FLIGHT DYNAMICS OF
HIGHLY FLEXIBLE FLYING WINGS

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Abstract: The paper presents a theory for flight dynamic analysis of highly flexible flying wing configurations. The analysis takes into account large aircraft motion coupled with geometrically nonlinear structural deformation subject only to a restriction to small strain. A large motion aerodynamic loads model is integrated into the analysis. The analysis can be used for complete aircraft analysis including trim, stability analysis linearized about the trimmed-state, and nonlinear simulation. Results are generated for a typical high-aspect-ratio “flying wing” configuration. The results indicate that the aircraft undergoes large deformation during trim. The flight dynamic characteristics of the deformed aircraft are completely different as compared to a rigid aircraft. For the example aircraft, the phugoid mode is unstable and the classical short-period mode does not exist. Furthermore, nonlinear flight simulation of the aircraft indicates that the phugoid instability leads to catastrophic consequences.

1 Introduction

The analysis and design of very light, and thus highly flexible, flying wing configurations is of interest for the development of the next generation of high-altitude, long-endurance (HALE), unmanned aerial vehicles (UAV). The flexibility of such aircraft leads to large deformation, so that linear theories are not relevant for their analysis. For example, the trim shape of a large flexible aircraft is highly dependent on the flight mission (payload) as well as on the flight condition; the deformed shape is significantly different from the undeformed shape. Thus, the flight dynamic response based on the actual trim shape can be quite different from that calculated based on linear, small deformation assumptions. In fact, the designer can use the deformation of the aircraft to positively affect the flight stability and control characteristics. The paper presents a theoretical basis for the flight dynamic response estimation of a highly flexible flying wing. Various realistic design space requirements including, concentrated payload pods, multiple engines, multiple control surfaces, vertical surfaces, discrete dihedral, and continuous pretwist, are taken into account. The code based on the theoretical development presented here can be used in preliminary design as well as in control synthesis. This work is a continuation of work conducted by the authors over the past decade in the area of nonlinear aeroelasticity\textsuperscript{1,2}. The focus of the present work is on flight dynamics.
2 Theory

The modeling of a flexible aircraft undergoing large deformation requires a geometrically-exact structural model coupled with a consistent large motion aerodynamic model. The present work is based on a flying wing concept and is modeled structurally as a beam undergoing large displacement and rotation. The governing equations are the geometrically-exact equations of motion from Hodges3 written in their intrinsic form (i.e. without displacement and rotational variables). However, instead of being augmented by the displacement- and rotation-based kinematical relations given therein, they are instead augmented by the intrinsic kinematical equations, derived in Ref. 4 by eliminating the displacement and rotational variables from the kinematical equations of Ref. 3. A 2-D airfoil model is appropriate for the very high-aspect-ratio wing (without fuselage interference) being analyzed here. The airloads are here based on the finite-state airloads model presented by Peters and Johnson5.

Before presenting the structural and aerodynamic theory used for the present research, there is a need to present the supporting preliminaries, including the frames and variables. The next section presents details of the nomenclature, which are essential to understand the theoretical details presented later.

2.1 Preliminaries

The axis system is as shown in Figure 1. The following frames of reference are used:

- $i$: inertial frame ($-z$ pointing in the direction of gravity)
- $b$: undeformed beam frame ($x$ along the beam axis and $y$ pointing towards the front of the airplane; $y$ and $z$ are the axes in which the cross-sectional stiffness and inertia matrices are calculated)
- $B$: deformed/moving beam frame
- $a$: aerodynamic frame in which the aerodynamic lift and moment is defined ($y$ and $z$ are defined in the airfoil frame with $z$ pointing perpendicular to the aerodynamic surface; $y$ and $z$ are the axes in which the aerodynamic coefficients are calculated)

The following rotation matrices are used to transform vectors in different frames:
• \( C_a = C_{ba} \): transfers a vector from the aerodynamic frame of reference to the undeformed beam frame of reference. Since the aerodynamic frame is defined relative to the structural frame, \( C^{BA} = C_{ba} \), where \( A \) is the aerodynamic frame of the deformed beam.

• \( C_{lr} \): the rotation matrix at a node with a slope discontinuity that transfers variables from a reference frame on the right of the node to one on the left of the node.

• \( C = C^{Bb} \): determines the deformed beam cross-sectional frame of reference. For the intrinsic formulation presented here, \( C \) is not required to solve the equations but may be used in post-processing.

The primary beam variables are

• \( F \): internal force measures in the \( B \) frame
• \( M \): internal moment measures in the \( B \) frame
• \( V \): velocity measures in the \( B \) frame
• \( \Omega \): angular velocity measures in the \( B \) frame

• \( g \): the measure numbers of the gravity vector in the \( B \) frame

The secondary beam variables are

• \( \gamma \): internal strain measures
• \( \kappa \): internal curvature measures
• \( P \): linear momentum measures
• \( H \): angular momentum measures

### 2.2 Structural Model

The geometrically-exact, intrinsic equations for the dynamics of a general, non-uniform, twisted, curved, anisotropic beam, undergoing large deformation, are given as

\[
F' + (\tilde{k} + \tilde{\kappa})F + f = \dot{P} + \tilde{\Omega}P \\
M' + (\tilde{k} + \tilde{\kappa})M + (\tilde{\epsilon}_1 + \tilde{\gamma})F + m = \dot{H} + \tilde{\Omega}H + \tilde{V}P \\
V' + (\tilde{k} + \tilde{\kappa})V + (\tilde{\epsilon}_1 + \tilde{\gamma})\Omega = \dot{\gamma} \\
\Omega' + (\tilde{k} + \tilde{\kappa})\Omega = \dot{\kappa}
\]

where \( k = [k_1 \; k_2 \; k_3] \) is the initial twist/curvature of the beam, \( \epsilon_1 = [1 \; 0 \; 0]^T \), and \( f \) and \( m \) are the external forces including gravity \((f_g, m_g)\), aerodynamic loads \((f_{aero}, m_{aero})\), and thrust \((f_T, m_T)\). The first two equations in the above set are the equations of motion\(^3\) while the latter two are the intrinsic kinematical equations\(^4\) derived from the generalized strain-displacement and generalized velocity-displacement equations.
2.2.1 Cross-sectional constitutive laws

The secondary beam variables are linearly related to the primary variables by the cross-sectional constitutive laws (flexibility and inertia matrices), such that

\[
\begin{bmatrix}
\gamma \\
\kappa 
\end{bmatrix} = \begin{bmatrix} R & S \\ S^T & T \end{bmatrix} \begin{bmatrix} F \\ M \end{bmatrix} \tag{5}
\]

\[
\begin{bmatrix}
P \\
H 
\end{bmatrix} = \begin{bmatrix} \mu \Delta & -\mu \tilde{\xi} \\ \mu \tilde{\xi} & I \end{bmatrix} \begin{bmatrix} V \\ \Omega \end{bmatrix} \tag{6}
\]

where \(R\), \(S\), and \(T\) are \(3 \times 3\) matrices of cross-sectional flexibility coefficients; and \(\mu\), \(\xi\), \(I\) are the mass per unit length, mass center offset, and mass moment of inertia per unit length, respectively. These relations are derived based on the assumptions of small strain and slenderness.

2.2.2 Finite-element discretization

To solve the above set of equations, the beam is discretized into finite elements. The equations for each element are obtained by discretizing the differential equations such that energy is conserved. For example, consider a variable \(X\). Let the nodal values of the variable after discretization be represented by \(\hat{X}_i^n\) and \(\hat{X}_r^n\), where the superscript denotes the node number, the subscript denotes the left or right side of the node, and the hat denotes that it is nodal value. The need to define two values of the variable at a node is clarified in later section. For the element \(n\)

\[
X' = \frac{\hat{X}_{i+1}^n - \hat{X}_r^n}{dl} \tag{7}
\]

\[
X = \overline{X}^n = \frac{\hat{X}_{i+1}^n + \hat{X}_r^n}{2} \tag{8}
\]

In a discretized form the equations of motion can be written as

\[
\frac{\hat{F}_{i+1}^n - \hat{F}_r^n}{dl} + (\overline{\kappa^n} + \tilde{\kappa^n})\overline{F}^n + \overline{F}^n - \overline{\Omega^n}\overline{P}^n = 0 \tag{9}
\]

\[
\frac{\hat{M}_{i+1}^n - \hat{M}_r^n}{dl} + (\overline{\kappa^n} + \tilde{\kappa^n})\overline{M}^n + (\overline{\epsilon_1^n} + \tilde{\epsilon_1^n})\overline{P}^n + \overline{m}^n - \overline{\Omega^n}\overline{H}^n - \overline{\Omega^n}\overline{H}^n - \overline{\tilde{\Omega}^n}\overline{P}^n = 0 \tag{10}
\]

\[
\frac{\hat{V}_{i+1}^n - \hat{V}_r^n}{dl} + (\overline{\kappa^n} + \tilde{\kappa^n})\overline{V}^n + (\overline{\epsilon_1^n} + \tilde{\epsilon_1^n})\overline{\Omega}^n + \overline{\Omega}^n - \overline{\kappa^n} = 0 \tag{11}
\]

\[
\frac{\hat{\Omega}_{i+1}^n - \hat{\Omega}_r^n}{dl} + (\overline{\kappa^n} + \tilde{\kappa^n})\overline{\Omega}^n - \overline{\kappa^n} = 0 \tag{12}
\]

where, as defined above, the barred quantities correspond to the values of the variables in the element interior while the hatted quantities are nodal values. The barred and hatted
quantities of the primary variables are related as

\begin{align*}
\mathbf{F}^n &= \frac{\mathbf{F}_{i}^{n+1} + \mathbf{F}_{r}^{n}}{2} \\
\mathbf{M}^n &= \frac{\mathbf{M}_{i}^{n+1} + \mathbf{M}_{r}^{n}}{2} \\
\mathbf{V}^n &= \frac{\mathbf{V}_{i}^{n+1} + \mathbf{V}_{r}^{n}}{2} \\
\mathbf{\Omega}^n &= \frac{\mathbf{\Omega}_{i}^{n+1} + \mathbf{\Omega}_{r}^{n}}{2}
\end{align*}  

(13)  

(14)  

(15)  

(16)

The barred secondary variables are related to the barred primary variables as stated above in the cross-sectional constitutive law.

2.2.3 Gravity loads

The force term in the equations of motion include gravitational forces. The gravitational force and moment are

\begin{align*}
f_g &= \mu g \\
m_g &= \mu \tilde{\xi} g
\end{align*}  

(17)  

(18)

where \( g \) is the gravity vector.

The measure numbers of \( g \) are known in the \( i \)-frame. The measure numbers of the gravity vector \( g \) in the \( B \) frame at all the nodes can be calculated using the following equations:

\begin{align*}
g' + (\tilde{\kappa} + \tilde{k})g &= 0 \\
\dot{g} + \tilde{\Omega}g &= 0
\end{align*}  

(19)

which in the discretized form can be written as

\begin{align*}
\frac{\hat{g}_{i}^{n+1} - \hat{g}_{r}^{n}}{dl} + (\tilde{k}^{n} + \tilde{k}^{n})\hat{g}^{n} &= 0 \\
\dot{\hat{g}} + \tilde{\Omega}\hat{g} &= 0
\end{align*}  

(20)

The second equation above, the time-differentiated one, is satisfied at one node; while the first equation, the spatially-differentiated one, is used to obtain the \( g \) vector at other nodes. Both equations are matrix equations, i.e. a set of three scalar equations. The three equations together can be shown to satisfy a constraint of constant length for the \( g \) vector.

One can thus replace any one of the equations by the static form of this length constraint. This will remove the artificial root caused by the differentiation of a constraint. Also the constraint is satisfied for the steady-state calculation when the dynamic terms are neglected. So, the equation can be written as

\begin{align*}
(e_{1}e_{1}^{T} + e_{2}e_{2}^{T})\dot{\tilde{g}}^{n} + (e_{1}e_{1}^{T} + e_{2}e_{2}^{T})\tilde{\Omega}^{n}\tilde{g}^{n} + (e_{3}e_{3}^{T})|\tilde{g}^{n}| &= e_{3}
\end{align*}  

(21)

For the case of symmetric flight at the central node, \( \tilde{\Omega}_{2} = \tilde{\Omega}_{3} = 0 \). Thus, the above equations become

\begin{align*}
\dot{\hat{g}}_{1} &= 0 \quad \Rightarrow \quad \hat{g}_{1} = 0 \\
\dot{\hat{g}}_{2} - \tilde{\Omega}_{1}\hat{g}_{3} &= 0 \\
(\hat{g}_{2})^{2} + (\hat{g}_{3})^{2} &= 1
\end{align*}  

(22)  

(23)  

(24)
From the second equation above it is clear that for steady-state, symmetric flight, \( \hat{\Omega}_1 = 0 \) because \( \hat{g}_3 \neq 0 \) (wing vertical).

### 2.2.4 Engines, nodal masses, vertical surfaces and slope discontinuities

Due to nodal mass, nodal force (thrust) and slope discontinuities the force on one side of the node is different from the force on the other side of the node. Thus,

\[
\begin{align*}
\hat{F}_r^n - \hat{C}_{ir}^n \hat{F}_i^n + \hat{f}_T^n + \hat{\mu}^n \hat{g}_r^n + \hat{f}_{\text{aero}}^n - \hat{\Omega}_r^n \hat{P}_r^n &= 0 \\
\hat{M}_r^n - \hat{C}_{ir}^n \hat{M}_i^n + \hat{m}_r^n + \hat{\mu}^n \hat{\xi}^n \hat{g}_r^n + \hat{m}_{\text{aero}}^n - \hat{\Omega}_r^n \hat{H}_r^n - \hat{V}_r^n \hat{P}_r^n &= 0
\end{align*}
\]

(25) (26)

where \( \hat{f}_T^n \) is the discrete nodal thrust force defined in the \( \hat{\cdot} \) reference frame, \( \hat{m}_r^n \) is the corresponding nodal moment, \( \hat{\mu}^n \) is the concentrated nodal mass, \( \hat{\xi}^n \) is the corresponding mass offset, \( f_{\text{aero}}^n \) and \( m_{\text{aero}}^n \) are the aerodynamic forces due to vertical surfaces/pods, and \( \hat{P}_r^n \) and \( \hat{H}_r^n \) are the linear and angular momenta of the concentrated modal mass, given by

\[
\begin{align*}
\begin{pmatrix}
\hat{P}_r^n \\
\hat{H}_r^n
\end{pmatrix} &= \begin{pmatrix}
\hat{\mu}^n & -\hat{\mu}^n \hat{\xi}^n \\
\hat{\mu}^n \hat{\xi}^n & \hat{I}_r^n
\end{pmatrix} \begin{pmatrix}
\hat{V}_r^n \\
\hat{\Omega}_r^n
\end{pmatrix} \begin{pmatrix}
0 \\
\hat{H}_{\text{engine}}^n
\end{pmatrix}
\end{align*}
\]

(27)

where \( \hat{I}_r^n \) is the mass moment of inertia matrix of the concentrated mass and \( \hat{H}_{\text{engine}}^n \) is the angular momentum of the engine.

A slope discontinuity in the beam will also change all the other variables, such that

\[
\begin{align*}
\hat{V}_l^n &= \hat{C}_{ir}^n \hat{V}_r^n \\
\hat{\Omega}_l^n &= \hat{C}_{ir}^n \hat{\Omega}_r^n \\
\hat{g}_l^n &= \hat{C}_{ir}^n \hat{g}_r^n
\end{align*}
\]

(28) (29) (30)

The \( \hat{\cdot} \) variables can be used to replace \( \hat{\cdot} \) variables for \( V \), \( \Omega \) and \( g \), thus reducing the number of variables.

### 2.2.5 Final structural equations

The development presented above leads to the following primary equations:

\[
\begin{align*}
\hat{F}_r^n - \hat{C}_{ir}^n \hat{F}_i^n + \hat{f}_T^n + \hat{\mu}^n \hat{g}_r^n + \hat{f}_{\text{aero}}^n - \hat{\Omega}_r^n \hat{P}_r^n &= 0 \\
\hat{M}_r^n - \hat{C}_{ir}^n \hat{M}_i^n + \hat{m}_r^n + \hat{\mu}^n \hat{\xi}^n \hat{g}_r^n + \hat{m}_{\text{aero}}^n - \hat{\Omega}_r^n \hat{H}_r^n - \hat{V}_r^n \hat{P}_r^n &= 0 \\
\frac{d}{dl} \left( \hat{M}_r^n + (\hat{\kappa}_r^n + \hat{k}_r^n) \hat{M}_i^n + (\hat{\epsilon}_1 + \hat{\gamma}_1^n) \hat{F}_i^n + \hat{m}_{\text{aero}}^n + \hat{\mu}^n \hat{\xi}^n \hat{g}_r^n - \hat{H}_i^n - \hat{\Omega}_r^n \hat{H}_i^n - \hat{V}_i^n \hat{P}_i^n \right) &= 0 \\
\frac{d}{dl} \left( \hat{C}_{ir}^n \hat{V}_r^n + \hat{V}_i^n + (\hat{\kappa}_r^n + \hat{k}_r^n) \hat{V}_i^n + (\hat{\epsilon}_1 + \hat{\gamma}_1^n) \hat{\Omega}_i^n - \hat{\gamma}_i^n \right) &= 0 \\
\frac{d}{dl} \left( \hat{C}_{ir}^n \hat{\Omega}_r^n + \hat{\Omega}_i^n + (\hat{\kappa}_r^n + \hat{k}_r^n) \hat{\Omega}_i^n - \hat{\gamma}_i^n \right) &= 0 \\
\frac{d}{dl} \left( \hat{C}_{ir}^n \hat{g}_r^n + \hat{g}_i^n + (\hat{\kappa}_r^n + \hat{k}_r^n) \hat{g}_i^n \right) &= 0
\end{align*}
\]

(31) (32) (33) (34) (35) (36) (37)
and, the following secondary equations:

\[
\begin{align*}
\{\pi^n\} &= \begin{bmatrix} R^n & S^n T^n \end{bmatrix} \{\bar{F}^n\} \\
\{\bar{F}^n\} &= \begin{bmatrix} \mu^n \Delta & -\mu^n \xi^n \\ \mu^n \xi^n & I^n \end{bmatrix} \{\bar{V}^n\}
\end{align*}
\]

\[
F^n = \frac{\bar{F}^n_{l+1} + \bar{F}^n_r}{2} \quad (38)
\]

\[
M^n = \frac{\bar{M}^n_{l+1} + \bar{M}^n_r}{2} \quad (39)
\]

\[
\bar{V}^n = \frac{\bar{V}^n_{l+1} + \bar{V}^n_r}{2} \quad (40)
\]

\[
\bar{\Omega}^n = \frac{\bar{\Omega}^n_{l+1} + \bar{\Omega}^n_r}{2} \quad (41)
\]

\[
\bar{g}^n = \frac{\bar{\mu}^n_{l+1} \bar{\xi}^n_{l+1} + \bar{\xi}^n_r}{2} \quad (42)
\]

2.2.6 Boundary conditions

The following are the boundary conditions for the problem:

\[
\begin{align*}
\hat{F}^1_l &= 0 \quad (46) \\
\hat{M}^1_l &= 0 \quad (47) \\
\hat{F}^{N+1}_r &= 0 \quad (48) \\
\hat{M}^{N+1}_r &= 0 \quad (49)
\end{align*}
\]

\[
(e_1 e_1^T + e_2 e_2^T) \hat{g}^n + (e_1 e_1^T + e_2 e_2^T) \hat{\Omega}^n \hat{g}^n + (e_3 e_3^T) |\hat{g}^n| = 0 \quad (50)
\]

where \( N \) denotes the total number of elements and \( n_g \) denotes the reference node for gravity.

2.3 Aerodynamic Model

The airloads are calculated based on 2-D aerodynamics using the known airfoil parameters.

First the velocities in the aerodynamic frame at the mid-chord are written as

\[
\bar{V}_a^n = C_a^n \bar{V}^n - \bar{y}_{mc} C_a^n \bar{\Omega}^n \quad (51)
\]

\[
\bar{\Omega}_a^n = C_a^n \bar{\Omega}^n \quad (52)
\]

where \( \bar{y}_{mc} \) is the vector from the beam reference axis to the mid-chord and can be written in terms of the aerodynamic center (at the quarter chord) location as \( \bar{y}_{mc} = [0 \quad \frac{y_{ac}}{2} \quad 0] \).

The lift, drag and pitching moment at the quarter-chord are given by:

\[
\begin{align*}
L_{aero}^n &= \rho b^n V_T^n \left( C_{l_0}^n + C_{l_2}^n \alpha^n + C_{l_3}^n \beta^n \right) + \rho b^n V_T^n V_{a_2} V_{l_2}^n \alpha_{\nu m} \cos \alpha^n \quad (53)
\end{align*}
\]
\[ D_{\text{aero}}^n = \rho b^n V_T^n C_{\text{d}_d}^n + \rho b^n V_T^n V_a^n C_{\text{i}_{\text{d}_d}}^n \alpha_{\text{rot}} n \sin \alpha^n \]  
(54)
\[ M_{\text{aero}}^n = 2\rho b^n V_T^n \left( C_{\text{m}_0}^n + C_{\text{m}_a}^n \alpha^n + C_{\text{m}_{\beta}}^n \beta^n \right) - \rho b^n V_T^n V_a^n C_{\text{i}_{\text{a}}}^n \alpha_{\text{rot}} n / 2 \]  
(55)
where
\[ V_T^n = \sqrt{V_{a_2}^n + V_{a_3}^n} \]  
(56)
\[ \alpha^n \approx \sin \alpha^n = -V_{a_3}^n / V_T^n \]  
(57)
\[ \alpha_{\text{rot}} n = \frac{\Omega a_1 b^n / 2}{V_T^n} \]  
(58)
and, \( V_{a_2}^n \) and \( V_{a_3}^n \) are the measure numbers of \( V_{a}^n \). \( \beta^n \) is the flap deflection of the \( n \)th element.

The lift, drag and pitching moment are aerodynamic forces which can be written in the \( a \)-frame as:
\[ f_a^n = \left\{ \begin{array}{c}
L_{\text{aero}} V_{a_3}^n / V_T^n - D_{\text{aero}} V_{a_2}^n / V_T^n \\
L_{\text{aero}} V_{a_2}^n / V_T^n - D_{\text{aero}} V_{a_3}^n / V_T^n
\end{array} \right\} \]  
(59)
\[ m_a^n = \left\{ \begin{array}{c}
M_{\text{aero}}^n \\
0
\end{array} \right\} \]  
(60)

Finally, the forces derived above are transformed to the \( B \) frame and transferred to the beam reference axis to give the applied aerodynamic forces as
\[ \bar{f}_{\text{aero}}^n = C_{\alpha}^f a^n f_a^n \]  
(61)
\[ \bar{m}_{\text{aero}}^n = C_{\alpha}^m a^n m_a^n + C_{\alpha}^m a^n \gamma_{\alpha} f_a^n \]  
(62)

2.3.1 Unsteady Effects

The above aerodynamic model is a quasi-steady one with neither wake (inflow) effects nor apparent mass effects. To add those effects into the model we have to firstly add the inflow \( \lambda_0 \) and acceleration terms in the force and moment equation. Secondly we have to include an inflow model that calculates \( \lambda_0 \). Here the Peters 2-D inflow theory of Ref. 6 is used.

The force and moment expressions with the unsteady aerodynamics effects are
\[ f_a^n = \rho b^n \left\{ \begin{array}{c}
0 \\
-(C_{\text{m}_0}^n + C_{\text{m}_a}^n \beta^n) V_{a_3}^n V_{a_3}^n + C_{\text{m}_a}^n (V_{a_3}^n + \lambda_0^n)^2 - C_{\text{d}_d}^n V_T^n V_{a_2}^n \\
(C_{\text{m}_0}^n + C_{\text{m}_a}^n \beta^n) V_T V_{a_2}^n - C_{\text{i}_{\text{a}}}(V_{a_3}^n + \lambda_0^n - \Omega a_1 b^n / 2 - C_{\text{i}_{\text{a}}}^n V_T V_{a_2}^n)
\end{array} \right\} \]  
(63)
and
\[ m_a^n = 2\rho b^n \left\{ \begin{array}{c}
(C_{\text{m}_0}^n + C_{\text{m}_a}^n \beta^n) V_T^n - C_{\text{m}_a}^n V_T V_{a_3}^n - b^n (C_{\text{i}_{\text{a}}} / 8 + C_{\text{p}_{\text{pitch}}} / 2) V_{a_3}^n \Omega a_1^n - b^n C_{\text{i}_{\text{a}}} \Omega a_1^n / 32 + b^n C_{\text{i}_{\text{a}}} V_{a_2}^n / 8 \\
0
\end{array} \right\} \]  
(64)
The inflow model can be written as:

\[
[A_{\text{inflow}}] \{\dot{\lambda}^n\} + \left( \frac{V_T^n}{b^n} \right) \{\lambda^n\} = \left( -\dot{V}_{a_3}^n + \frac{b^n}{2} \dot{\Omega}_{a_1}^n \right) \{c_{\text{inflow}}\}
\]  \hspace{1cm} (65)

and

\[
\lambda^n_0 = \frac{1}{2} \{b_{\text{inflow}}\}^T \{\lambda^n\}
\]  \hspace{1cm} (66)

where \(\lambda^n\) is a vector of inflow states for the \(n^{th}\) element, and \([A_{\text{inflow}}]\), \(\{c_{\text{inflow}}\}\), \(\{b_{\text{inflow}}\}\) are constant matrices derived in Ref. 6.

### 2.4 Aeroelastic System

An aeroelastic model is obtained by coupling the aerodynamic force definition given in the previous section with the set of equations presented in the section on the structural model. The aerelastic equations are nonlinear equations in terms of the primary variables \((F^n_l, F^n_r, M^n_l, M^n_r, V^n_r, \Omega^n_r, g^n_r)\).

The set of aeroelastic equations is solved using Newton-Raphson method to obtain the steady-state (trim) solution. The Jacobian calculated for this solution is needed to assess the stability of the linearized system at the trim state.

### 2.5 Trimming

The trim conditions are the same as steady-state conditions, i.e. all the time-derivatives are zero. If all the controls are given, then the solution gives the steady-state (trim) corresponding to that flight condition. On the other hand, more often it is desired to trim the flight at a specific trim state. To do so, the trim state in terms of the airspeed and flight angle (climb/descent indicator) are prescribed and the controls (thrust and flap angle) are determined. Thus two additional equations and two variable (controls) are appended to the system. The two equations are given below. The equations are sufficient for a symmetric trimmed state of flight. For asymmetric trimmed flight, the equations would be modified and additional equations relating the radius of turn and side-slip would be added. The symmetric trim equations are

\[
\hat{g}_2 \hat{V}_2 + \hat{g}_3 \hat{V}_3 - \tan \phi (\hat{g}_3 \hat{V}_2 - \hat{g}_2 \hat{V}_3) = 0
\]  \hspace{1cm} (67)

\[
\hat{V}_2^2 + \hat{V}_3^2 - V_\infty^2 = 0
\]  \hspace{1cm} (68)

where \(\phi\) is the prescribed flight angle and \(V_\infty\) is the prescribed airspeed. The first equation is derived from the fact that \(\phi = \theta - \alpha\), where \(\theta\) is the pitch angle \((\tan \theta = \frac{\hat{g}_3}{\hat{g}_2})\) and \(\alpha\) is the angle of attack \((\tan \alpha = -\frac{\hat{V}_3}{\hat{V}_2})\).

### 2.6 Post-processing and Graphics

The solution for the intrinsic equations described above can be used to determine and plot the deformation. The following equations relate the strains (and curvatures) to displacements (and rotations):

\[
r_i' = C^{ib} e_1
\]  \hspace{1cm} (69)

\[
C^{bi'} = -\bar{k}C^{bi}
\]  \hspace{1cm} (70)

\[
(r_i + u_i)' = C^{iB}(\gamma + e_1)
\]  \hspace{1cm} (71)

\[
C^{Bi'} = -(\bar{k} + \bar{k})C^{Bi}
\]  \hspace{1cm} (72)
where \( r_i \) is the position vector of the beam axis from the origin of the reference frame \( i \), and \( u_i \) is the deformation in the \( i \) frame. The first two equations determine the geometry of the undeformed wing. The undeformed beam axis could be plotted if \( r_i \) is known at all the nodal locations, while the deformed beam axis could be plotted if \( r_i + u_i \) is known.

To plot the wing surface one needs the vector defining the position vector of the points on the wing surface from the beam axis. For plotting it is assumed that the cross section is rigid, the undeformed and deformed surface can be generated by plotting \( r_i + C^i b \zeta \) and \( r_i + u_i + C^i B \zeta \), respectively, where \( \zeta \) is the cross-sectional position vector.

The discretized strain-displacement equations are

\[
\hat{C}^{bi}_{n+1} = \left( \Delta \left( \frac{d}{dl} + \frac{\tilde{k}_{n}}{2} \right) \right)^{-1} \left( \Delta \left( \frac{d}{dl} - \frac{\tilde{k}_{n}}{2} \right) \right) \hat{C}^{bi}_{n} \tag{73}
\]

\[
r^{n+1}_i = r^n_i + \bar{C}^{ib^n} e_i dl \tag{74}
\]

\[
\hat{C}^{Bi}_{n+1} = \left( \Delta \left( \frac{d}{dl} + \frac{\tilde{k}_{n}}{2} + \frac{\tilde{\kappa}_{n}}{2} \right) \right)^{-1} \left( \Delta \left( \frac{d}{dl} - \frac{\tilde{k}_{n}}{2} - \frac{\tilde{\kappa}_{n}}{2} \right) \right) \hat{C}^{Bi}_{n} \tag{75}
\]

\[
r^{n+1}_i + u^{n+1}_i = r^n_i + u^n_i + \bar{C}^{iB^n} (\gamma^n + e_i) dl \tag{76}
\]

3 Example

Consider the aircraft as illustrated in Figure 2. The example aircraft has a span of 238.78 ft and a constant chord of 8 ft. \( 1/6 \)th of the span at each end has a dihedral of \( 10^\circ \). The inertial, elastic and aerodynamic properties of the wing cross section are given in Table 1.

There are five propulsive units; one at the mid-span and two each at \( 1/3 \)rd and \( 2/3 \)rd semi-span distance from the mid-span. There are three vertical surfaces (pods) which act as the landing gear. Two of the pods weigh 50 lb each and are located at \( 2/3 \)rd semi-span distance from the mid-span. The central pod also acts as a bay for payload and weighs between 60 lb (‘empty’) and 560 lb (‘full’). The pod/payload weight is assumed to be a point mass hanging 3 ft under the wing. The aerodynamic coefficients for the pods are \( Cl_u = 5 \) and \( C_{d0} = 0.02 \).
Table 1: Wing cross-sectional properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic (reference) axis</td>
<td>25% chord</td>
</tr>
<tr>
<td>Torsional rigidity</td>
<td>$0.4 \times 10^6$ lb ft^2</td>
</tr>
<tr>
<td>Bending rigidity</td>
<td>$2.5 \times 10^6$ lb ft^2</td>
</tr>
<tr>
<td>Bending rigidity (chordwise)</td>
<td>$30 \times 10^6$ lb ft^2</td>
</tr>
<tr>
<td>Mass per unit length</td>
<td>6 lbs/ft</td>
</tr>
<tr>
<td>Center of gravity</td>
<td>25% chord</td>
</tr>
<tr>
<td>Centroidal Mass Mom. Inertia:</td>
<td></td>
</tr>
<tr>
<td>about x-axis (torsional)</td>
<td>30 lb ft</td>
</tr>
<tr>
<td>about y-axis</td>
<td>5 lb ft</td>
</tr>
<tr>
<td>about z-axis</td>
<td>25 lb ft</td>
</tr>
<tr>
<td>Aerodynamics Coefficients (25% chord):</td>
<td></td>
</tr>
<tr>
<td>$C_{f\alpha}$</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>$C_{f\delta}$</td>
<td>1</td>
</tr>
<tr>
<td>$C_{d0}$</td>
<td>0.01</td>
</tr>
<tr>
<td>$C_{m0}$</td>
<td>0.025</td>
</tr>
<tr>
<td>$C_{m2}$</td>
<td>-0.25</td>
</tr>
</tbody>
</table>

3.1 Trim Results

The flight dynamic analysis of a flexible aircraft begins with trim analysis. The trim solution involves the steady-state solution of the complete nonlinear equations. For a symmetric aircraft, one could calculate the trim solution (the airspeed and rate of climb) at a specified thrust and flap deflection. Here we use a trimming algorithm to calculate the thrust (assumed constant for each of the five motors) and flap deflection (assumed constant throughout the span) to achieve a specified trim state. For most of the results in this paper, the trim solution is calculated for a level flight condition of 40 ft/s at sea-level.

Figure 3 shows the control values, the flight angle of attack and the structural deformation as a function of the payload weight for the aircraft. The results for a rigid aircraft with the same configuration are also shown.

Figure 3(d) shows the trim shape of the aircraft. The given aircraft, for the empty configuration, has an almost evenly distributed mass (gravitational forces) balanced by aerodynamic forces and thus the equivalent loads and deformation of the aircraft are small. With the addition of the concentrated payload at the center we get significantly higher aerodynamic loads. Such loads lead to large deformation in the highly flexible aircraft. The ‘U’ shape of the ‘full’ configuration has very different structural as well as flight dynamic characteristics as will be seen throughout the rest of the paper.

Figure 3(a) shows the thrust required for the specified trim condition. The change in required thrust is insignificant for payload changes. This is because the primary source of drag for such aircraft is the profile drag and the skin-friction drag. This drag does not change with the aircraft weight. The induced drag which is proportional to the lift (and thus aircraft gross weight) is minor for very high-aspect-ratio aircraft.

The flap deflection required for trim is shown in Figure 3(b). The flap deflection is used for pitch control of the aircraft. As the aircraft deforms to a ‘U’-shape, the center of gravity position moves forward relative the aerodynamic center and thus lower flap deflection is required to provide the pitch-down moment.

The (root) angle of attack at the trim condition is shown in Figure 3(c). The angle of attack increases with payload as expected. The difference in the rigid and flexible angle
of attack comes from two sources, the aeroelastic deformation and the direction of the lift. The aircraft in its undeformed shape has negligible aeroelastic coupling since the aerodynamic center and the shear center are coincident. But as the aircraft deforms the drag on the wings lead to twist-up moment at the root leading to aeroelastic deformation. Thus lower root angle of attack is sufficient to provide the required lift. On the other hand, the lift generated is perpendicular to the airfoil, and as the deformation increases the direction the lift acts departs more and more from the vertical direction. Thus, for large deformation, because the aerodynamic vertical force is reduced, a larger angle of attack is required to generate the same amount of vertical force.

<table>
<thead>
<tr>
<th></th>
<th>‘empty’ payload</th>
<th>‘full’ payload</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rigid model</td>
<td>Phugoid</td>
<td>$-0.106 \pm 0.146 , i$</td>
</tr>
<tr>
<td></td>
<td>Short Period</td>
<td>$-2.84 \pm 1.82 , i$</td>
</tr>
<tr>
<td>Flexible model</td>
<td>Phugoid</td>
<td>$-0.108 \pm 0.142 , i$</td>
</tr>
<tr>
<td></td>
<td>Short Period</td>
<td>$-2.74 \pm 1.76 , i$</td>
</tr>
</tbody>
</table>

Table 2: Flight dynamics roots
3.2 Linear Stability Estimation

The aircraft as a flight dynamic/aeroelastic system is nonlinear. Once the nonlinear trim is calculated, one could then simulate the aircraft for various control settings and/or external disturbances using the complete nonlinear equations. It is prudent though to gauge the response of the system by investigating the linear dynamics of the aircraft about the trim condition. This is achieved by linearizing the aircraft at the trim condition. By analyzing the linearized system we can gauge the stability as well as the response for small disturbances.

The linear symmetric flight dynamics performance of the aircraft is normally discussed in terms of the phugoid and short-period modes of the aircraft. The two modes are the only symmetric modes of a rigid aircraft. It should be noted that the flexible aircraft has a large number of modes which have flexible as well as aircraft motion. For most conventional aircraft one would still be able to separate the modes into two flight dynamic modes which are dominated by aircraft motion and the rest as flexible modes dominated by structural deformation. For the present aircraft the low frequency flexible modes of the aircraft are in the same frequency range as the flight dynamic modes and thus there is strong coupling between the modes.

Table 2 presents the short period and phugoid modes for the rigid and flexible aircraft at the ‘empty’ and ‘full’ payload configuration. The frequency of the phugoid mode increases in frequency while the damping decreases with added payload mass. For the flexible aircraft the damping crosses the imaginary axis and the mode becomes unstable for payload above 260 lb. The root locus of the phugoid mode for the range of payload mass is shown in Figure 4(a).

![Figure 4: Root locus for a rigid as well as flexible aircraft as a function of payload; the trim condition is 40 ft/s at sea-level (green star represents ‘empty’ configuration while green circle represents ‘full’)](image)

The short-period mode root locus is shown in Figure 4(b). The short-period mode for the rigid aircraft does not change significantly with payload. The flexible aircraft
model shows a drastic change in eigenvalues of the short-period mode. The root moves rapidly with added mass; and, for payload above 95 lb, the pair of complex-conjugate short period roots merges to become two real roots. This can be expected because for increase in payload there is a corresponding increase in the deformation. The deformed ‘U’-shape leads to an order-of-magnitude increase in the pitch moment of inertia and so there is corresponding decrease in the frequency. Thus, this highly flexible aircraft does not show a classical short-period mode in its deformed state.

The phugoid mode of the aircraft can lead to instability and is thus investigated in detail. Figure 5 shows the unstable phugoid mode shape for the ‘full’ configuration. The phugoid mode shows the classical coupling of pitch and airspeed. The expected exchange of kinetic and potential energy is also seen in Figure 5(b). The aircraft loses potential energy as it loses altitude but gains kinetic energy (airfoils are further apart). In the present case, the strain energy due to deformation is also involved. As seen from Figure 5(d), the elastic deformation is significant, but it is not the dominant factor in the motion of the aircraft. The mode can be clearly seen by observing Figure 5(c). Here, the aircraft motion due to trimmed flight is removed so as to focus on the perturbations about the trimmed flight condition. The figure shows the classical elliptical motion of the aircraft about its expected trim position.

Figure 6 shows the four flexible modes with the lowest eigenvalues. Though some of these modes can be said to be dominant in bending, the others are coupled modes with torsion, bending (both directions), and aircraft motion (pitch and plunge). In fact, one of the modes develops from the real eigenvalue of the short-period mode.

3.3 Nonlinear Simulation

Linear stability analysis estimates the response of the system to small disturbances. Linear analysis does not provide information regarding response to large excitation or the response of an unstable system. To estimate the large deformation response of the system we have to solve the complete dynamic nonlinear equations in time. In the present we use a simple, second-order, central-difference, time marching algorithm with high frequency damping.

Figure 7 shows the nonlinear response of ‘full’ aircraft at 40 ft/s. The aircraft simulation is initiated at the trim state. An external disturbance is introduced by adding a flap deflection. A maximum 5° flap deflection is added to trim flap deflection. The shape of the excitation is a ramp up between 1 s and 2 s, and a ramp down between 2 s and 3 s. After 3 s the flap deflection is maintained at the trim value. A time step of 0.02 seconds is used for the simulation. The simulations for a time step of 0.01 seconds as well as 0.05 second are practically the same.

As expected the unstable phugoid mode gets excited and the amplitude of oscillation increases. The exchange of potential and kinetic energy is seen in Figure 7(a). Within two-three cycles of the oscillations the aircraft starts experiencing very high angles of attack at the highest altitudes. Since stall is not modeled in the simulation, the results after stall do not resemble the real aircraft motion. The motion of the aircraft is visualized in Figure 7(b). It is obvious from this figure that once dynamic stall is modeled one would see a slightly different response at the highest altitudes.

The simulation is robust enough to run for all possible large aircraft motion as far as the relative structural deformations are moderate. If the simulation is run for longer time then one sees the aircraft perform pitch loops (Figure 7(c)). During the pitch loops, however, the angle of attack stays small; and stall is not expected. As expected from an
unstable mode, there is loss of energy for a constant thrust and aircraft loses altitude.

4 Conclusion

A complete theoretical methodology for the analysis of a highly flexible flying wing has been presented. The analysis methodology is based on geometrically exact beam theory for elastic deformation coupled with large motion airfoil aerodynamic theory. The rigid-body degrees of freedom are accounted by including the gravity vector in the formulation. The equations for the complete analysis, including trim, linear stability and nonlinear simulation are ‘intrinsic’, i.e. the equations do not require displacement and rotation variables. The analysis accounts for realistic design space requirements including, concentrated payload pods, multiple engines, multiple control surfaces, vertical surfaces, discrete dihedral, and continuous pre-twist.

An example of a typical, flexible, flying-wing aircraft is presented and analyzed. The aircraft is very flexible and undergoes large deformation for ‘full’ payload configuration since the payload is concentrated at the center (rather than being evenly distributed over the wing). The trim shape and the corresponding trim control requirements are calculated using a trim algorithm. The trim shape as expected is a ‘U’ shape. Due to the change in the shape, the flight dynamic modes as well as the flexible modes change significantly. The classical phugoid mode becomes unstable with increase in aircraft deformation (trim). The classical short-period mode does not exist at trim because of the large pitch moment of inertia of the deformed configuration. Nonlinear simulation of the aircraft confirm that the unstable phugoid mode can be catastrophic for such aircraft.

References


Figure 5: Phugoid mode of the flexible configuration with 500 lb payload
(a) First bending (dominant): $-2.8583 \pm 2.1057i$

(b) Coupled bending-torsion-aircraft plunge:
   $-3.6045 \pm 2.9881i$

(c) Second bending (dominant): $-2.0735 \pm 5.1086i$

(d) Coupled bending-torsion-aircraft pitch:
   $-1.3762 \pm 6.3915i$

Figure 6: Flexible modes of the aircraft
Figure 7: Nonlinear simulation of aircraft response to initial flap excitation

(a) Variation in aircraft parameters

(b) Inertial 2-D view of the mid-span section: time 0-25s (aircraft moving to the left)

(c) Inertial 2-D view of the mid-span section: time 25-50s (showing pitch loops)