Decoupled Second-Order Equations and Modal Analysis of a General Nonconservative System

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Abstract

The paper presents the theoretical basis for modal analysis of a general nonconservative system. The modal analysis uses only real modes in terms of the displacements as well as velocity coordinates and converts the system to a real, uncoupled, second-order form. This is the first time that such a transformation has been presented in the literature. Earlier works presented modal analysis in the first-order state-space, of which, only the gyroscopic conservative system analysis was conducted with real modes. The modal analysis presented herein is a general framework which can be specialized for specific systems, e.g., self-adjoint conservative systems or gyroscopic conservative systems. For a self-adjoint conservative system the modal analysis reduces to the normal modal analysis. A byproduct of the modal analysis is the generation of transformations from the original physical coordinates to the real, uncoupled modal space. Such transformations are shown to be quite useful in understanding the behaviour of a nonconservative system. It is used in the present paper to shed light on the destabilizing effect of damping in circulatory system.

Introduction

Modal analysis is a primary tool that structural engineers use regularly. It has become the backbone of linear structural dynamics and related fields like aeroelasticity. The popularity of modal analysis is due to its numerous benefits. One of the most important advantage is its ability to reduce the order of a system by choosing the most important modes of the system. Also, the natural modes decouple the (conservative) structural system and further simplifies the system analysis. Finally, the most important advantage of using natural modes and modal analysis is the tremendous insight gained into the characteristics and behavior of the system. The only disadvantage of modal analysis is the computationally expensive calculation of the normal modes. But with the help of efficient eigenvalue search algorithms and the advent of powerful computers this obstacle is becoming easier to handle.

Modal analysis as described above has been completely developed for the case of self-adjoint conservative systems. For nonconservative systems the self-adjoint conservative part of the system can still be represented by modal analysis. One could reduce the order of the structural part of the system and then use the modal representation to compute the complete system. But, since the equations of motion are no longer uncoupled any possibility of analytical solutions or in general the modal perspective and insight is lost. This is a great loss which the present paper attempt to overcome.

As an example one could consider aeroelasticity. Aeroelastic problems exhibit a wide range of nonconservative behavior, and is one of the field which would benefit drastically by a general modal analysis. Apart from the insight gained, modal analysis of the complete system would be helpful in computing the aeroelastic modes, not just structural or aerodynamic modes. It would thus be helpful in reducing the complete aeroelastic system instead of just the structural portion. This would be especially helpful considering the current interest in eigenvalue analysis of the aerodynamics. The modal analysis of the complete system would help in reducing the order of the aerodynamic degrees of freedom as well.

Modal analysis of self-adjoint conservative structural systems is very well-known and can be found it almost all structural dynamics textbooks. There have been a few extensions. Caughey and O’Kelly1 derived a necessary and sufficient condition under which one could uncouple the equations of motion of a damped system using the undamped normal modes. Foss2 has derived uncoupled first-order equations of motion of a general
damped system using the state-space system eigenvalues and eigenmodes. The solution is in terms of complex modes and thus not very attractive.

Modal analysis of non-self-adjoint but conservative systems, e.g., gyroscopic systems, has been presented by Meirovitch. The method for modal analysis included representing \( n \) second-order differential equations in a 2\( n \) first-order state-space form, followed by eigenvalue analysis and then representation of the system in terms of real and imaginary parts of the complex eigenvectors, leading to \( n \) uncoupled pairs of coupled first-order equations. One could then apply analytical solution for each pair of coupled first-order equations. This analysis is good for modal analysis because it uses only real quantities. Such an analysis can easily be extended to a general system. The only changes would be that the form of the coupling in the coupled first-order equations will be different and thus the analytical form of the modal solution will have to be derived accordingly.

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Transformation to decoupled second-order set

Consider a general system with \( n \) degrees of freedom. The system can be represented as

\[
[M] \{\ddot{q}\} + [C + G] \{\dot{q}\} + [K + H] \{q\} = \{Q\}
\]

where, \( \{q\} \) are the generalized coordinates of the systems. \( [M], [C], [G], [K], \) and \( [H] \) are the inertial, damping, gyroscopic, stiffness and circulatory matrices. Inertial, damping and stiffness matrices are symmetric, while gyroscopic and circulatory forces are antisymmetric. \( \{Q\} \) is the generalized force vector.

First-Order Form

Transforming the above equation to a first-order form we get

\[
\begin{bmatrix}
I_n & 0_n \\
0_n & M
\end{bmatrix}
\begin{bmatrix}
\dot{q} \\
\ddot{q}
\end{bmatrix}
=
\begin{bmatrix}
0_n & I_n \\
-(K + H) & -(C + G)
\end{bmatrix}
\begin{bmatrix}
q \\
\dot{q}
\end{bmatrix}
+ \begin{bmatrix}
0_{n \times 1} \\
Q
\end{bmatrix}
\]

where, \( 0_n \) denote a \( n \times n \) matrix of zeros and \( I_n \) denotes an identity matrix of size \( n \times n \).

The above equation can be represented in a first-order form as

\[
[B] \{\dot{x}\} = [A] \{x\} + \{X\}
\]

where

\[
[A] = \begin{bmatrix}
0_n & I_n \\
K + H & C + G
\end{bmatrix}
\]

\[
[B] = \begin{bmatrix}
I_n \\
0_n \\
0 & M
\end{bmatrix}
\]

\[
\{x\} = \begin{bmatrix}
q \\
\dot{q}
\end{bmatrix}, \quad \{X\} = \begin{bmatrix}
0_{n \times 1} \\
Q
\end{bmatrix}
\]

Now, the aim is to convert the above set of \( 2n \) coupled, first-order differential equations into a set of \( n \) uncoupled second-order equations.

Complex Eigenvalue Problem

Consider the eigenvalue problem based on the homogeneous part of the above equation given by

\[
\lambda_i \ [B] \{u_i\} = [A] \{u_i\} \quad (i = 1, 2, \cdots, 2n)
\]

and the adjoint eigen-problem given by

\[
\lambda_i \ [B^T] \{v_i\} = [A]^T \{v_i\} \quad (i = 1, 2, \cdots, 2n)
\]

where, \( \lambda_i \) are the eigenvalues (same for both eigenvalue problems), and, \( \{u_i\} \) and \( \{v_i\} \) are the right and left eigenvectors of (\( [A], [B] \)) system.

The left and right eigenvectors are biorthogonal and can be normalized to satisfy the following conditions (similar to that accomplished Meirovitch for \( B = I_{2n} \))

\[
\{v_i\}^T [B] \{u_j\} = 2\delta_{ij}
\]

\[
\{v_i\}^T [A] \{u_j\} = 2\lambda_i \delta_{ij}
\]

The 2 in the above equations is used so that it will give a good normality condition using real basis vectors as shown later in the paper.

Since \( [A] \) and \( [B] \) are a real matrices, the eigenvalues are real or complex conjugate pairs. In most of the structural systems one would have complex conjugate
roots, thus we shall for the present consider the eigenvalues to be \( n \) complex conjugate pairs. The present analysis will also work for real pairs but the methodology is slightly different and will be presented later. The \( n \) pairs of complex conjugate eigenvalues have to \( n \) pairs of complex conjugate eigenvectors (both right and left). Thus writing the eigenvalues and eigenvectors in terms of their real and imaginary part

\[
\lambda_j = \alpha_j + i\beta_j \\
\lambda_{n+j} = \overline{\lambda_j} = \alpha_j - i\beta_j
\]

\( \{u_j\} = \{y_j\} + i\{z_j\} \) \( \{u_{n+j}\} = \{\overline{y}_j\} = \{y_j\} - i\{z_j\} \)

\( \{v_j\} = \{r_j\} + i\{s_j\} \) \( \{v_{n+j}\} = \{\overline{r}_j\} = \{r_j\} - i\{s_j\} \)

(10)

**Basis of Real Vectors**

Now, consider a basis of vectors given by

\[
[\mathbf{U}] = \begin{bmatrix} y_1, z_1, y_2, z_2, \cdots, y_n, z_n \end{bmatrix} \\
[\mathbf{V}] = \begin{bmatrix} r_1, -s_1, r_2, -s_2, \cdots, r_n, -s_n \end{bmatrix}
\]

(11)

For the set of vectors defined above we have the new biorthonormality relations given by

\[
[\mathbf{V}]^T [\mathbf{B}] [\mathbf{U}] = I_{2n} \\
[\mathbf{V}]^T [\mathbf{A}] [\mathbf{U}] = [\mathbf{A}]
\]

(12)

where, \( [\mathbf{A}] \) is a block diagonal matrix consisting of \( n \) blocks of size \( 2 \times 2 \). Each block can be represented as

\[
[\mathbf{A}_r] = \begin{bmatrix} \alpha_r & \beta_r \\ -\beta_r & \alpha_r \end{bmatrix} \quad (r = 1, 2, \cdots, n)
\]

(13)

From the above equation it is clear that now if one represents the variables of the state-space system in terms of the modes \( [\mathbf{U}] \) we have

\[
\{x\} = [\mathbf{U}] \{\xi_r, \eta_r, \xi_2, \eta_2, \cdots, \xi_n, \eta_n\}^T
\]

(14)

and, substituting the above modal expansion into the state-space equation (Eq. (3)) and premultiplying by \([\mathbf{V}]^T\), we get \( n \) uncoupled pairs of first-order equation given by

\[
\begin{bmatrix} \xi_r \\ \eta_r \end{bmatrix} = \begin{bmatrix} \alpha_r & \beta_r \\ -\beta_r & \alpha_r \end{bmatrix} \begin{bmatrix} \xi_r \\ \eta_r \end{bmatrix} + \begin{bmatrix} \mathbf{X}_r^\xi \\ \mathbf{X}_r^\eta \end{bmatrix}
\]

(15)

where

\[
\mathbf{X}_r^\xi = \{r_r\}^T \{X\} \\
\mathbf{X}_r^\eta = \{-s_r\}^T \{X\}
\]

(16)

The set of equations can be further explored for modal analysis similar to Meirovitch. This is not the aim of the present work and so is not followed further. But the equations presented above are the starting point to combine each pair of first-order equations into one second-order equation.

**Second-Order Form**

To transform the pair of first-order equations to a second-order equation one needs a similarity transform which converts the above given form of each block diagonal \([\mathbf{A}_r]\) to a form such that

\[
[A_r] = \begin{bmatrix} 0 & 1 \\ -(\beta_r^2 + \alpha_r^2) & 2\alpha_r \end{bmatrix} \quad (r = 1, 2, \cdots, n)
\]

(17)

Such a transformation is possible and is given below

\[
\begin{bmatrix} 1 & 0 \\ \alpha_r & \beta_r \end{bmatrix} \begin{bmatrix} \alpha_r & \beta_r \\ -\beta_r & \alpha_r \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\alpha_r/\beta_r & 1/\beta_r \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\beta_r^2 + \alpha_r^2) & 2\alpha_r \end{bmatrix}
\]

(18)

and

\[
\begin{bmatrix} 1 & 0 \\ \alpha_r & \beta_r \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\alpha_r/\beta_r & 1/\beta_r \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

(19)

Keeping the above relationships in mind we can rewrite the equations of motion in terms of a new set of assumed modes given by

\[
[U] = [y_1 - (\alpha_1/\beta_1)z_1, (1/\beta_1)z_1, \cdots, y_n - (\alpha_n/\beta_n)z_n, (1/\beta_n)z_n]
\]

(20)

\[
[V] = [r_1, \alpha_1 r_1 - \beta_1 s_1, \cdots, r_n, \alpha_n r_n - \beta_s s_n]
\]

now, the biorthonormality conditions become

\[
[V]^T [B] [U] = I_{2n} \\
[V]^T [A] [U] = [A^*]
\]

(21)

where, \([A^*]\) is a block diagonal matrix consisting of \( n \) blocks of size 2. Each block has the form

\[
[A^*] = \begin{bmatrix} 0 & 1 \\ -(\beta_r^2 + \alpha_r^2) & 2\alpha_r \end{bmatrix} \quad (r = 1, 2, \cdots, n)
\]

(22)

Thus now using a modal expansion in terms of \([U]\)

\[
\{x\} = [U] \{\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_n, \eta_n\}^T
\]

(23)

and, substituting into the state-space equation we get \( n \) uncoupled sets of 2 first-order equation given by

\[
\begin{bmatrix} \xi_r \\ \eta_r \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\beta_r^2 + \alpha_r^2) & 2\alpha_r \end{bmatrix} \begin{bmatrix} \xi_r \\ \eta_r \end{bmatrix} + \begin{bmatrix} \mathbf{X}_r^\xi \\ \mathbf{X}_r^\eta \end{bmatrix}
\]

(24)
where
\[ X_r^\xi = \{ r_r \}^T \{ X \} \]
\[ X_r^\eta = \{ \alpha r_r - \beta r_s \}^T \{ X \} \]  
(25)

The equations are in a form which can be transformed into a set of uncoupled second-order equations. For free vibration and response to initial condition nothing more is required. For forced response there is still one obstacle to converting the above set of equations to uncoupled second-order equations, viz., \( X_r^\xi \neq 0 \). If \( X_r^\xi = 0 \), then the \( \eta_r \)'s and \( \xi_r \)'s cannot be related as \( \eta_r = \xi_r \), which is required for a building a second-order system.

**Calculation of Modal Forces**

To transform the above equations to second-order form and to calculate the modal force one needs to define the eigenvectors so that \( X_r^\xi = 0 \). Let us turn our attention back to the beginning of this derivation when the original set of eigenvalues and eigenvectors were calculated (Eqs. (7) and (8)). \( \{ \nu_r \} = \{ r_r \} + i \{ s_r \} \), for \( r = 1, 2, \cdots, n \) were the left eigenvectors. Now even though the eigenvectors were so defined it should be noted that any vector obtained by multiplying an eigenvector by a complex constant, i.e., \( (c_R + i c_I) \{ \nu_r \} \) is also an eigenvector of the system. The right eigenvector will have to be multiplied by the complex conjugate of the constant to satisfy the orthogonality condition (Eq. (9)). Furthermore, if \( (c_R^2 + c_I^2) = 1 \) then the new set of eigenvectors would also satisfy the normality condition. Thus, multiplying the original set of left eigenvectors by a complex constant (and the right eigenvectors by its complex conjugate) leads to a different set of eigenvectors on which to base the modal analysis. Now, it is trivial to show that one such complex constant leads to a eigenvector such that \( X_r^\xi = 0 \). For each mode one can find \( c \) such that \( (c_R^2 + c_I^2) = 1 \) and \( (c_R \{ r_r \}^T - c_I \{ s_r \}^T) \{ X \} = 0 \). For the present case such a constant is
\[ c_r = \frac{1}{\left( X_r^\eta + i X_r^\xi \right)^{0.5}} \left( X_r^\eta - i X_r^\xi \right) \]  
(26)

Thus knowing the above constant, one can start with eigenvectors given by \( (c_R + i c_I) \{ \nu_r \} \) instead of the original \( \{ \nu_r \} \) and based on those eigenvectors we have all the \( X_r^\xi \) equal to zero. And we get the modal force as \( X_r^\eta \).

**Second-Order Uncoupled Equations of Motion**

With \( X_r^\xi = 0 \), we can use the relation relation \( \eta_r = \xi_r \) from Eq. (24) and thus get \( n \) uncoupled second-order equations of motion as
\[ \ddot{\xi}_r - 2\alpha_r \xi + (\alpha_r^2 + \beta_r^2) \xi = X_r^\eta, \quad (r = 1, 2, \cdots, n) \]  
(27)

The above set of equations can be obtained by using a particular set of eigenvectors (corresponding to a given force vector). Now, since the base eigenvectors chosen for modal analysis are a function of the force vector, if there are more than one force vectors then it is quite likely that \( X_r^\xi \) will not be zero for all force vectors using just one set of basis vectors. To uncouple the equations in terms of a single set of eigenvectors is a bit involved. Such an analysis leads to complex force vectors given by \( X_r^\eta + i X_r^\xi \). Such an analysis will be presented later.

**Transformation for System Parameter Changes**

In a general nonconservative system the effect of parameter changes on the overall system performance is not as intuitive as that for self-adjoint conservative system. For example, addition of damping in a circulatory system can make the system unstable, or, change in stiffness can lead to destabilizing of the system. With the uncoupling of the system in the modal basis as presented herein, it is now possible to determine the effect of changes in the system parameters on the system response easily by representing the changes in the uncoupled modal space. As such, this insight will be very helpful in system design and modification for performance improvement.

Let us begin by considering how the system parameters are transferred to the modal decoupled equations. To get a physical insight into the problem let us represent the modal matrices in terms of representative blocks as
\[ [U] = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \]  
(28)
such that
\[ \{q\} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \{ \xi \} \]  
(29)
and correspondingly
\[ [V] = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \]  
(30)

Now substituting the above representation of the modal matrices into the original first-order equation
We can get the analytical form of the modal equations as

\[
\begin{bmatrix}
V_{11}^{T}U_{11} + V_{21}^{T}MU_{21} \\
V_{11}^{T}U_{12} + V_{21}^{T}MU_{22}
\end{bmatrix}
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix}
+ \begin{bmatrix}
-V_{11}^{T}U_{11} + V_{21}^{T}(K + H)U_{11} + V_{21}^{T}(C + H)U_{12} \\
-V_{11}^{T}U_{21} + V_{21}^{T}(K + H)U_{11} + V_{21}^{T}(C + G)U_{21}
\end{bmatrix}
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix}
+ \begin{bmatrix}
-V_{11}^{T}U_{22} + V_{21}^{T}(K + H)U_{12} + V_{21}^{T}(C + G)U_{22} \\
-V_{11}^{T}U_{12} + V_{21}^{T}(K + H)U_{12} + V_{21}^{T}(C + G)U_{22}
\end{bmatrix}
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix}
= \begin{bmatrix}
V_{21}^{T}Q \\
V_{22}^{T}Q
\end{bmatrix}
\]

(31)

The above equation for properly chosen basis vectors leads to decoupling of the problem. But it is easy to see why the dynamics of general nonconservative systems is not obvious. It is seen that for a general system all the parameters in \([C], [G], [K],\) and \([H]\) can directly affect both the frequency and damping of the system.

Also from the above break up of the system now it is easy to see the simplification of the system for specialized cases. For example, for a self-adjoint conservative system we have \([U_{11}] = [U_{22}] = [V_{11}] = [V_{22}]\) and \([U_{12}] = [U_{21}] = [V_{21}] = [V_{21}] = [0s].\) For such a system we have \([K]\) affecting only the first order of the system.

Let us now consider a system with a nominal model as considered up to now but with a possibility of parameter changes represented here as

\[
[M + \Delta m] \{q\} + [C + G + \Delta cg] \{\dot{q}\}
+ [K + H + \Delta kh] \{\ddot{q}\} = \{Q\}
\]

(32)

where, \(\Delta m, \Delta cg\) and \(\Delta kh\) represents the possible parameter change in the original system. Now the changes to the system matrices in the first-order form in the modal space can be written as

\[
[\Delta B] = \begin{bmatrix}
V_{21}^{T}\Delta m U_{21} & V_{21}^{T}\Delta m U_{22} \\
V_{22}^{T}\Delta m U_{21} & V_{22}^{T}\Delta m U_{22}
\end{bmatrix}
\]

(33)

\[
[\Delta A] = \begin{bmatrix}
V_{21}^{T}\Delta kh U_{11} + V_{21}^{T}\Delta cg U_{21} & V_{21}^{T}\Delta kh U_{12} + V_{21}^{T}\Delta cg U_{22} \\
V_{22}^{T}\Delta kh U_{11} + V_{22}^{T}\Delta cg U_{21} & V_{22}^{T}\Delta kh U_{12} + V_{22}^{T}\Delta cg U_{22}
\end{bmatrix}
\]

(34)

The above equation gives the analytical form of the change in first-order modal basis equations with changes in the original system parameters. The above form does not give any more insight into the effect of parametric change as compared to the original system equations. More understanding would be derived if the effect on the second-order decoupled equation is calculated. Transformation of the matrices corresponding to the parameter change into the second-order form is possible but has to be done for each linearly independent column of \(\Delta m, \Delta cg\) and \(\Delta kh\) separately as given below.

For a column of change in either of \(\Delta m, \Delta cg\) and \(\Delta kh\) one can find a basis vectors derived such that \([V_{21}] \text{ (Column)} = 0.\) This is done in exactly the same way in which the modal force was calculated, i.e., changing the phase of the original eigenvectors and rederiving the new set of basis. Once that is established one can find the effect on the second-order decoupled system from the above equations. All such effects are then added. This can be tedious if the a given parameter affects many columns independently but the results may be worth the effort. In general, we have a parameter affecting only one column and its effect in the second-order modal space can be easily calculated.

Thus we have the second-order system mass, stiffness and damping matrices as the sum of the uncoupled part and the addition due to parameter change as given below

\[
\begin{align*}
\mathbf{M} &= I_n + V_{22}^{T}\Delta m U_{22} \\
\mathbf{K} &= \text{Diag}(\beta^2 + \alpha^2) + V_{22}^{T}\Delta kh U_{11} + V_{22}^{T}\Delta cg U_{21} \\
\mathbf{C} &= \text{Diag}(-2\alpha) + V_{22}^{T}\Delta cg U_{22} + V_{22}^{T}\Delta kh U_{12} + V_{22}^{T}\Delta m U_{21}
\end{align*}
\]

(35)

## Modal Analysis

The previous section described a way to choose modeshape so as to convert the original system (Eq. (1)) into a set of \(n\) uncoupled equations. Thus representing the system states in the modal form we have

\[
\{ q \} = \{ x \} \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix}
\]

(36)

The inverse relationship is given by

\[
\begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix} = [V]^T \begin{bmatrix} q \\ \dot{q} \end{bmatrix}
\]

(37)

And, the system of equations are

\[
\ddot{\xi}_r - 2\alpha_r \dot{\xi}_r + (\alpha_r^2 + \beta_r^2)\xi_r = X_r, \quad (r = 1, 2, \cdots, n)
\]

(38)

where

\[
X_r = \{ \alpha_r \dot{r}_r - \beta_r s_r \}^T \begin{bmatrix} 0 \\ Q \end{bmatrix}
\]

(39)

again it is important to select the original eigenvectors such that \(X_\xi = 0\)
Before proceeding further, it would be interesting to see the similarities and differences of the modal analysis presented here with that used for conservative self-adjoint systems. There are two major differences, i) the modal expansion (and thus inversion) relates the generalized displacements as well as velocities to both modal displacements and velocities, ii) the uncoupled equations has a part proportional to velocity (damping term).

**Free Vibration Response to Initial Condition**

Consider the response of the system given in Eq. (1), with initial condition given by \( q(0) = q^0 \) and \( \dot{q}(0) = q^0_e \). The initial conditions for our set of uncoupled equations can be written as

\[
\begin{align*}
\{ \xi(0) \\ \xi(0) \} &= \{ \xi^0 \\ \eta^0 \} = [V]^T \{ q^0 \\ q^0_e \} \\
\end{align*}
\]

(40)

Now the response of a second-order system to an initial disturbance is given by

\[
\begin{align*}
\xi_r(t) &= e^{\alpha_r t} \left( \xi^0_r \cos \beta_r t + \frac{\eta^0_r - \alpha_r \xi^0_r}{\beta_r} \sin \beta_r t \right) \\
\dot{\xi}_r(t) &= \eta_r(t) \\
&= \alpha_r e^{\alpha_r t} \left( \xi^0_r \cos \beta_r t + \frac{\eta^0_r - \alpha_r \xi^0_r}{\beta_r} \sin \beta_r t \right) \\
&+ \beta_r e^{\alpha_r t} \left( -\xi^0_r \sin \beta_r t + \frac{\eta^0_r - \alpha_r \xi^0_r}{\beta_r} \cos \beta_r t \right)
\end{align*}
\]

(41)

Closed form expressions for the actual generalized coordinates (and their time derivative) can be expressed using the above expressions in Eq. (36). An interesting point is that the closed form expression is simpler in terms of the intermediate modeshapes given in \( \{ U \} \) (Eq. (37)) as

\[
\begin{align*}
\{ \eta \} &= \sum_{r=1}^{n} e^{\alpha_r t} \left( \xi^0_r \cos \beta_r t + \frac{\eta^0_r - \alpha_r \xi^0_r}{\beta_r} \sin \beta_r t \right) \{ y_r \} \\
&+ e^{\alpha_r t} \left( -\xi^0_r \sin \beta_r t + \frac{\eta^0_r - \alpha_r \xi^0_r}{\beta_r} \cos \beta_r t \right) \{ z_r \}
\end{align*}
\]

(42)

**Harmonic Response**

The harmonic response of the system can similarly be written in terms of the harmonic responses of the individual second-order systems. The response of a second-order system to harmonic excitation of amplitude \( X_r \) and frequency \( \omega \) is given by

\[
\begin{align*}
\xi_r(\omega) &= \frac{X_r}{(\alpha_r^2 + \beta_r^2) - \omega^2 - i2\alpha_r \omega} \\
\dot{\xi}_r(\omega) &= \eta_r(\omega) = \frac{iX_r \omega}{(\alpha_r^2 + \beta_r^2) - \omega^2 - i2\alpha_r \omega}
\end{align*}
\]

(43)

Thus the frequency response of the complete system can be written as

\[
\begin{align*}
\{ q \} &= \sum_{r=1}^{n} \frac{X_r}{(\alpha_r^2 + \beta_r^2) - \omega^2 - i2\alpha_r \omega} \{ y_r \} \\
&+ \frac{i\omega - \alpha_r}{\beta_r} \{ z_r \}
\end{align*}
\]

(44)

from the above equation the phase and amplitude of the response can be easily calculated.

**Examples**

**Modal Analysis of a Damped Gyroscopic System**

The example considered is a dual-spin flexible spacecraft with structural viscous damping. The degrees of freedom considered are the nutation angles \( \theta_1 \) and \( \theta_2 \), and the elastic modes represented by \( \zeta_1 \) and \( \zeta_2 \). This spacecraft model is a reduced order model based on Ref. 6. The damping as suggested by Meirovitch and Barnh6 is used. The system generalized coordinates are \( \{ q \} = \{ \theta_1, \theta_2, \zeta_1, \zeta_2 \} \). The system matrices are given by

\[
[M] = \begin{bmatrix} 2500 & 0 & 0 & 5.29 \\
0 & 8000 & -54.71 & 0 \\
0 & -54.71 & 0.4979 & 0 \\
5.29 & 0 & 0 & 0.06499 \end{bmatrix}
\]

(45)

\[
[C + G] = \begin{bmatrix} 0 & 7536 & 0 & 0 \\
-7536 & 0 & 0 & 0 \\
0 & 0 & 0.0998 & 0 \\
0 & 0 & 0 & 0.0361 \end{bmatrix}
\]

(46)

\[
[K + H] = \begin{bmatrix} 6.31 \times 10^4 & 0 & 0 & 0 \\
0 & 6.31 \times 10^4 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2 \end{bmatrix}
\]

(47)

Modal analysis is conducted based on the theory presented in this paper. Using, the modal analysis one can find an analytical expression for the free vibration
response (based on Eq. (43)). The initial conditions are assumed to be
\[
\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.0001 \\ 0.02 \\ 0.0002 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \xi_r \\ x_{1r} \end{bmatrix} = \begin{bmatrix} 0.06414 \\ -0.4028 \\ 0.01643 \\ -0.2045 \\ -0.03356 \end{bmatrix}
\]

The response of \( \dot{\theta}_2 \) can be written as
\[
\dot{\theta}_2(t) = e^{-0.3055t} \left( 6.989 \times 10^{-5} \cos(8.659t) 
+ 4.253 \times 10^{-5} \sin(8.659t) \right)
+ e^{-0.3152t} \left( 3.49 \times 10^{-5} \cos(5.917t) 
+ 7.528 \times 10^{-6} \sin(5.917t) \right)
+ e^{-0.0721t} \left( 1.638 \times 10^{-5} \cos(4.032t) 
- 2.795 \times 10^{-5} \sin(4.032t) \right)
+ e^{-0.04605t} \left( -2.117 \times 10^{-5} \cos(1.67t) 
+ 7.262 \times 10^{-7} \sin(1.67t) \right)
\]

For a disturbance in the first flexible mode the frequency response of the nutation parameter is assumed to be stiffness proportional. From the above system the critical load parameter, \( \beta \), is found that for no damping in the system (\( \mu = 0 \)) the critical load parameter is \( \beta_{cr} = 20.1045 \). Even the smallest amount of damping destabilizes the system at a very low critical load of \( \beta_{cr} = 10.6804 \).

Effect of Damping on a Circulatory System

A follower force problem is considered here as an example of circulatory system. A cantilevered beam is subjected to a compressive follower force at the free end of the beam. The exact solution is given in the book by Bolotin. Here a two mode approximation of the cantilever beam gives the system equations as

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \{ \ddot{q} \} + \mu \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \{ \dot{q} \}
+ \begin{bmatrix} k_1 + 0.8582 \beta & -11.74 \beta \\ 1.874 \beta & k_2 - 13.30 \beta \end{bmatrix} \{ q \} = 0
\]

where, \( k_1 = 12.3623 \) and \( k_2 = 485.5225 \) are the nondimensional stiffness parameters, \( \mu \) is the nondimensional damping parameter and \( \beta \) is the nondimensional load parameter. Damping in the above equation is assumed to be stiffness proportional. From the above system the critical load parameter, \( \beta_{cr} = 20.1045 \). Even the smallest amount of damping destabilizes the system at a very low critical load of \( \beta_{cr} = 10.6804 \).

To investigate the effect of addition of damping in this circulatory system, a \( \beta = 15 \) case is considered. Using the methodology developed earlier in the paper we can now calculate the effect of addition of damping on the system parameters in the second-order decoupled modal space. It is seen that in the decoupled modal space the addition of damping leads to change in the damping matrix. But there is a drastic difference between the added damping with respect to the original coordinate and the damping matrix in the modal space. For the case considered

\[
\Delta_{cg} = \mu \begin{bmatrix} 12.3623 & 0 \\ 0 & 485.5225 \end{bmatrix}
\]

leads to the modal damping matrix

\[
\mathcal{C} = \mu \begin{bmatrix} -32.05 & 308.13 \\ -74.61 & 529.94 \end{bmatrix}
\]

Thus, the system in the modal space can be written
Thus, it can be seen that the damping characteristics of the system in the modal space are completely different from the original damping. It is now very clear that the first mode will go unstable even for the smallest amount of stiffness proportional damping. Thus even for the positive definite damping matrix (in the original co-ordinates space) leads to destabilization.

One can explain the damping characteristics further by investigating analytically the effect of parameter changes on a circulatory system. For a circulatory system without damping and below the critical load, the eigenvalues are purely imaginary and the modeshapes are such that $[U_{12}] = [U_{21}] = [V_{12}] = [V_{21}] = [0_n]$. In addition, $[U_{11}] = [U_{22}]$ and $[V_{11}] = [V_{22}]$. The only difference as compared to the self-adjoint conservative system is that $[U_{11}] \neq [V_{11}]$. Under these conditions we can easily write the effect of damping for the circulatory system from the transformation as

$$\bar{C} = V_{22}^T \Delta_{cg} U_{22} = U_{22}^{-1} \Delta_{cg} U_{22}$$

(58)

Thus it can be seen that even for circulatory system (like the self-adjoint system) the eigenvalues of the original matrix are retained in the transformation (since it is a similarity transformation). But, in the self-adjoint system the transformation retains symmetry while in the circulatory system it does not. Thus for the self-adjoint conservative system the positive definiteness of the damping matrix is conserved in the modal space but for the circulatory system it is the eigenvalues of the symmetric part of $V_{22}^T \Delta_{cg} U_{22}$ that determines the positive definiteness and the stability of the system.

**Conclusions**

A modal analysis for a general nonconservative system is presented. Modal analysis is based on a transformation that converts a general system into a set of uncoupled second-order systems. One can then use analytical solutions for each of the uncoupled second-order system. Also, one can then reduce the order of the system based on the frequency and the damping of each mode (unlike just frequency based reduction of regular modal analysis). Unlike the modal analysis of self-adjoint systems, in the modal analysis of general system the generalized displacements (and velocities) are functions of both the modal displacements and velocities.

The theoretical basis established for modal analysis of general nonconservative systems is likely to help in understanding some complex non-conservative phenomenon. In the present paper, the understanding derived from modal analysis is applied to explain the destabilizing effects of damping in circulatory systems.

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**References**


Figure 1: Free vibration response to initial condition

Figure 2: Frequency response of nutational angle $\theta_2$ to disturbance in the flexible mode $\zeta_1$