Intersections of phase volumes bounded by invariant manifolds

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Introduction

- Invariant manifolds of unstable bound orbits act as **separatrices** (codimension 1 surfaces)

- Determine **transport**, e.g., collisions, transitions

- Use **analytical map** of phase space where appropriate
Value-added: results apply to similar problems in e.g., chemistry, biomechanics, boat capsize
3-Body Problem

- Circular restricted 3-body problem
- Two important landmarks, the unstable points $L_1, L_2$
Equations of motion in rotating frame describe $P$ moving in an effective potential plus coriolis force (goes back to work of Jacobi, Hill, etc)
Motion in energy surface

- Hamiltonian function $H(q, p)$

- **Energy surface** of energy $E$ is codim-1

  $\mathcal{M}(E) = \{(q, p) \mid H(q, p) = E\}$

- In 2 d.o.f., 3D surfaces foliating the 4D phase space (in 3 d.o.f., 5D energy surfaces)
Realms of possible motion

- $\mathcal{M}_\mu(E)$ partitioned into three **realms**
  - e.g., Earth realm = phase space around Earth
- Energy $E$ determines their connectivity
Realms of possible motion

Case 1 : $E < E_1$

Case 2 : $E_1 < E < E_2$

Case 3 : $E_2 < E < E_3$

Case 4 : $E_3 < E < E_4$

Case 5 : $E > E_4$
Realms and tubes

- Realms connected by **tubes** in phase space $\simeq S^k \times \mathbb{R}$
  - Conley & McGehee, 1960s, found these locally for planar case, speculated on use for **“low energy transfers”**
Multi-scale dynamics

- Slices of energy surface: Poincaré sections $U_i$
- Tube dynamics: evolution between $U_i$
- What about evolution on $U_i$?
Multi-scale dynamics: Part 1

- Slices of energy surface: Poincaré sections $U_i$
- Tube dynamics: evolution between $U_i$ ←
- What about evolution on $U_i$?
Near $L_1$ or $L_2$, linearized vector field has eigenvalues

$$\pm \lambda \text{ and } \pm i\omega_j, \ j = 2, \ldots, N$$

Under local change of coordinates

$$H(q, p) = \lambda q_1 p_1 + \sum_{i=2}^{N} \frac{\omega_i}{2} (p_i^2 + q_i^2)$$
Equilibrium point is of type saddle \( \times \) center \( \times \cdots \times \) center \((N - 1\) centers\) (i.e., rank 1 saddle)
For energy $h$ just above saddle pt, $(q_1, p_1) = (0, 0)$ is normally hyperbolic invariant manifold of bound orbits

$$M_h = \sum_{i=2}^{N} \frac{\omega_i}{2} \left( p_i^2 + q_i^2 \right) = h > 0.$$
Note that $\mathcal{M}_h \simeq S^{2N-3}$

- $N = 2$, the circle $S^1$, a single periodic orbit
- $N = 3$, the 3-sphere $S^3$, a set of periodic and quasi-periodic orbits

The $N$ canonical planes
Motion near saddles

□ Note that $\mathcal{M}_h \simeq S^{2N-3}$

- $N = 2$, the circle $S^1$, a single periodic orbit
- $N = 3$, the 3-sphere $S^3$, a set of periodic and quasi-periodic orbits

□ Four “cylinders” or tubes of asymptotic orbits: stable, unstable manifolds, $W^s_\pm(\mathcal{M}_h), W^u_\pm(\mathcal{M}_h), \simeq S^3 \times \mathbb{R}$. 
Motion near saddles: 3-body problem

- **B**: bounded orbits (periodic/quasi-periodic): $S^3$
- **A**: asymptotic orbits to 3-sphere: $S^3 \times \mathbb{R}$ (tubes)
- **T**: transit and **NT**: non-transit orbits.

Projection to configuration space.
Tube dynamics: inter-realm transport

○ **Tube dynamics**: All motion between realms connected by necks around saddles must occur through the interior of tubes\(^1\)

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\(^1\)Koon, Lo, Marsden, Ross [2000,2001,2002], Gómez, Koon, Lo, Marsden, Masdemont, Ross [2004]
Tube dynamics

- Motion between Poincaré sections on $\mathcal{M}(E)$
- System reduced to $k$-map dynamics between the $k$ $U_i$
Tube dynamics

- Motion between Poincaré sections on $\mathcal{M}(E)$
- System reduced to $k$-map dynamics between the $k U_i$
Construction of orbits

- search for an initial condition with a given itinerary
- first in 2 d.o.f., then in 3 d.o.f.
Consider how tubes connect the $U_i$
Construction of orbits — 2 d.o.f.

\[ y = (T[J], S U_3) U(L[J], S) = (T_X[J], S U_3) U(X, [J]) \]

Poincare Section \( U_3 \)
\( \{x = 1 - \mu, y > 0, \dot{x} < 0\} \)
Denote the intersection \( (X, [J]) \cap ([J], S) \) by \( (X, [J], S) \).
Construction of orbits — 2 d.o.f.

- Forward and backward numerical integration

![Diagram showing orbit construction with labels for X Realm, S Realm, Forbidden Realm, J Realm, and Sun.](image)
**Construction of orbits — 3 d.o.f.**

- **Similar for 3 d.o.f.**: Invariant manifold tubes $S^3 \times \mathbb{R}$
- Poincaré section of energy surface
  - at $x = $ constant, $(y, \dot{y}, z, \dot{z}) \subset \mathbb{R}^4$
Similar for 3 d.o.f.: Invariant manifold tubes $S^3 \times \mathbb{R}$

Poincaré section of energy surface

- at $x = \text{constant}$, $(y, \dot{y}, z, \dot{z}) \subset \mathbb{R}^4$

Tube cross-section is a topological 3-sphere $S^3$ of radius $r$

- $S^3$ projection: disk $\times$ disk
Determining interior of $S^3$

$S^3$ projection: disk $\times$ disk

\[
y^2 + \dot{y}^2 + z^2 + \dot{z}^2 = r^2
\]
\[
r_y^2 + r_z^2 = r^2
\]
For fixed \((z, \dot{z})\), projection onto \((y, \dot{y})\) is a closed curve

\[
\begin{align*}
y^2 + \dot{y}^2 &= r^2 - (z^2 + \dot{z}^2) \\
\frac{r_y^2}{r_z^2} &= r^2 - \frac{r_y^2}{r_z^2}
\end{align*}
\]
Determining interior of $S^3$

For different $(z, \dot{z})$, a different **closed curve** in $(y, \dot{y})$

\[
\begin{align*}
    y^2 + \dot{y}^2 &= r^2 - (z^2 + \dot{z}^2) \\
    r_y^2 &= r^2 - r_z^2
\end{align*}
\]

(y, \dot{y}) Plane \hspace{2cm} (z, \dot{z}) Plane
Determining interior of $S^3$

Cross-section of tube effectively reduced to a two-parameter family of closed curves

\[ y^2 + \dot{y}^2 = r^2 - (z^2 + \dot{z}^2) \]

Plane $(y, \dot{y})$ Plane

Plane $(z, \dot{z})$
Determining interior of $S^3$

- Can be demonstrated numerically: $\{\text{int}(\gamma_{z\bar{z}})\} (z, \bar{z})$

- Provides nice way to calculate interior of tube, intersections of tubes, etc.

\[(y, \dot{y}) \text{ Plane} \quad (z, \dot{z}) \text{ Plane}\]
Intersection of phase volumes

- Find (X,J,S) orbit via tube intersection

Poincaré Section
Intersection of phase volumes

Find \((X,J,S)\) orbit via tube intersection
All orbits in intersection correspond to transition

Gómez, Koon, Lo, Marsden, Masdemont, Ross, Nonlinearity [2004]
Other orbits obtained this way
On the tubes, rather than in the tubes

An $L_1$-$L_2$ heteroclinic connection
Transition and collision

- Interpret relative phase volumes as probabilities\(^2\)

- Transition between realms and/or collision.

\(^2\)Ross [2003] Statistical theory of interior-exterior transition and collision probabilities for minor bodies in the solar system
Example: Comet transport between outside and inside of Jupiter (i.e., Oterma-like transitions)
Phase volume ratio gives the relative probability to pass from outside to inside Jupiter’s orbit.
Transition probabilities

Jupiter family comet transitions: X → S, S → X
Capture time determined by tube dynamics

Temporary capture time profiles are structured

\[ \epsilon = 0.58 \]

\[ \epsilon = 0.6 \]
Collision probabilities

- eg, **Shoemaker-Levy 9** and **Earth-impacting asteroids**
- Compute from tube intersection with planet on Poincaré section
- Planetary diameter is a parameter
Collision probabilities

Poincaré section through planet showing collision portion of tube
Probability for comet collision with Jupiter

Collision Probability for Jupiter Family Comets

- **L1**
- **L2**

Energy vs. Collision probability (%)

- **SL9**
Collision Probability for Near-Earth Asteroids

Energy (scaled)
Collision probability (%)

Collision Probability for Near-Earth Asteroids

L1
L2

-4 -3.5 -3
$10^{-4}$

x $10^{-4}$
Coming from direction of sun; harder to detect; **surprise!**
Multi-scale dynamics: Part 2

- Slices of energy surface: Poincaré sections $U_i$
- Tube dynamics: evolution between $U_i$
- What about evolution on $U_i$?
Kicks at periapsis

Key idea: model particle motion as "kicks" at periapsis

In rotating frame where $m_1, m_2$ are fixed

Kicks at periapsis

- Sensitive dependence on argument of periapse $\omega$

In rotating frame where $m_1, m_2$ are fixed

Construct **update map** \((\omega_1, a_1, e_1) \mapsto (\omega_2, a_2, e_2)\)
using average perturbation per orbit by smaller mass.
Construct update map \( (\omega_1, a_1, e_1) \mapsto (\omega_2, a_2, e_2) \)
using average perturbation per orbit by smaller mass
Nearly integrable Hamiltonian

- Particle assumed on near-Keplerian orbit around $m_1$
- Hamiltonian in nearly integrable action-angle form

$$H(I, \theta) = H_0(I) + \mu H_1(I, \theta), \quad \mu \ll 1,$$

i.e.,

$$H(L, G, l, \omega) = K(L) - G + \mu R(L, G, l, \omega)$$

in Delaunay (action-angle) variables
Change in orbital elements over one particle orbit

- Evolution of $G$ (angular momentum)
  \[ \frac{dG}{dt} = -\mu \frac{\partial R}{\partial \omega}, \]

- Approximate change in $G$ over an orbit
  \[ \Delta G = -\mu \int_{\text{one orbit}} \frac{\partial R}{\partial \omega} \, dt \]

- $\Delta K = \text{Keplerian energy change}$ over an orbit
  \[ \Delta K = \Delta G - \mu \Delta R \]
Energy kick function

Changes have form

$$\Delta K = \mu f(\omega),$$

$f$ is the energy kick function with parameters $K, E$
Changes have form
\[ \Delta K = \mu f(\omega), \]

\( f \) is the **energy kick function** with parameters \( K, E \).
The periapsis kick map (Keplerian Map)

- Cumulative effect of **consecutive passes** by perturber

- Can construct an **update map**
  \[(\omega_{n+1}, K_{n+1}) = F(\omega_n, K_n)\] on the cylinder \(\Sigma = S^1 \times \mathbb{R}\),
  i.e., \(F : \Sigma \rightarrow \Sigma\) where
  \[
  \begin{pmatrix}
  \omega_{n+1} \\
  K_{n+1}
  \end{pmatrix} =
  \begin{pmatrix}
  \omega_n - 2\pi(-2(K_n + \mu f(\omega_n)))^{-3/2} \\
  K_n + \mu f(\omega_n)
  \end{pmatrix}
  \]

- **Area-preserving (symplectic twist) map**

- Ex.: particle in Jupiter-Callisto system, \(\mu = 5 \times 10^{-5}\)
Identify Keplerian map as Poincaré return map

\[ F(\omega, K) \]

Poincaré map at periapsis in orbital element space

\[ F : \Sigma \rightarrow \Sigma \text{ where } \Sigma = \{ l = 0 \mid H = E \} \]
Verification of Keplerian map: phase portrait

Keplerian map
Verification of Keplerian map: phase portrait

Keplerian map = fast orbit propagator

- preserves phase space features
  - but breaks left-right symmetry present in original system
  - can be removed using another method (Hamilton-Jacobi)
Dynamics of Keplerian map

Resonance zone

Structured motion around resonance zones

\[^{3}\text{in the terminology of MacKay, Meiss, and Percival [1987]}\]
Dynamics of Keplerian map

Structured motion around resonance zones

Resonance zone

Structured motion around resonance zones

in the terminology of MacKay, Meiss, and Percival [1987]
Large orbit changes via multiple resonance zones

Multiple flybys for orbit reduction or expansion\textsuperscript{5}

\textsuperscript{5}Grover & Ross, J. Guid. Cont. Dyn. [2009]
Large orbit changes, $\Gamma_n = F^n(\Gamma_0)$
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Reachable orbits and diffusion

Diffusion in semimajor axis

... increases with $E$ (larger kicks)
Reachable orbits: upper boundary for small $\mu$

A rotational invariant circle (RIC)

RIC found in Keplerian map for $\mu = 5 \times 10^{-6}$
Exit: where tube of capture orbits intersects $\Sigma$

Orbits reaching exit are ballistically captured, passing by $L_2$
Relationship to capture around perturber

Exit: where tube of capture orbits intersects $\Sigma$

Orbits reaching exit are **ballistically captured**, passing by $L_2$
Relationship to capture around perturber

Exit: where tube of capture orbits intersects $\Sigma$

- Orbits reaching exit are **ballistically captured**, passing by $L_2$
Relationship to capture from infinity

\[ K > 0 \quad \text{Hyperbolic orbits} \]

\[ \omega / \pi \]

\[ K < 0 \quad \text{Elliptical orbits} \]

Captured from infinity after previous periapsis

Jerg, Junge, Ross [2009]
Final word about Keplerian map

Extensions:

- out of plane motion \((4D \text{ map})\)
- control in the presence of uncertainty
- eccentric orbits for the perturbers
- multiple perturbers
  - transfer from one body to another

- Consider other problems with spatially localized perturbations?
  - chemistry, vortex dynamics, ...
Conclusions

- **Invariant manifold tubes** are related to transport across rank 1 saddles (saddle $\times$ center $\times \cdots \times$ center)

- In the restricted 3-body problem:

- **Tube dynamics**: the interior of tube manifolds — related to capture, escape, transition, collision

- **Keplerian map** provides analytical expression approximating a Poincaré map
Thank you!

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