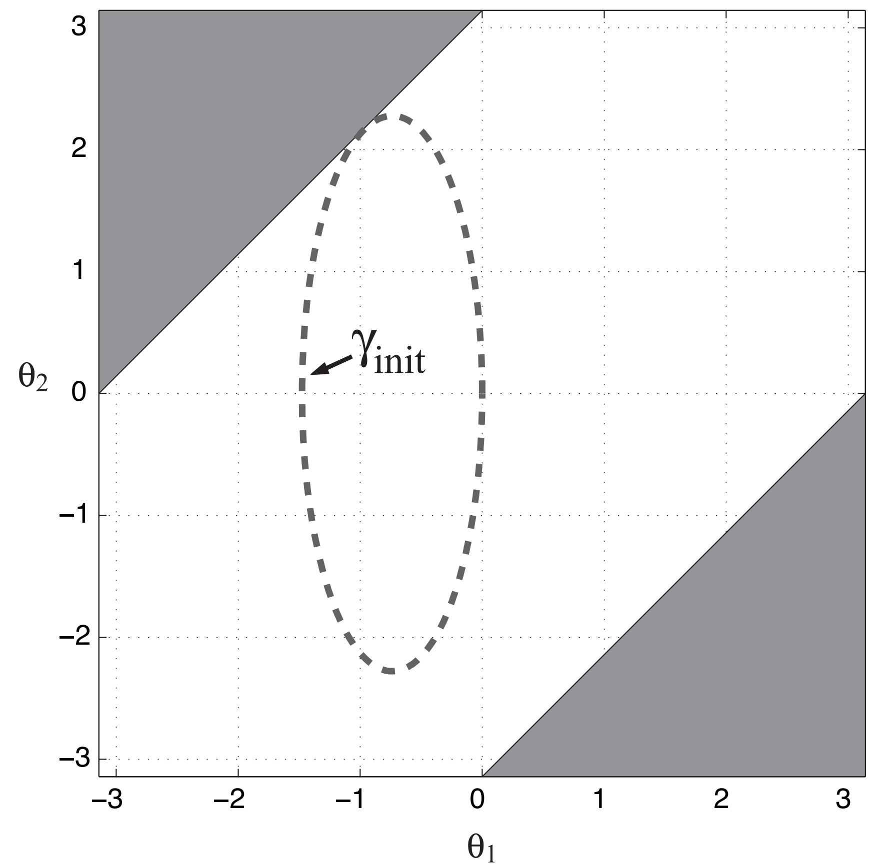
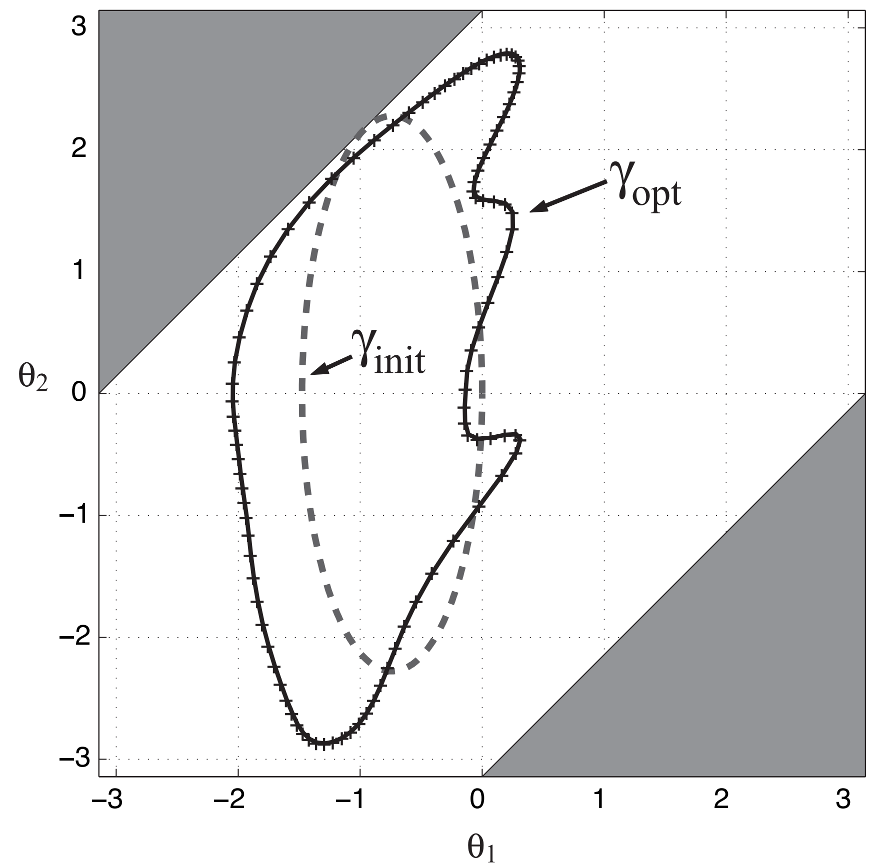



$\gamma_{\text{init}} \Longrightarrow$ Minimize discrete version of $\delta(\gamma)$ subject to discrete forced Euler-Lagrange eqs $\Longrightarrow \gamma_{\text{opt}}$

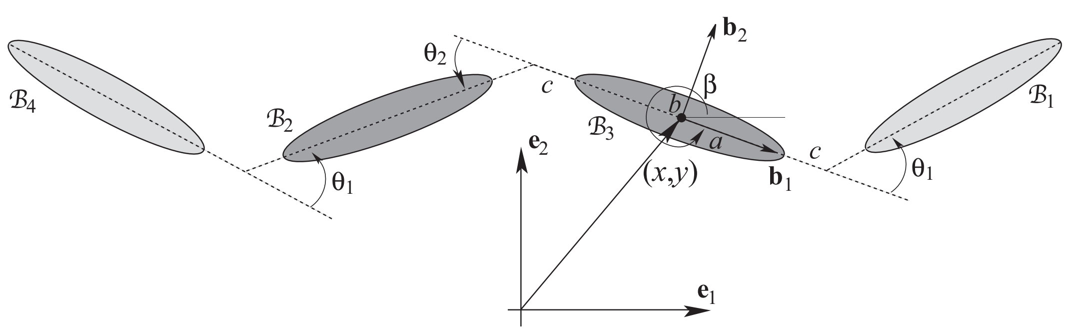
# Optimization via DMOC




$$\gamma_{\text{init}} \Longrightarrow \boxed{\begin{array}{l}\text{Minimize discrete version of } \delta(\gamma)\\ \text{subject to discrete forced Euler-Lagrange eqs}\end{array}} \Longrightarrow \gamma_{\text{opt}}$$

# Summary

- Developed a jointed four-link model of self-propulsion via cyclic strokes in a 2D perfect fluid.

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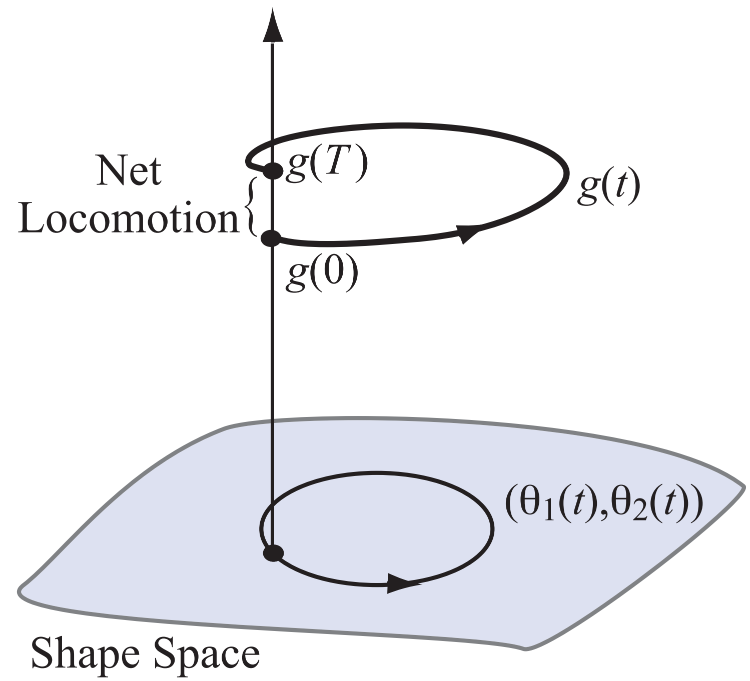
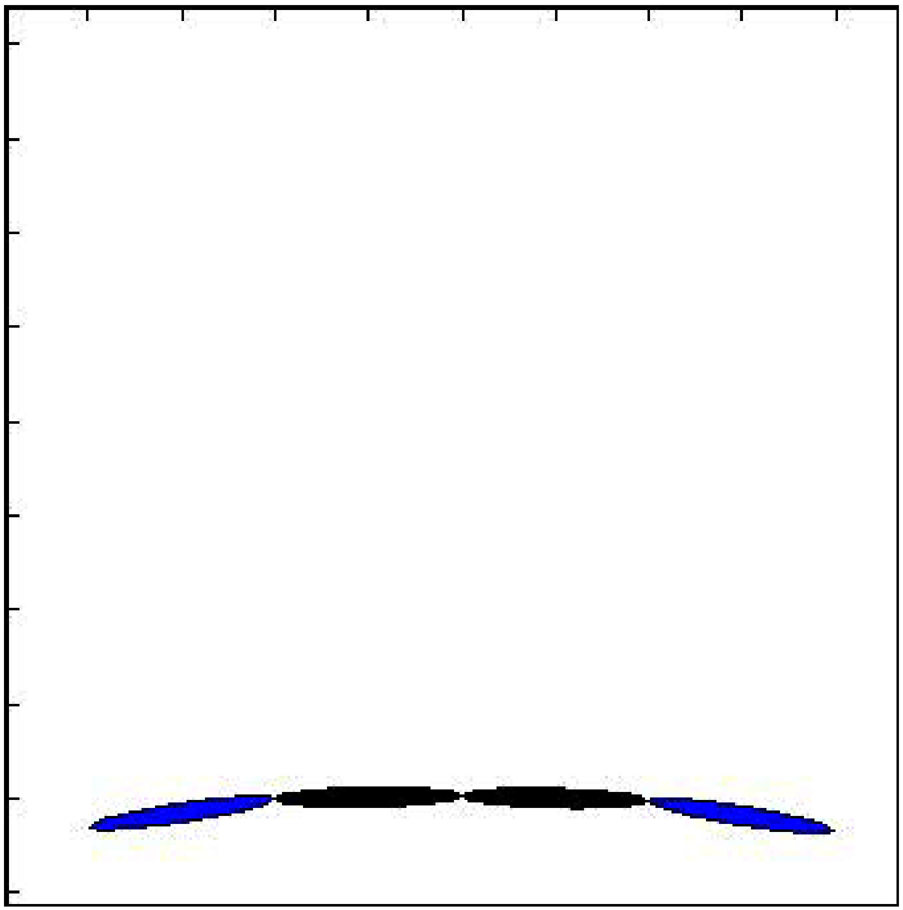
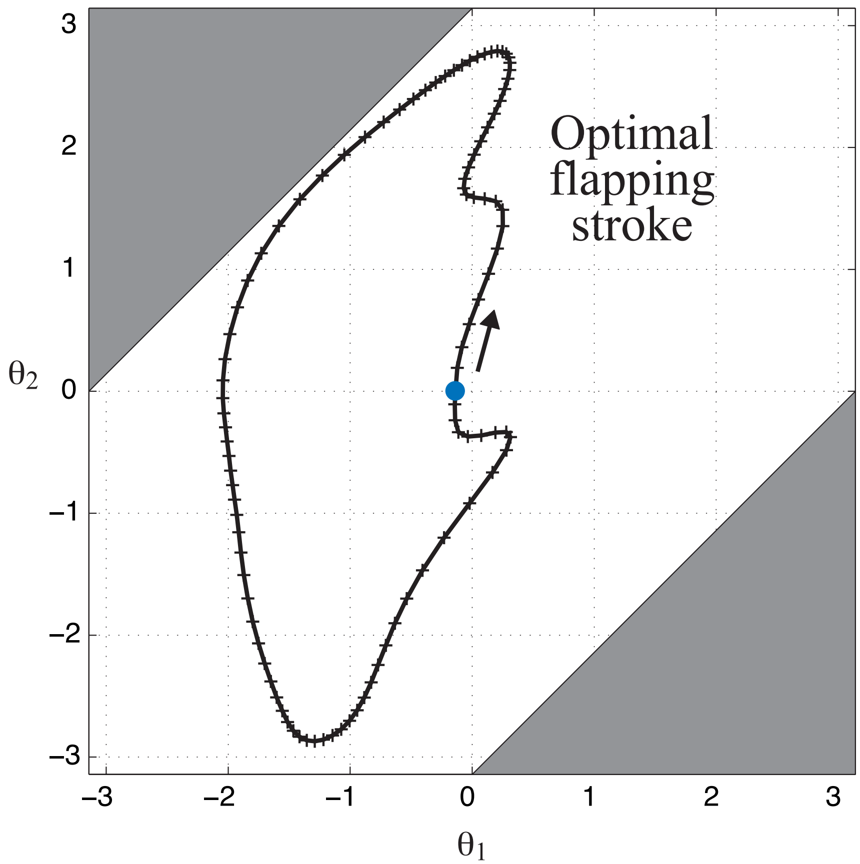


# Summary

- Determined which stroke yields the greatest locomotive efficiency, minimizing the control effort (muscle activity) per unit distance traveled.





# Future directions

- Comparison with biological data

- Exploit maneuverability
  — include turning, 3D
  — many vehicles
  — collision avoidance

Thank you