Multi-Flexible-Body Analysis for Application to Wind-Turbine Control Design

Dewey H. Hodges* and Mayuresh Patil†
Georgia Institute of Technology, Atlanta, Georgia 30338-0150

Abstract

A methodology is presented for the modeling of flexible horizontal-axis, wind-turbine systems. The multi-rigid-body portions of the system are modeled as a system of interconnected rigid bodies using Autolev, a symbolic manipulator ideally suited for dynamics of multi-rigid-body systems. The flexible portions of the wind-turbine system are represented as beam finite elements. Equations are presented for the beam element, and the choices of generalized coordinates and generalized speeds for the multi-rigid-body portion of the system are discussed. Finally, a method for the explicit linearization of the system equations is presented, along with a method for the unification of these equations into one framework.

Introduction

Wind is one of the most environmentally friendly, as well as abundant, sources of energy. Wind turbines are being designed for power generation on land and off shore. Though wind turbines do not require any form of fuel, the infrastructure required is quite expensive. To make wind energy conversion cost effective requires i) the wind turbines to operate at the best possible performance, and ii) the wind turbine’s operational life to be as long as possible.

The wind turbine performance is related to various design parameters and control system design [1]. To achieve the above-mentioned goals requires firstly a systematic analysis and design of the complete wind turbine system followed by efficient control design to achieve high performance and gust load alleviation. Thus, a realistic model of the wind turbine system and its interaction with wind is very important.

Dynamic analysis of wind turbines has been the topic of many earlier works. Lobitz [2] conducted dynamic response analysis of HAWT and VAWT wind turbines, modeling the HAWT wind turbine system as a constant-speed rotor attached to a flexible tower. The blades and tower were modeled as a collection of beam finite elements. The analysis was conducted using a DMAP-modified version of NASTRAN to couple the components where rotation is involved. Garrad and Quarton [3] have used symbolic computing to obtain equations of motion of the entire wind turbine system. Results are presented for a simplified three degree of freedom tower and blade model. Sheinman and Rosen [4] have presented a complete analysis of a horizontal axis wind turbine (HAWT) by modeling the rotor as well as the gear-box, generator, shafts, couplings and brakes. The analysis was used for performance calculations. Bir and Butterfield [5] have analyzed a wind turbine with a flexible rotor and soft tower to investigate the modal dynamics of a HAWT.

A wind-turbine model, if sufficiently low-order and high-fidelity, can be used for control design. This is the challenge of control design for a complex system. If the order of the model is not sufficiently low, then certain control design techniques cannot be used at all. On the other hand, if the model is not sufficiently high-fidelity, then the designed controller may not work at all. De La Salle et al. [6] give an in-depth review of control methodology used for wind turbine control. It is clear from their discussion that the wind turbine model should be as simple as possible while retaining all significant dynamic components and model the different components to comparable degree of complexity. That is exactly what the authors are proposing. The wind turbine model should model the flexible and rigid-body parts with the same degree of accuracy. Later, when an aerodynamic model is added, it too should have an equivalent predictive ability. The wind turbines of the future are likely to be light and flexible. To understand
and predict the dynamics of wind turbines one needs to accurately characterize the flexible components. The wind turbine modeling approach outlined herein will lead to a low-order, high-fidelity model which can be easily modified to predict the performance at various operating conditions to desired accuracy.

Present Work

A new methodology for wind turbine code development is presented in this paper. The methodology is based on breaking a generic, 2-bladed, teetering-rotor HAWT into flexible and multi-rigid-body subsystems. Specifically because of the intended use of the code, the analysis is tailored to minimize the complexity associated with the multi-rigid-body part of the system, which is accomplished by using Kane’s method [7]. This is known to lead to simpler equations of motion than conventional methods [8]. Simultaneously, the methodology must provide the needed fidelity for the flexible elements. These elements are geometrically-exact, beam finite elements [9], capable of representing accurately initially curved and twisted composite beams undergoing large deformation. Some control design methods require explicit expressions for the elements of the dynamic matrices, and this method provides that.

The flexible members consist of a flexible tower, shaft, and blades; whereas, the multi-rigid-body portion consists of a rigid hub and of a collection of interconnected bodies which make up the nacelle. The hub has the teetering degree of freedom relative to the rotor shaft, and the nacelle bodies have yaw, pitch, and rotor spin degrees of freedom. For the flexible subsystems (tower, shaft, and blades), the tower and blades are represented by geometrically-exact mixed finite elements, whereas the shaft is a single, linear compliance finite element connected to rigid bodies at its ends representing its mass and inertia. The equations for the rigid bodies and connectivity among them are derived by use of Autolev [10]. Other than the use of Rodrigues parameters for finite rotation of the structural nodes (see [11, 12, 9]), the choice of generalized coordinates is quite standard. However, a specific choice of generalized speeds is adopted for rotations about an axis fixed in each of two bodies which leads to much simpler equations than would be the case were the derivative of the rotation angle used [13].

As is well known, Kane’s method facilitates the recovery of constraint forces in the multi-rigid-body parts of the system without the use of Lagrange multipliers. Moreover, the use of the mixed method allows for (1) direct determination of constraint forces and moments within the beam elements and at their boundaries and (2) for simple connectivity between the finite elements and the rigid bodies. Indeed, the generalized coordinates and generalized speeds at the connection points are the boundary values of the corresponding quantities used in the mixed finite element approach. It should be noted that the mixed finite element equations must be put into a specific form to achieve this simplicity.

The discrete portions of the model (the hub and the bodies that make up the nacelle) are modeled using the symbol manipulator Autolev. Autolev produces a complete FORTRAN code for the discrete part which is coupled with equations for the flexible components. Autolev’s use is limited to the interconnected rigid bodies which make up the nacelle subsystem, the compliance model for the shaft, the hub, and the details of the connection between the beam elements and the multi-rigid-body subsystem. The rest of the tower and blade elements are modeled by using mixed finite elements in the way they’ve been used in other applications to date by the first author and his co-workers (for example, see [14]). The primary goal of the present paper is to demonstrate the feasibility of the present modeling approach for dynamics analysis of a wind-turbine system.

Theoretical Basis

In this section the theoretical basis for the modeling approach is described in detail. In particular, the beam finite element method and the method for handling the multi-rigid-body part of the system are described.

Treatment of Flexible Subsystems

All flexible elements are represented as beams using mixed finite elements. Here we present the derivation of the equations from the spatially weakest form. The lower the order of the highest-order derivative in the variational statement, the weaker it is said to be. In the spatially weakest form, no unknowns are spatially differentiated. A beam is assumed to be clamped at one of its ends, and can represent either the blade or the tower with an appropriate choice of conditions at the other end.

We start with the weakest variational formulation given in Eq. (74) of [9]. The equation is reduced for application to a straight, untwisted beam. Moreover, the weak form is integrated by parts in time and the time integration is removed from the form as given in
This way only the spatial dependence is accounted for in the finite element modeling. The weak form then reduces to

\[
\int_0^\ell \left\{ \delta q^T \epsilon - \delta \bar{q}^T \kappa - \delta \widetilde{\psi}^T (\bar{e}_1 + \bar{\gamma}) \right\} F - \delta q^T mg_B \\
+ \delta \bar{q}^T \dot{P} + \left( \delta \bar{q}^T \Omega + \delta \widetilde{\psi}^T \dot{V} \right) P + \left( \delta \widetilde{\psi}^T \kappa - \delta \bar{q}^T \dot{\kappa} \right) M \\
+ \delta \bar{\psi}^T \left( \ddot{H} + \ddot{\Omega} + \ddot{\psi}^T \dot{V} \right) [e_1 - C^T (e_1 + \gamma)] \\
- \delta \bar{F}^T u - \delta \bar{M}^T \left( \Delta + \frac{1}{2} \overline{\theta} + \frac{1}{4} \overline{\theta}^T \overline{\theta} \right) \kappa - \delta \bar{M}^T \theta \\
- \delta \bar{P}^T \left[ v + \tilde{\omega} (x_1 e_1 + u) - C^T V + \dot{u} \right] \\
- \delta \bar{H}^T \left[ \left( \Delta + \frac{1}{2} \overline{\theta} + \frac{1}{4} \overline{\theta}^T \right) \left(C \omega - \Omega \right) + \dot{\theta} \right] \right] dx_1 \\
= \left( \delta q^T F + \delta \bar{q}^T M - \delta \bar{F}^T \dot{u} - \delta \bar{M}^T \dot{\theta} \right)_1^\ell 
\]

where \( u \) is the column matrix of displacement measures of the beam reference line in the \( b \) basis (the undeformed beam cross-sectional frame basis), \( \theta \) is the column matrix of Rodrigues parameters relating the \( B \) basis (the deformed beam cross-sectional frame basis) to the \( b \) basis so that the matrix of direction cosines \( C \) relating the bases of \( B \) and \( b \) is given in terms of \( \theta \) by

\[
C = \frac{\Delta (1 - \frac{1}{2} \overline{\theta} T \overline{\theta}) - \tilde{\theta} + \frac{1}{4} \overline{\theta} \overline{\theta} T}{1 + \frac{1}{2} \overline{\theta} T \overline{\theta}} 
\]

\( F \) is the column matrix of section force resultant measures in the \( B \) basis, \( M \) is the column matrix of section moment resultant measures in the \( B \) basis, \( P \) is the column matrix of section linear momentum measures in the \( B \) basis, \( \gamma \) is the column matrix of force strains, \( \kappa \) is the column matrix of moment strains, \( V \) is the column matrix of velocity measures of the beam reference line in the \( B \) basis, \( \Omega \) is the column matrix of cross-sectional angular velocity measures in the \( B \) basis, \( \gamma \) is the column matrix of virtual displacement measures in the \( B \) basis, \( \dot{\psi} \) is the column matrix of virtual rotation measures in the \( B \) basis, \( \bar{\psi}^T \) is the column matrix of virtual linear momentum measures transformed to the \( b \) basis, \( \bar{M}^T \) is a column matrix of virtual angular momentum test functions, all of which are completely defined in [9] and

\[
g_B = \begin{cases} 
\delta q_B \\
\delta \bar{q}_B \\
\delta \bar{p}_B 
\end{cases} 
\]

where \( g_B = -g_n \cdot B_1 \), and \( n_1 \cdot B_1 = C_{ij}^B n_1 \) and the vectors \( n_i \) and \( B_i \) are the basis vectors for the Newtonian frame \( n \) and \( B \), respectively. Overbars on the test functions (i.e., the \( \delta \)-quantities) indicate that these quantities are not the variations of functions. As discussed in [9], for example, the virtual displacement \( \delta \bar{q} = C \delta u \), where \( \delta u \) is the variation of \( u \). Hats over certain quantities indicate values at the ends of the beam. Finally, the tildes over certain quantities indicate the dual matrix of the column matrix over which the tilde is placed. For example,

\[
\hat{\theta} = \begin{bmatrix} \theta_1 \\
\theta_2 \\
\theta_3 
\end{bmatrix} \\
\tilde{\theta} = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\
\theta_3 & 0 & -\theta_1 \\
-\theta_2 & \theta_1 & 0 
\end{bmatrix} 
\]

Also, for simplicity the first mass moments of inertia are assumed to be zero, so that \( \bar{\xi}_2 = \bar{\xi}_3 = 0 \). Upon this simplification the sectional linear and angular momenta are given by

\[
\begin{bmatrix} P \\
H \end{bmatrix} = \begin{bmatrix} m \Delta & 0 \\
0 & \rho I \end{bmatrix} \begin{bmatrix} V \\
\Omega \end{bmatrix} 
\]

where \( m \) is the constant mass per unit length of the beam element, \( \rho \) is the material density, and \( I \) is the inertia matrix of the cross-sectional area given by

\[
I = \begin{bmatrix} I_2 + I_3 & 0 & 0 \\
0 & I_2 & 0 \\
0 & 0 & I_3 
\end{bmatrix} 
\]

By virtue of Eq. (5), the quantities \( P \) and \( H \) are eliminated in favor of \( V \) and \( \Omega \), the “generalized speeds” of the beam element. Moreover, \( F \) and \( M \) are related to \( \gamma \) and \( \kappa \) in accordance with the 1-D constitutive law for an isotropic beam:

\[
\begin{bmatrix} F \\
M \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\
0 & GK_2 & 0 \\
0 & 0 & GK_3 \end{bmatrix} \gamma \\
\begin{bmatrix} GJ & 0 & 0 \\
0 & EI_2 & 0 \\
0 & 0 & EI_3 \end{bmatrix} \kappa 
\]

where \( EA \) is the axial rigidity, \( GK_2 \) and \( GK_3 \) are the shear rigidities along the directions \( b_2 \) and \( b_3 \), \( GJ \) is the torsional rigidity, and \( EI_2 \) and \( EI_3 \) are the bending rigidities about the directions \( b_2 \) and \( b_3 \). The transverse shear stiffnesses may be replaced with numbers sufficiently large to essentially lock out transverse shear without significant loss of accuracy in low-frequency modes of slender beams.
For finite element implementation, we first note that certain quantities are never differentiated with respect to the spatial coordinate. For example, the test functions $\delta F$ and $\delta H$ and the unknown quantities $u, \theta, V,$ and $\Omega$ may be approximated as functions of time only (i.e., spatially constant) within each element, regardless of the number of finite elements. The resulting equations provide the kinematical differential equations for each element in terms of constant trial functions within the $i^{th}$ element, viz.,

$$
\begin{align*}
  u &= \overline{u}_i \\
  \theta &= \overline{\theta}_i \\
  V &= \overline{V}_i \\
  \Omega &= \overline{\Omega}_i
\end{align*}
$$

where the overbar on the unknowns indicates the spatially constant value of that unknown within the $i^{th}$ finite element. The resulting equations can be solved for the time derivatives of $\overline{u}_i$ and $\overline{\theta}_i$, the element “generalized coordinates” in this framework, in terms of the following quantities: $\overline{u}_i$ and $\overline{\theta}_i$; $\overline{V}_i$ and $\overline{\Omega}_i$, the element generalized speeds; and $v$ and $\omega$, the generalized speeds of the body/frame to which the element is attached. This is typical of the form of kinematical differential equations in Kane’s method [7]. Thus, one finds that the element kinematical differential equations become

$$
\begin{align*}
  \overline{u}_i &= \overline{C}_i^T \overline{V}_i - v - \overline{\varphi} \left( \frac{\tau}{2} e_1 + \overline{u}_i \right) \\
  \overline{\theta}_i &= \left( \Delta + \frac{1}{2} \overline{\theta}_i + \frac{1}{4} \overline{\theta}_i \overline{\gamma} \right) \left( \overline{C}_i - \overline{C} \overline{\gamma} \right)
\end{align*}
$$

where $\overline{C}_i$ is the spatially constant value of $C$ within the $i^{th}$ finite element.

Now focusing on the remaining terms in the weak form, we group them according to the test functions, resulting in

$$
\int_0^t \left\{ \delta q^T F + \delta \overline{q}^T \left( \overline{P} + \overline{\Omega} P - \overline{\kappa} F - m g \overline{B} \right) + \delta \overline{\psi}^T \overline{M} \\
+ \delta \overline{\psi}^T \left[ \overline{H} + \overline{\Omega} \overline{H} + \overline{P} - \overline{\kappa} M - (\overline{e}_1 + \overline{\gamma}) F \right] \\
- \delta F^T u + \delta \overline{F}^T \left[ e_1 - C^T (e_1 + \gamma) \right] \\
- \delta M^T \theta - \delta \overline{M}^T \left( \Delta + \frac{1}{2} \overline{\theta} + \frac{1}{4} \bar{\theta} \overline{\gamma} \right) \right\} \overline{d}x_1 \\
- \left( \delta q^T \overline{F} + \delta \overline{\psi}^T \overline{M} - \delta \overline{F}^T \overline{u} - \delta \overline{M}^T \overline{\theta} \right) \bigg|_0^t = 0
$$

There are 4 test functions which are spatially differentiated within this weak form; none of the unknowns is ever differentiated spatially, nor are the remaining test functions. Thus, the differentiated test functions can be assumed to be linear within the element, so that (for the $i^{th}$ element)

$$
\begin{align*}
  \delta q &= \overline{q}_i (1 - \eta) + \overline{q}_{i+1} \eta \\
  \delta \overline{\psi} &= \overline{\psi}_i (1 - \eta) + \overline{\psi}_{i+1} \eta \\
  \delta F &= \overline{F}_i (1 - \eta) + \overline{F}_{i+1} \eta \\
  \delta M &= \overline{M}_i (1 - \eta) + \overline{M}_{i+1} \eta
\end{align*}
$$

where $\eta$ is the dummy variable within the element. The remaining quantities are set equal to functions of time only (spatially constant quantities within the $i^{th}$ element), so that

$$
\begin{align*}
  \gamma &= \overline{\gamma} \\
  \kappa &= \overline{\kappa} \\
  F &= \overline{F} \\
  M &= \overline{M}
\end{align*}
$$

Finally, in accordance with the bi-discontinuous Galerkin method, we assume distinct values of $u, \theta, F,$ and $M$ at the element boundaries. With this simplification the integration in Eq. (10) can be performed in closed form to yield

$$
\sum_{i=0}^{n} \left\{ \left( \overline{q}_{i+1} - \overline{q}_i \right)^T \overline{F} + \left( \overline{\psi}_{i+1} - \overline{\psi}_i \right)^T \overline{M} \\
+ \left( \overline{q}_i + \overline{q}_{i+1} \right)^T \frac{\Delta x}{2} \left( \overline{P} + \overline{\Omega} \overline{P} - \overline{\kappa} F - m g \overline{B} \right) \\
+ \left( \overline{\psi}_i + \overline{\psi}_{i+1} \right)^T \frac{\Delta x}{2} \left( \overline{H} + \overline{\Omega} \overline{H} - \overline{\kappa} M \right) \\
- \left( \overline{e}_1 + \overline{\gamma} \right) F - \left( \overline{\delta F}_{i+1} - \overline{\delta F}_i \right)^T \overline{\pi} \\
- \left( \overline{\delta M}_{i+1} - \overline{\delta M}_i \right)^T \overline{\theta} \\
+ \left( \overline{\delta F}_{i+1} + \overline{\delta F}_i \right)^T \frac{\Delta x}{2} \left[ e_1 - \overline{C}^T (e_1 + \gamma) \right] \\
- \left( \overline{\delta M}_{i+1} + \overline{\delta M}_i \right)^T \frac{\Delta x}{2} \left( \Delta + \frac{1}{2} \overline{\theta} + \frac{1}{4} \bar{\theta} \overline{\gamma} \right) \right\} = 0
$$

where $n_e$ is the number of elements used to model the beam. From this equation, taking into account the arbitrariness of the $\delta$-quantities, we can readily obtain the equations relating various hatted and barred quantities, once appropriate boundary conditions are specified.

**Treatment of Blades** Keeping in mind that the blades are clamped to the hub $H$ at $x = 0$ and free at $x = \ell$
and that $v$ and $\omega$ are the generalized speeds for the hub, $H$, we obtain the following equations:

$$\delta q_1 : = - F_1 + \frac{\Delta x}{2} \left( \tilde{p}_1 + \bar{\tilde{p}}_1 \bar{p}_1 - \tilde{p}_1 \bar{p}_1 - m \bar{g} \bar{p} \right)$$

$$+ \tilde{F}_0 = 0$$

$$\delta q_i : = - F_{i+1} + \frac{\Delta x}{2} \left( \tilde{p}_{i+1} + \bar{\tilde{p}}_{i+1} \bar{p}_{i+1} \right.$$  

$$- \tilde{p}_{i+1} \bar{p}_{i+1} - m \bar{g} \bar{p} + \bar{F}_i$$  

$$+ \frac{\Delta x}{2} \left( \tilde{p}_i + \bar{\tilde{p}}_i \bar{p}_i - \tilde{p}_i \bar{p}_i - m \bar{g} \bar{p} \right) = 0$$

$$\delta \psi_1 : = - M_1 + \frac{\Delta x}{2} \left[ \tilde{h}_1 + \bar{\tilde{h}}_1 \bar{h}_1 - \tilde{h}_1 \bar{h}_1 \right.$$  

$$- (\tilde{e}_1 + \bar{\tilde{e}}_1) \bar{F}_1 \left] + \tilde{M}_0 = 0 \right.$$

$$\delta \psi_i : = - M_{i+1} + \frac{\Delta x}{2} \left[ \tilde{h}_{i+1} + \bar{\tilde{h}}_{i+1} \bar{h}_{i+1} \right.$$  

$$- \tilde{h}_{i+1} \bar{h}_{i+1} - (\tilde{e}_i + \bar{\tilde{e}}_{i+1}) \bar{F}_{i+1} \left] + \bar{M}_i = 0 \right.$$

$$\delta \psi_{i+1} : = M_n + \frac{\Delta x}{2} \left[ \tilde{h}_n + \bar{\tilde{h}}_n \bar{h}_n - \tilde{h}_n \bar{h}_n \right.$$  

$$- (\tilde{e}_1 + \bar{\tilde{e}}_n) \bar{F}_n \right] = 0$$

$$\delta F_1 : = \bar{u}_1 + \frac{\Delta x}{2} \left[ \tilde{e}_1 - C_i \bar{e}_1 + \bar{\tilde{e}}_1 \bar{e}_{i+1} \right] = 0$$

$$\delta F_i : = \bar{u}_{i+1} + \frac{\Delta x}{2} \left[ \tilde{e}_i - C_{i+1} \bar{e}_i + \bar{\tilde{e}}_{i+1} \bar{e}_{i+1} \right]$$

$$- \bar{u}_i + \frac{\Delta x}{2} \left[ \tilde{e}_i - C_i \bar{e}_i + \bar{\tilde{e}}_i \bar{e}_{i+1} \right] = 0$$

$$\delta F_{n+1} : = - \bar{u}_n + \frac{\Delta x}{2} \left[ \tilde{e}_i - C_n \bar{e}_i + \bar{\tilde{e}}_n \bar{e}_n \right] + \dot{u}_{n+1} = 0$$

$$\delta M_i : = \bar{\theta}_i - \frac{\Delta x}{2} \left( \Delta + \bar{\tilde{\theta}}_{i+1} \bar{\theta}_i \right) \kappa_{i+1} = 0$$

$$\delta M_i : = \bar{\theta}_{i+1} - \frac{\Delta x}{2} \left( \Delta + \bar{\tilde{\theta}}_{i+1} \bar{\theta}_{i+1} \right) \kappa_{i+1}$$

$$- \bar{\theta}_i - \frac{\Delta x}{2} \left( \Delta + \bar{\tilde{\theta}}_i \bar{\theta}_i \right) \kappa_i = 0$$

$$\delta \tilde{M}_{i+1} : = - \bar{\theta}_{i+1} - \frac{\Delta x}{2} \left( \Delta + \bar{\tilde{\theta}}_{i+1} \bar{\theta}_{i+1} \right) \kappa_{i+1}$$

$$+ \dot{\theta}_{i+1} = 0$$

(14)

In addition we note that the constitutive law, Eq. (7), gives a way to relate the element section force ($\bar{F}$) and moments ($\bar{M}$) to $\bar{\tau}$ and $\bar{\pi}$. Hence, the element force and moment strains can be eliminated in terms of element section forces and moments. Coupling with kinematic equations (see Eq. 9) a complete set of equations is obtained. It should be noted here that the root forces ($\bar{F}_0$) and moments ($\bar{M}_0$) can either be explicitly written in terms of the other variables or considered as independent variables. In either case they can be easily transferred to the discrete portion of the system.
rotor shaft, are treated via Autolev. Autolev command files generate both the nominal and perturbation equations and corresponding FORTRAN codes.

The multi-rigid-body portion of the wind-turbine model is comprised of the collection of rigid bodies in Fig. 1. This model consists of six rigid bodies (A, Y, P, R, M, and H) and a massless beam finite element to represent the shaft, S. Body A is attached rigidly to the end of the tower at point $A_T$. Body $Y$ yaws with respect to $A$ about an axis common to both which is parallel to $a_1 = y_1$ and passes through a point fixed in both bodies, $A_Y$. Body $P$ pitches with respect to $Y$ about an axis common to both which is parallel to $y_2 = p_2$ and passes through a point fixed in both bodies, $Y_P$. Body $R$ rotates as part of the rotor shaft with respect to $P$ about an axis common to both which is parallel to $p_3 = r_3$ and passes through a point fixed in both bodies, $P_R$. The shaft $S$ is a massless beam finite element which is clamped to $R$ at the point $R_S$ (which lies along the axis of rotation of $R$ in $P$); see Fig. 2 for additional details about the layout associated with the rotor shaft and rigid bodies associated therewith. At the other (i.e., the hub) end of $S$, a rigid body $M$ is rigidly attached to $S$ which represents the rigid mast to which the body $H$, the hub, is attached with a hinge; $H$ teeters with respect to $M$ about an axis which is parallel to $m_2 = h_2$ and passes through $M_H$. Blades are attached to $H$ at points fixed in $H$, the locations of which depend on the number of blades. For the two-bladed case, these points are designated as $H_{U_1}$ and $H_{D_0}$, associated with blades $U$ and $D$ – $\omega$ (1) and $\phi$ (2), respectively.

The generalized coordinates for the discrete portion are defined in terms of the three displacement measures of the point $A_T$ in the $n$ basis ($n$ is the Newtonian frame); the three Rodrigues parameters which define the rotation of $A$ relative to $n$; a yaw angle $q_6$ of $Y$ relative to $A$; a pitch angle $q_6$ of $P$ relative to $Y$; an angle of rotation for $R$ relative to $P$ which is equal to $q_7 + P \omega^R t$ where $P \omega^R$ is a given constant part of the total angular speed of $R$ in $P$; $q_{i}$, for $i = 1, 2, \ldots, 6$ which represent the usual finite element displacements and rotations at the end node of $S$ (the point $M_S$) in $R$ (where 1 and 2 are lateral translations, 3 is axial translation, 4 and 5 are rotations induced by bending, and 6 is rotation induced by torsion); and $q_6$ which represents the teetering angle of $H$ relative to $M$.

The generalized speeds are $u_i = n v^A_T \cdot a_i$, $i = 1, 2, \text{ and } 3$, the inertial velocity measures of $A_T$ in the $A$ basis; $u_{3+i} = n \omega^A_3 \cdot a_i$, $i = 1, 2, \text{ and } 3$, the inertial angular velocity measures of $A$ in the $A$ basis; $u_7 = n \omega^Y \cdot y_1$, the dot product of $y_1$ with the inertial angular velocity of $Y$; $u_8 = n \omega^P \cdot p_2$, the dot product of $p_2$ with the inertial angular velocity of $P$; $u_9 = n \omega^R \cdot r_3$, the dot product of $r_3$ with the inertial angular velocity of $R$; $u_{9+i} = n v^R_H \cdot h_i$, $i = 1, 2, \text{ and } 3$, the inertial velocity measures of $H_O$ in the $H$ basis; $u_{13} = n \omega^H \cdot m_2$, the dot product of $m_2$ with the inertial angular velocity of $M$, and $u_{13+i} = n \omega^H \cdot h_i$, $i = 1, 2, \text{ and } 3$, the inertial angular velocity measures of $H$ in the $H$ basis. In particular, generalized speeds $u_7$, $u_8$, and $u_9$ are those suggested by [13].

The kinematical differential equations in conventional methods of analytical dynamics are quite simple. Namely, the generalized speeds are simply the time-derivatives of the generalized coordinates. This normally results in very complex equations of motion. With modest increase in the complexity of the kinematical differential equations, one can many times greatly simplify the resulting equations of motion. With the above definitions of generalized speeds, the kinematical differential equations can be easily obtained. The way this is done is to write the velocity of a point or the angular velocity of a body in two ways: (1) in terms of the derivatives of generalized coordinates and (2) in terms of the chosen generalized speeds. These two expressions can then be equated to solve for the time-derivatives of the generalized coordinates in terms of the generalized coordinates and generalized speeds. The invertibility of these relations (i.e., the possibility to write the generalized speeds in terms of time-derivatives of the generalized coordinates and vice-versa) is a necessary condition for generalized speeds to be well-defined. This is done for the present case by starting with the points and bodies attached to the Newtonian frame and working out toward the blades.

The shaft is axisymmetric and exerts reaction forces at the points $S_M$ and $R_S$ which are linear in the shaft generalized coordinates and expressed in the $R$ basis (recalling that $R$ is the body to which the shaft is clamped). These are given by

$$\mathbf{F}_{M_S} = -\left(12 \frac{EI_s}{\ell_s} q_{s_1} - 6 \frac{EI_s}{\ell_s^2} q_{s_2}\right) \mathbf{r}_1 - \left(12 \frac{EI_s}{\ell_s} q_{s_2} + 6 \frac{EI_s}{\ell_s^2} q_{s_3}\right) \mathbf{r}_2 - \frac{EA_s}{\ell_s} q_{s_3} \mathbf{r}_3$$

and

$$\mathbf{F}_{R_S} = \left(12 \frac{EI_s}{\ell_s} q_{s_1} - 6 \frac{EI_s}{\ell_s^2} q_{s_2}\right) \mathbf{r}_1 + \left(12 \frac{EI_s}{\ell_s} q_{s_2} + 6 \frac{EI_s}{\ell_s^2} q_{s_3}\right) \mathbf{r}_2 + \frac{EA_s}{\ell_s} q_{s_3} \mathbf{r}_3$$
respectively. Also, the shaft exerts reaction torques on the bodies \( M \) and \( R \). These also are linear in the shaft generalized coordinates and expressed in the \( R \) basis. These are given by

\[
\begin{align*}
\mathbf{T}_M &= -\left(\frac{EI_s}{\ell_s^2} q_s + 4 \frac{EI_s}{\ell_s} q_s r_1 \right) \mathbf{r}_1 \\
&\quad - \left( -6 \frac{EI_s}{\ell_s} q_s + 4 \frac{EI_s}{\ell_s} q_s \right) \mathbf{r}_2 - \frac{GJ_s}{\ell_s} q_s \mathbf{r}_3
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{T}_R &= -\left(6 \frac{EI_s}{\ell_s} q_s - 2 \frac{EI_s}{\ell_s} q_s \right) \mathbf{r}_1 + \\
&\quad \left(6 \frac{EI_s}{\ell_s} q_s - 2 \frac{EI_s}{\ell_s} q_s \right) \mathbf{r}_2 + \frac{GJ_s}{\ell_s} q_s \mathbf{r}_3
\end{align*}
\]

respectively. All mass and inertia properties of \( S \) are included in the bodies to which the shaft is rigidly attached, namely \( R \) and \( M \). Body \( R \) is axisymmetric about \( \mathbf{r}_3 \) and \( M \) is axisymmetric about \( \mathbf{m}_3 \). Fig. 2 shows the various points associated with \( R \), \( S \), and \( M \) in detail.

The equations of motion are formed via Kane’s method, which requires the formation of the generalized active forces and generalized inertia forces. To facilitate this, the velocity, acceleration, angular velocity, and angular acceleration vectors are continually expressed in the current body basis. It is noted that the interaction forces at the roots of the blades and at the tip of the tower contain the inertial forces of the blades and tower. These quantities are normally found in the generalized inertia forces, but this way they will appear in the generalized active forces for the system. This is in no way a problem, however, as both types of generalized forces are combined anyway to determine the equations of motion. In addition to these forces, the interaction forces at the root and tip of the shaft will contribute to the generalized active forces, as will the gravitational forces acting on all the rigid bodies. The generalized inertia forces for the discrete portion of the system are determined in terms of the accelerations of the mass centers of all bodies and the angular accelerations of all bodies. The resulting equations are quite lengthy by virtue of the number of subsystems and the complicated geometry including the various offset vectors and rotational transformations.

**Coupling of Subsystems**

The coupling of the beam finite elements to rigid bodies modeled by Kane’s method is undertaken by making use of the following two sets of rules:

i) In the mixed finite element model for the tower, the position and orientation coordinates at the tower root are set equal to zero. The position coordinates at \( A_T \), the point where the nacelle is rigidly attached to the tower, are used as the tip boundary values for the column matrix of displacement measures \( \mathbf{u}_{n+1} \) of the tower beam element model. The orientation measures (Rodrigues parameters) of \( A \), the portion of the nacelle which is rigidly attached to the tower, are used as the tip boundary values for the column matrix of Rodrigues parameters \( \mathbf{\theta}_{n+1} \) for the tower beam element model. The mixed method allows one to solve for the nodal forces and moments \( A_T \) in terms of the internal variables of the tower finite element. This set of forces is applied to the discrete model at \( A_T \) and the moment is applied to \( A \). Finally, the generalized speeds of the body \( A \) are directly related to the tower element generalized speeds (so that only one set is needed in the final set of independent generalized speeds).

ii) In the mixed finite element model for the blades, the blade root is clamped to \( H \) at points \( H_{U_0} \) and \( H_{D_0} \), and thus the blade root displacement \( \mathbf{u}_0 \) and orientation variables \( \mathbf{\theta}_0 \) are set equal to zero. The blade tip force \( \mathbf{F}_{n+1} \) and moment and \( \mathbf{M}_{n+1} \) are set equal to zero. The inertial velocity of \( H_0 \) and angular velocity of \( H \) define the motion of the frame to which the blades are clamped and thus determine the variables \( u \) and \( \omega \) needed in the mixed finite element formulation. One can solve for the blade root nodal force \( \mathbf{F}_0 \) and moment \( \mathbf{M}_0 \) in terms of the element internal variables. This set of forces is applied at the points where the blades are attached to the hub, and the moment is applied to \( H \).

It should be noted that the use of explicit force and moment variables at the interfaces with the tower and blades should make the resulting equations for the discrete portion of the model somewhat simpler. It has been verified that the above procedure works for models with single-element representations for the tower and blades. Equations for multiple element implementations show the procedure will also work with multiple element models for the tower and blades.

**Linearization**

Linearization of the equations is considered an important part of this work for modal reduction and control design. The way linearization is accomplished is as follows. First, within the mixed formulation one can write the linearized equations directly given the simplicity of the governing equations. This is facilitated by use of a symbolic manipulator (such as Autolev or Mathematica). The discrete part of the system, however,
is intended to be done using Autolev. It was therefore necessary to demonstrate that the Autotaylor feature of Autolev was up to the task of linearizing the model just obtained. So far we have only showed that it is up to the task for linearization about a constant steady-state solution. There is some uncertainty about its capability to handle linearization about a time-dependent steady-state solution, because of the size of the expressions expected in that operation.

The form of the final linearized equations obtained is

$$\begin{bmatrix} \Delta & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{u} \end{bmatrix}$$

(25)

where the matrices $M$, $A$, $B$, $C$, and $D$ are matrices which may be functions of time $t$, $0$ is a matrix of zeros, and $\Delta$ is the identity matrix. The unknowns $\dot{q}$ and $\dot{u}$ represent perturbation values of all generalized coordinates and speeds, respectively, and the overdot means time derivative. The first set of equations represented by Eqs. (25) corresponds to the linearized kinematical differential equations, and the second set to the linearized equations of motion.

**Time-Finite-Element Form of Kane’s Equations**

The equations of the mixed method are already in a form suitable for solution via finite elements in time. Time-marching and other forms of solution have been developed in [15]. However, Kane’s equations are derived in a completely different manner. Therefore, a means to integrate the Autolev-based equations into the mixed finite element framework needed to be developed. The method developed is based on putting Kane’s equations into the mixed variational formulation presented by Mingori [16]. The mixed formulation uses displacement, rotation, velocity, and angular velocity variables. The last two fall into the category of generalized speeds. It is helpful, though not essential, to eliminate the nodal force and moment variables. This way, the nonlinear form of the above linearized equations can be written as

$$\dot{q} = Bu + x$$

$$M \dot{u} + b = 0$$

(26)

where $M$ and $B$ are the same matrices as above and are, in general along with the column matrix $x$, functions of $q$ and $t$. Here, however, we can generalize the formulation to include all $N$ generalized coordinates and generalized speeds in a multi-element formulation, so that $M$ and $B$ are $N \times N$ matrices. The $N \times 1$ column matrix $b$ is a function of $q$ and $u$, as well as the time $t$. Adjoining these equations and appropriate initial/final conditions to any existing weak form (such as that pertaining to the mixed finite element treatment of the tower and blades) with column matrices of Lagrange multipliers which consist of $N \times 1$ column matrices of the generalized coordinates $q$ and the generalized conjugate momenta $p$, one obtains for the contribution of Kane’s equations and the kinematical differential equations to the weak form of the system equations

$$\int_{t_1}^{t_2} \left[ \delta \dot{p}^T (\dot{q} - Bu - x) - \delta q^T (M \dot{u} + b) \right] dt$$

$$+ \left[ \delta p^T (\dot{q} - q) - \delta q^T (M \dot{u} - Mu) \right] \bigg|_{t_1}^{t_2} = 0$$

(27)

where $\dot{q}$, $M$, and $\dot{u}$ are the values of $q$, $M$, and $u$ at the ends of the time interval. Normally the initial time values are known, and the final time values can be obtained after solution of the equations for the time finite element. After integration by parts, one finds

$$\int_{t_1}^{t_2} \left[ \frac{d}{dt} (\delta q^T M) u - \delta q^T b - \delta \dot{p}^T q - \delta p^T (Bu + x) \right] dt$$

$$+ \left( \delta p^T \dot{q} - \delta q^T M \dot{u} \right) \bigg|_{t_1}^{t_2} = 0$$

(28)

or

$$\int_{t_1}^{t_2} \left[ \delta \dot{q}^T Mu + \delta q^T (M \dot{u} - b) - \delta \dot{p}^T q - \delta p^T (Bu + x) \right] dt$$

$$\bigg|_{t_1}^{t_2} = 0$$

(29)

Note that Autolev can be used to express $M$ in terms of $u$, $q$, and $t$ only, viz.,

$$M_{ij} = \frac{\partial M_{ij}}{\partial t} + \sum_{k=1}^{N} \frac{\partial M_{ij}}{\partial q_k} q_k$$

$$= \frac{\partial M_{ij}}{\partial t} + \sum_{k=1}^{N} \frac{\partial M_{ij}}{\partial q_k} \left( x_k + \sum_{l=1}^{N} B_{kl} u_l \right)$$

(30)

For finite elements in the time domain, one can choose the simplest shape functions. We let

$$t = t_1 + \tau \Delta t$$

$$\Delta t = t_2 - t_1$$

(31)

where $\tau$ is the nondimensional time coordinate, varying from 0 to 1 within each element. Since only the variations of $q$ and $u$ are differentiated with respect to time, we can let

$$\delta q = \delta q_1 (1 - \tau) + \delta q_2 \tau$$

$$\delta p = \delta p_1 (1 - \tau) + \delta p_2 \tau$$

(32)
while all other quantities are constant within the element, viz.,
\[ q = \overline{q} \]
\[ u = \overline{u} \]  
(33)
and have distinct values at the element ends, so that
\[ q = \dot{q}_1 \text{ at } t = t_1 \]
\[ q = \dot{q}_2 \text{ at } t = t_2 \]
\[ u = \ddot{u}_1 \text{ at } t = t_1 \]
\[ u = \ddot{u}_2 \text{ at } t = t_2 \]  
(34)
The subscripts 1 and 2 refer to values at times \( t_1 \) and \( t_2 \), respectively. All this leads to four sets of equations:
\[ \delta q_1 : -\overline{M}\overline{u} + \frac{\Delta t}{2} \left( \overline{M}\overline{u} - \overline{b} \right) + \hat{M}_1 \ddot{u}_1 = 0 \]
\[ \delta q_2 : \overline{M}\overline{u} + \frac{\Delta t}{2} \left( \overline{M}\overline{u} - \overline{b} \right) - \hat{M}_2 \ddot{u}_2 = 0 \]
\[ \delta p_1 : \overline{\dot{q}} - \frac{\Delta t}{2} \left( \overline{\dot{q}} + \overline{\pi} \right) - \dot{q}_1 = 0 \]
\[ \delta p_2 : -\overline{\dot{q}} - \frac{\Delta t}{2} \left( \overline{\dot{q}} + \overline{\pi} \right) + \dot{q}_2 = 0 \]  
(35)
These equations can be used to “time-march” the solution, or they could be assembled over a time interval to obtain a direct periodic solution.

For time marching, for example, one has known values of \( \dot{q}_1 \) (and thus \( \hat{M}_1 \)) and \( \ddot{u}_1 \). Eqs. (35) lead to two sets of equations for \( \overline{\dot{q}} \) and \( \overline{\pi} \):
\[ \overline{M}\overline{u} - \frac{\Delta t}{2} \left( \overline{M}\overline{u} - \overline{b} \right) = \hat{M}_1 \ddot{u}_1 \]
\[ \overline{\dot{q}} - \frac{\Delta t}{2} \left( \overline{\dot{q}} + \overline{\pi} \right) = \dot{q}_1 \]  
(36)
These equations must be solved numerically via an iterative process, such as the Newton-Raphson method. Upon finding \( \overline{\dot{q}} \) and \( \overline{\pi} \), one can then find the values at \( t = t_2 \) directly as
\[ \dot{q}_2 = 2\overline{\dot{q}} - \dot{q}_1 \]
\[ \ddot{u}_2 = \hat{M}_2^{-1} \left( 2\overline{M}\overline{u} - \hat{M}_1 \ddot{u}_1 \right) \]  
(37)

For assembly over a large time interval and enforcing periodicity, the resulting equations are very sparse, and one should take advantage of that sparsity to create an efficient solution algorithm. The Harwell library [17] offers subroutine MA28, for example, which allows one to restrict storage to only the nonzero elements of coefficient matrices.

Thus, it is shown to be possible to put Kane’s equations along with the mixed formulation in the time-finite-element form. This makes the integration of the two approaches with multiple mixed finite elements quite seamless, since Autolev is able to symbolically manipulate Kane’s equations and put them in time-finite-element form.

**Concluding Remarks**

A methodology has been presented for modeling a HAWT based on the use of Kane’s equations for the multi-rigid-body portions of the system and mixed beam finite elements for the flexible portions. The governing equations are in state-space form, and for specific configurations explicit expressions for the matrix elements can be obtained. The formulation is believed to offer certain advantages in control design, since both rigid and flexible portions of the model are represented by use of techniques which provide a reasonable balance between low order and high fidelity. Coding is underway, and results will be presented in later papers.

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**References**


Figure 1: Schematic of Wind-Turbine Model

Figure 2: Schematic of Shaft with Bodies $R$ and $M$