Chaos in Space and Time

- Chris Mehrvarzi, Alireza Sedighi, and Mu Xu
- Faculty Sponsor: Prof. M. Paul
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Why study chaos in space and time?

Many phenomena in nature are systems far-from equilibrium and exhibit chaotic dynamics in both space and time.
Presentation Outline

Lattice map with “diffusive” coupling
  – Difference equations
Calculating Lyapunov vectors
  – System of ODEs
Transport in complex flow
  – Governing PDEs
Conclusions
Future directions
Coupled Map Lattice

\[ x_{i,j}^{(n+1)} = f(x_{i,j}^n) + D \left[ \frac{1}{2} (f(x_{i+1,j}^n) + f(x_{i-1}^n) \\
+ f(x_{i,j-1}^n) + f(x_{i,j+1}^n)) - f(x_{i,j}^n) \right] \]

\[ \delta x_{i,j}^{(n+1)} = f'(x_{i,j}^n) \delta x_{i,j}^{(n)} + D \left[ \frac{1}{2} (f'(x_{i+1,j}^n) \delta x_{i+1,j}^{(n)} \\
+ f'(x_{i-1,j}^n) \delta x_{i-1,j}^{(n)} + f'(x_{i,j-1}^n) \delta x_{i,j-1}^{(n)} \\
+ f'(x_{i,j+1}^n) \delta x_{i,j+1}^{(n)} - f'(x_{i,j}^n) \delta x_{i,j}^{(n)} \right] \]

\[ f(x_{i}^n) = ax_{i}^n (1 - x_{i}^n) \]
Coupled Map Lattice

Gram-Schmidt Method

In a chaotic system, each vector tends to fall along the local direction of most rapid growth.

To overcome the problem, Gram-Schmidt method was implemented:

\[
\delta x_1'(n) = \frac{\delta x_1(n)}{\|\delta x_1(n)\|},
\]

\[
\delta x_2'(n) = \frac{\delta x_2(n) - \left(\delta x_2(n), \delta x_1'(n)\right)\delta x_1'(n)}{\|\delta x_2(n) - \left(\delta x_2(n), \delta x_1'(n)\right)\delta x_1'(n)\|},
\]

\[
\vdots
\]

\[
\delta x_n'(n) = \frac{\delta x_n(n) - \left(\delta x_n(n), \delta x_{n-1}'(n)\right)\delta x_{n-1}'(n) - \cdots - \left(\delta x_n(n), \delta x_1'(n)\right)\delta x_1'(n)}{\|\delta x_n(n) - \left(\delta x_n(n), \delta x_{n-1}'(n)\right)\delta x_{n-1}'(n) - \cdots - \left(\delta x_n(n), \delta x_1'(n)\right)\delta x_1'(n)\|}.
\]

\[
\lambda_1^* = \lim_{n \to \infty} \frac{1}{n\Delta t} \sum_{i=1}^{n} \ln \|\delta x_1(n)\|,
\]

\[
\lambda_2^* = \lim_{n \to \infty} \frac{1}{n\Delta t} \sum_{i=1}^{n} \ln \|\delta x_2^*(n)\|,
\]

\[
\vdots
\]

\[
\lambda_n^* = \lim_{n \to \infty} \frac{1}{n\Delta t} \sum_{i=1}^{n} \ln \|\delta x_n^*(n)\|.
\]
Coupled Map Lattice

One-dimensional

\[ D_\lambda = j + \sum_{i=1}^{j} \lambda_i \]

\[ D_\lambda = 91.04e^{(-1.584D)} \]

\[ a = 3.7 \]

\[ D = 0.4 \]
Coupled Map Lattice

Two-dimensional

\[ \Sigma x_i \]

\[ D_x \]

\[ a = 3.7 \]

\[ D = 0.4 \]

\[ D_\lambda \propto \Gamma^d \]
Lyapunov Vectors

Covariant Lyapunov Vectors

Pros:
• True direction in phase space.
• Reflect the direction of perturbation
• Test hyperbolicity

Cons:
• Difficult to calculate
• Algorithm only recently available (Ginelli (2007) and Pazo (2007))

Orthogonal Lyapunov Vectors

Pros:
• Easy to calculate
• Leading order Lyapunov vector is in correct direction
• Can calculate fractal dimension

Cons
• Lose all direction except leading order
Lorenz System

\[ \frac{dx}{dt} = \sigma(x - y) \]
\[ \frac{dy}{dt} = x(\rho - z) - y \]
\[ \frac{dz}{dt} = xy - \beta z \]

\[ \sigma = 10 \]
\[ \rho = 28 \]
\[ \beta = \frac{8}{3} \]
Results of Covariant Lyapunov Vectors

The direction of the second covariant Lyapunov vector and the direction of the tangent vector should be same.
The Lyapunov exponents from different algorithm should agree with each other.

\[ \lambda(\delta \tilde{x}_0) = \lim_{t \to +\infty} \frac{1}{t} \ln(M(\tilde{x}_0, t) \delta \tilde{x}_0) \]
The size of system does not influence the hyperbolicity of the system.
Boussinesq Equations

\[ \sigma^{-1} \left( \frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} \right) = -\nabla p + \nabla^2 \bar{u} + RT \bar{z} \]

\[ (\frac{\partial T}{\partial t} + (\bar{u} \cdot \nabla) T) = \nabla^2 T \]

\[ \nabla \cdot \bar{u} = 0 \]

Advection-Diffusion Equation

\[ \frac{\partial c}{\partial t} + (\bar{u} \cdot \nabla) c = L \nabla^2 c \]

R = \frac{\alpha gd^3}{\nu \kappa} \Delta T \quad L = \frac{D}{\kappa}

Ning et al. (2009)
Direct Numerical Simulations

Pr = 1

Ra = 3000, Le = 0.1, Aspect Ratio = 10

Pr = 1

Ra = 3000, Le = 0.001, Aspect Ratio = 10
Spreading of Species

\[ [L]^2 \propto D[T] \]

Normal Diffusive Transport

\[ \frac{V_q^{2/q}}{V} = \text{const.} \]

\[ \frac{\partial \tilde{c}}{\partial t} = L^* \frac{\partial^2 \tilde{c}}{\partial r^2} \]
Enhanced Transport

\[ \Delta = \frac{L^* - L}{L} \]

\[ P = \frac{||\bar{u}||}{L} \]

\[ L^* - L \propto \left( \frac{R - R_c}{R_c} \right)^{1/2} \]
Conclusions and Future Directions

- Fractal dimension proportional to map lattice size
- Hyperbolicity was not influenced by lattice size
- Two transport enhancement regimes due to spatiotemporal chaotic flow field
- Calculate covariant Lyapunov vectors in Rayleigh-Bénard convection
- Conduct formal study on influence of system size