On locating saddle-points on a surface using experiment data

Yawen Xu*, Lawrence N. Virgin* and Shane D. Ross **
*Department of Mechanical Engineering and Materials Science, Duke University, Durham, NC 27708, USA
**Engineering Mechanics Program, Virginia Tech, Blacksburg, VA 24061, USA

Abstract. Two methods to estimate the location of equilibria are developed in this study. We are primarily interested in conservative systems governed by smooth potential but with the addition of a small amount of damping. Estimating the location of unstable equilibria (typically saddle-points) is a challenge. The first method is essentially a static approach in which the restoring force is measured, while the second approach is based on (dynamic) transient trajectories of the system. Experiments are conducted to verify these two methods. The experimental approach consists of a small ball rolling on a relatively shallow curved track, or surface under the influence of gravity: a direct mimicry of the potential energy function on one or two dimensions. For the first (static) method, the slop of the surface is extracted from a measurement of the magnitude of force in horizontal direction that is required to balance the ball on the surface. For the second (dynamic) method, the motion of the ball that is tracked with a digital camera and the generated data compares well with the output of numerical simulation. The experimental results suggest that both methods can effectively locate the equilibria in the system, and thus provide a starting point for the consideration of more complicated dynamical systems. Both methods have advantages, the dynamic method provides information on the local dynamics; while the static method can locate all equilibria in the system simultaneously. Given the relative simplicity of the system, it is a useful testing environment for system identification in a nonlinear and higher-order context.

Introduction

Conservative dynamical systems are typically associated with an underlying potential energy, and one can easily imagine in a nonlinear context a trajectory meandering through phase space, encountering and passing-by both stable and unstable equilibria as it goes [1]. Given a little positive energy dissipation we would expect a trajectory to end up in a position of stable equilibrium (a potential energy minimum), but which final state persists depends crucially on the role played by unstable equilibria (saddles of varying degrees of instability and local maxima) as well as the initial conditions. However, by their very nature, unstable equilibria are very difficult to directly observe in an experiment.

The Approach

It is quite natural to ask the question: under what circumstances might trajectories *escape* the local confines of a potential energy minimum. In nonlinear systems we must consider stability *in-the-large*, where perturbations are not necessarily small and co-existing equilibria come into play [2]. This situation is quite straightforward for a single-degree-of-freedom (DOF) system in which the location of an adjacent unstable equilibrium (potential energy hill-top) is easily identified, and a trajectory effectively has no choice but to transition over the hilltop if it is to escape. However, in multi-DOF systems, the exit of trajectories would likely be influenced by the location of the various other equilibria that might be present as it explores the phase space, and this presents considerable challenges especially in an experimental context.

In order to fix these ideas in a relatively simple dynamical system setting, it is compelling to develop the analogy of a small ball rolling along a single-valued curved surface under the influence of gravity, [3] and observe the behavior of trajectories that live in this 2D configuration space (X, Y): with vertical height representing directly a potential energy *surface*, or landscape. In a nonlinear system the potential energy surface would typically contain a variety of turning points (valleys hilltops, saddles), and we are especially interested in saddle-points, that is, equilibria characterized locally by positive and negative curvature. Despite the presence of stable manifolds, they are still unstable, and by definition, would not be easily identified in experiments. However, they typically play a crucial organizing role in the global dynamics of a system.

The main goal of this study is: can we estimate the locations of unstable equilibria (saddle-points) based on limited information associated with experiment data, like trajectories or information based on the restoring force. The specific system chosen has the advantage that it can be modeled unambiguously, and hence provides relatively high-fidelity data for comparative purposes. It is also relatively easy to conduct experiments on, especially with the ability to machine a surface shape to high accuracy, and exploiting non-invasive sensors, e.g. a high-speed camera to extract the position of the ball in each frame and hence form the dynamic profile for the ball as it rolls as a function of time. The procedure simply consists of an initial guess of the location of equilibrium is required. Then the portions of the trajectories that falls into an area around the initial guess of the location of equilibrium are collected and used to calculate the location of the saddle point based on a fitting process. For the static method, we design a *cage*, the four walls containing sensitive load cells. By moving the cage above the surface with a ball inside, in a finely controlled manner, the force that is required to maintain the position of the ball on the surface is measured, and then the slope of the surface at a given location on the surface is backed-out. Again a fitting process is used to extract the shape of the curve or surface.

Sample results

We use both 1D and 2D surface to test these methods. Considering the system in figure 1(a) we have a (3D-printed) double well potential shape, an archetypal form used extensively in nonlinear dynamics [1]. In this 1D case the ball is pushed along the surface with two load cells recording the pushing force, which is then manipulated and integrated to obtain the corresponding potential energy (with two minima separated by a maximum). The same

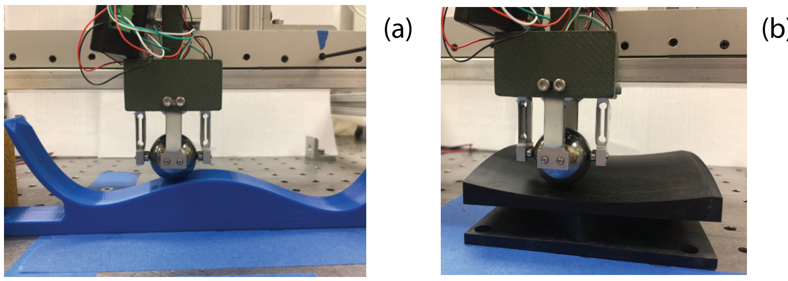

Figure 1: Pushing a ball along a surface in 1D, (b) similarly in 2D.

approach is sued for the 2D saddle shape in figure 1(b), an isolated (linear) saddle-point, but now with four load cells sued to extract the pushing force during traversing over the surface. Equilibria correspond to those points on the track or surface where the force drops to zero. In both of these cases, both approach gave very accurate estimates of the equilibrium locations.

Both methods were then applied to a system (in 2D) with a number of equilibria (in fact, two minima, two saddles and a hilltop). The results of using the static approach are shown in figure 2(a), and indicating the difference between numerical and experimental results. Finally, applying the dynamic method to this non-simple 2D surface

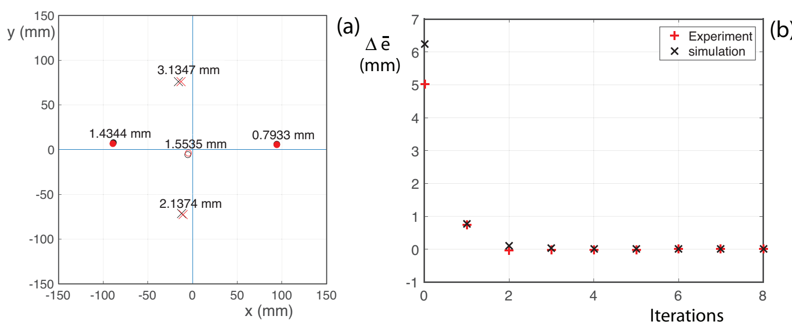

Figure 2: (a) Estimates of the equilibria locations using the static method, (b) convergence towards equilibria for the dynamic method.

led to the results shown in figure 2(b) in which the convergence of the estimates for a typical equilibrium position are shown as a function of the number of iterations used in the estimation procedure. Other aspects of the approaches were also tested, for example, the area of the *local* region, noise level (in the simulations), etc.

Conclusions

This paper presents techniques for estimating certain characteristics of a mass moving on a smooth track/surface, with a special focus on identifying the location of saddle-points, since they have a profound effect on global dynamic behavior.

Acknowledgement

This work was supported by the National Science Foundation under award number 1537349 and 1537425.

References

[1] Virgin, L. N. (2000) Introduction to Experimental Nonlinear Dynamics: a case study in mechanical vibration. Cambridge University Press.
[2] Thompson, J. M. T. and Stewart, H. B. (2002) Nonlinear Dynamics and Chaos. John Wiley & Sons.
[3] Virgin, L. N., Lyman, T. C. and Davis, R. B. (2010) Nonlinear dynamics of a ball rolling on a surface.*American Journal of Physics*, 78:250-57. 273-303.