A NUMERICAL FEASIBILITY STUDY OF A PARAMETRIC ROLL ADVANCE WARNING SYSTEM

Leigh Shaw McCue
Aerospace and Ocean Engineering
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061
Email: mccue@vt.edu

Gabriele Bulian
DINMA-University of Trieste
Trieste, Italy
Presently research fellow at:
Naval Architecture, Ocean and Environmental Engineering,
Graduate School of Engineering
Osaka University
Osaka, Japan
Email: gbulian@units.it

ABSTRACT
This work studies the practicality of using finite-time Lyapunov exponents (FTLEs) to detect the inception of parametric resonance for vessels operating in irregular longitudinal seas. Parametrically excited roll motion is modelled as a single degree-of-freedom system, with nonlinear damping and restoring terms. FTLEs are numerically calculated at every integration time step. Using this numerical model of parametric roll and through tracking trends in the FTLE time series behavior, warnings of parametric roll are identified. This work serves as a proof-of-concept of the FTLE technique’s viability in detecting parametric resonance. The ultimate aim of the research contained in this paper, along with future work, is the development of a real-time, on-board aid to warn of impending danger allowing for avoidance of severe, even catastrophic, vessel instabilities.

Keywords: capsize, chaos, Lyapunov exponents, parametric roll

INTRODUCTION
Parametric roll is a phenomenon arising, most typically, in waves of length approximately equal to the ship length at an encounter frequency, \( \omega_e \), twice that of the roll natural frequency, \( \omega_0 \). However, due to both nonlinear effects on the restoring term and Doppler effects, the most dangerous situations in terms of roll or roll accelerations very seldom occur at the exact 2:1 resonance, with respect to any characteristic frequency of the excitation spectrum at the encounter frequencies. Such uncertainty, however, reduces when the spectrum of the parametric excitation is sufficiently narrow banded. In ships subject to these conditions, particularly vessels with large bow flare and a wide transom stern, the roll restoring arm changes substantially due to changes in the underwater hull geometry from when the wave peak is amidships versus peaks at the bow and stern. In addition pitch affects the shape of the underwater hull geometry, and heave motion changes the instantaneous displacement, inducing additional parametric excitation with respect to the “pure geometrical” effect of waves. This periodic oscillation in the roll restoring arm can result in rapidly, and unexpectedly increasing roll motions. While its theoretical existence has been known of for decades, parametric roll has only recently garnered attention from a regulatory and prevention standpoint [1, 2]. Such incidents as the APL China [3], which involved litigation on the order of $100 mil, have proven that parametric roll exists beyond theory and carries substantial financial risk and grave potential for loss of life. In addition, several other cases of large amplitude roll motions, likely due to parametric excitation, seem to occur worldwide to other types of vessels [4], and this is raising concerns regarding safety and even the actual economic risk involved in transportation [5]. On the other hand, new future giant container designs are presently being tested specifically for parametric roll [6], and classification societies are trying to provide means for prevention and mitigation of this problem [2].

Today, parametric roll represents potentially as great a stability issue as the beam sea condition in the past. This is not to say that the beam sea condition is no longer a problem, but simply that, due the change in the ship typologies, and the almost constant environmental conditions (at least with respect to the “mean human time scale”), there are some ships more
likely to encounter environmental conditions able to trigger the parametric roll phenomenon. Often this type of ships are less endangered in beam sea, especially when they are particularly large. Of course, other types of ship, especially small ships, are still prone to suffer very dangerous motions, and even capsize, in beam sea. A correct definition of the most dangerous phenomena for every ship is thus of utmost importance: the designer should indeed provide information to the master of the vessel regarding the best way to respond to harsh weather conditions. The usual way of keeping bow/head sea at reduced speed in rough weather has been shown by facts to be sometimes not only ineffective, but even dangerous [3]. Experiments, in addition, have demonstrated that the increasing of speed, in certain conditions, can be an effective means for the reduction of the parametrically excited rolling motion [7, 8]. On the other hand, again, experiments, have shown that change of course could not be an efficient way of escaping from parametric resonance [7], especially in short crested irregular waves where the pure parametric (auto-parametric) excitation combines with a “beam sea type” excitation (excitation of the ideal source of energy type according to [9]). Of course an increase in speed is likely to increase undesired effects such as slamming or water on deck. For this reason, a correct ship handling could be thought as a sort of multi-objective optimization problem, where parametric roll phenomenon is to be taken into account due to its capability of inducing large amplitude motions and consequent accelerations. Accelerations, are, indeed, to be properly considered because of the fact that cargo shift, a typical initiating event [10] able to lead to capsize, is strongly related not only to the instantaneous rolling angle (that is, by definition, a static quantity), but also to the actual accelerations [11]. It is then important to get insights into the parametric roll phenomenon in realistic environmental conditions, i.e. mainly irregular sea, and for this reason analytical models are very effective in a preliminary phase of the research. Although they are not fully reliable for accurate predictions, they can give important information on the underlying physics of the phenomenon under analysis, tracing the main way to be followed by means of more sophisticated calculation tools. In addition, analytical models can more easily be tackled by means of theoretical tools, that can hardly be applied to fully numerical computations. In particular, in this paper, Lyapunov exponents are considered with respect to an analytical 1-DOF model for parametrically excited roll motion in irregular longitudinal long crested waves.

Lyapunov exponents measure the rate of convergence or divergence of nearby trajectories. A positive Lyapunov exponent indicates exponential divergence, or a strong sensitivity to initial conditions. This sensitivity is a standard sign of chaotic behavior. The aim of this research is to detect this sort of rapid divergence of the roll/roll-velocity trajectory. Therefore, measuring rates of divergence/convergence of neighboring trajectories while tracking trends of instabilities, can be used as an indicator of foreseeable danger.

The use of Lyapunov exponents to study capsize has been touched upon in the literature for both naval architecture and nonlinear dynamics. In recent years the asymptotic Lyapunov exponent has been calculated from equations of motion for the mooring problem [12], single degree of freedom capsize models [13, 14], single degree of freedom flooded ship models [15–17], works studying the effects of rudder angle while surf-riding as it leads to capsize [18], and investigation into the use of the tool for validating numerical simulation to experimental data [19, 20]. In order to anticipate chaotic vessel motions in real time, one can instead use the incremental variant, i.e. finite-time Lyapunov exponents (FTLEs) to detect instantaneous changes in ship behavior [21, 22].

MODELLING

ANALYTICAL MODEL

Analytical nonlinear modelling of parametric roll in regular sea has received, in the past, a considerable attention, both with respect to 1-DOF modelling (e.g. [23–25]) and multi-DOF approaches [26]. Much less effort seems to have been devoted to the development of nonlinear models for the more realistic irregular sea conditions. Nonlinearities in the analytical model are of fundamental importance both in damping and restoring, in order to allow the model for providing bounded statistics inside the stochastic instability region. In this paper, an analytical 1.5-DOF model is used. The model was developed in [8] and has been exploited in the past for the analysis of parametrically excited roll motion in longitudinal long crested irregular sea [25, 27]. The analytical model is based on the following fundamental simplifying assumptions:

1. The ship speed of advance is constant;
2. The actual sea surface is substituted by the Grim’s effective wave [28];
3. The pressure under the Grim’s effective wave is hydrostatic;
4. Heave and pitch motions are considered in a quasi-static way, i.e. the ship is in instantaneous equilibrium on the Grim’s effective wave with respect to heave and pitch;
5. Sway and yaw are neglected;

Under the aforementioned assumptions, it is possible to consider roll motion as a 1.5-DOF system, where the .5 recalls the fact that heave and pitch are instantaneously determined, in principle, by the quasi-static assumption. Although the used assumptions are quite crude, they allow for the expression of the roll dynamics by means of a simple second order differential equation, where the parametric excitation is given by the time evolution of the Grim’s effective wave amplitude $\eta_{eff}$:

$$\ddot{\phi} + d(\phi) + \frac{GZ(\phi, \eta_{eff}(t))}{GM} \dot{\phi} = 0$$

$$GZ(\phi, \eta_{eff}) = \sum_{n=0}^{N_2} K_n(\eta_{eff}) \phi^n$$
\[ K_n(\eta_{eff}) = \sum_{j=0}^{N_k} Q_j \eta_j \]  

\[ d(\phi) = 2\mu \dot{\phi} + \beta \dot{\phi} |\dot{\phi}| + \delta \dot{\phi}^3 \]

Damping parameters in Equation 1 are to be considered as speed dependent, in order to account for the increase of linear damping with speed due to lift effects, and the usual reduction of the nonlinear damping due reduced efficiency in vortex shedding.

The effective wave is, according to [28], a fictitious wave having length equal to the ship length, and its wave crest is assumed to be always amidship. If the wave amplitude and the mean value of the so defined substitute wave are determined, at each instant, by a least square fitting of the actual sea surface’s profile, it is possible to determine two transfer functions relating the spectra of the effective wave amplitude and of the mean value with the sea elevation spectrum. Because of the quasi static assumption for heave motion, only the amplitude \( \eta_{eff} \) is of interest, and its spectrum can be determined, in longitudinal long crested irregular sea, starting from the sea elevation spectrum at the real frequencies:

\[ S_{\eta}(\omega) = f^2_{\eta}(k(\omega)) \cdot S_Z(\omega) \quad (2) \]

\[ f_{\eta}(k) = 2 \frac{\pi Q \sin Q}{\pi^2 - Q^2} ; Q = \frac{L}{2k} = \frac{\sigma^2 L}{2g} \quad \text{in deep water} \]

According to the actual ship speed, Doppler effect is to be applied to \( S_{\eta}(\omega) \) in order to obtain the spectrum of the effective wave amplitude at the encounter frequencies.

The results of the analytical model in 1 have been compared, in [8], with experimental results on the same ship used in this paper, obtaining a general overestimation of roll motion from the analytical model in Equations 1. However, by introducing an empirical tuning coefficient \( k_c \) on the parametric excitation, a good agreement has been found in terms of predicted statistical, non-Gaussian, characteristics of roll motion, this meaning that the modelling in 1 could be considered a good archetypal model for the analyzed phenomenon, although the accuracy of the model is strongly dependent on the tuning factor \( k_c \). In particular, the tuning process consists in the substitution of the original Grim’s effective wave process \( \eta_{eff} \) with a corrected process \( \eta_{eff,c} = k_c \cdot \eta_{eff} \), where, due to the necessity of reducing the predicted roll amplitude, it is \( k_c < 1 \). Thanks to the linearity of the involved operations, this correction actually corresponds to a linear stretching of the scale of the significant wave heights when the used sea spectrum is proportional to the square of the significant wave height.

**SHIP AND ENVIRONMENTAL CONDITIONS**

The sample ship used in this paper has been tested in the past for parametric roll in regular and irregular long crested longitudinal waves (see [8,27,29–31]), showing the very easy inception of large amplitude motions even for very moderate wave amplitudes (both in regular and irregular sea). The bodyplan of the ship is reported in Figure 1 and main particulars are reported in Table 1. Damping coefficients have been determined from the direct analysis of experimental roll decays with advance speed, in order to allow for the speed dependence of linear and nonlinear damping. In particular, a linear+cubic model has been assumed as appropriate for this ship, due to the absence of bilge keels that usually make a linear+quadratic model more suitable.

<table>
<thead>
<tr>
<th>Hull</th>
<th>RoRo TR2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length [m]</td>
<td>132.22</td>
</tr>
<tr>
<td>Breadth [m]</td>
<td>19.00</td>
</tr>
<tr>
<td>Draught [m]</td>
<td>5.875</td>
</tr>
<tr>
<td>Volume [m³]</td>
<td>7714</td>
</tr>
<tr>
<td>Metacentric height ( \bar{GM} ) [m]</td>
<td>0.865</td>
</tr>
<tr>
<td>Natural roll frequency [rad/s]</td>
<td>0.397</td>
</tr>
</tbody>
</table>

Table 1. MAIN PARTICULARS OF THE SHIP TR2

The sea spectrum has been assumed of the Bretschneider type, \( i.e. \) Equation 3:
As a “nominal worst case” condition, the modal wave frequency of the spectrum, $\omega_m$, is assumed to be 0.683 rad/s, corresponding to a wave length equal to the ship length, leading to the regular wave 2:1 resonance occurring at a speed of advance of about 4.5 knots (2.315 m/s, $Fn=0.064$) in head seas. For this type of spectrum, a series of experimental results are available for different significant wave heights [30].

**LYAPUNOV EXPONENT CALCULATION**

If one considers a ball of points in $N$-dimensional phase space, in which each point follows its own trajectory based upon the system’s equations of motion, over time, the ball of points will collapse to a single point, will stay a ball, or will become ellipsoid in shape [32]. A common measure of the rate at which this infinitesimal ball collapses or expands is the Lyapunov exponent. For a system of equations written in state-space form $\dot{x} = u(x)$, small deviations from the trajectory can be expressed by the equation $\delta\dot{x}_i = (\partial u_i/\partial x_j)\delta x_j$ [33]. $\delta\dot{x}$ is a vector representing the deviation from the trajectory with components for each state variable of the system. Lyapunov exponents are asymptotic parameters taken in the limit as time approaches infinity. For the case of real-time detection of ship motions and/or observation of discrete events such as capsise, the FTLE can be used to investigate time-dependent behaviors with the aim of lending insight as to the nature of chaos in vessel motions. Initial studies indicated potential for advance detection of instabilities leading to capsise through tracking of variations in phasing and/or magnitude of the FTLE time series [19, 21, 22]. For the sake of reference, standard equations for the Lyapunov exponent and FTLE are given in Equations 4-5 [33], where $t$ represents time, $T$ is a finite time step, and $\lambda_{\infty}$ and $\lambda_T$ are Lyapunov exponents and FTLEs respectively.

$$\lambda_{\infty} = \lim_{t \to \infty} \frac{1}{t} \log \left( \frac{\|\delta x(t)\|}{\|\delta x(0)\|} \right)$$

$$\lambda_T(x(t), \delta x(0)) = \frac{1}{T} \log \left( \frac{\|\delta x(t+T)\|}{\|\delta x(t)\|} \right)$$

**RESULTS**

**Non-capsize**

The equation of motion given by Equation 1 was simulated over 5000 seconds for a number of runs featuring identical roll/roll velocity initial conditions (2.0 deg, 0.0 deg/s). The ship is set to travel with speed equal to 1.414 m/s ($Fn = 0.04$) in head seas emulated by a Bretschneider spectrum with 2.644 m significant wave height ($H_{1/3}/L_{pp} = 1/50$) and modal period of 9.2 s ($\lambda = L_{pp}$). Note, the selection of speed and modal period allows the ship to be excited with a spectrum of the parametric excitation process having mean encounter frequency equal to twice the roll frequency, i.e. traditional conditions for excitation of parametric roll. Each simulation was conducted with a different random seed resulting in varied random wave trains. The effective wave elevation spectrum Doppler adjusted for forward speed is given in Figure 2.

![Effective wave amplitude spectrum from Bretschneider spectrum with modal frequency $\omega_m = 0.683$ and $H_{1/3}/L_{pp} = 1/50$.](image)

Figure 2. EFFECTIVE WAVE AMPLITUDE SPECTRUM FROM BRETSCHNEIDER SPECTRUM WITH MODAL FREQUENCY $\omega_m = 0.683$ AND $H_{1/3}/L_{pp} = 1/50$.

Ten sample roll time histories are presented in Figure 3. For all cases, as a result of parametric roll initially small motions rapidly grow to dangerously large oscillations. These motions appear to arise unpredictably.

Typically, in the study of capsise, one finds distinct peaks in the FTLE time series prior to a capsise event [21, 22]. Applying the same methodology to parametric roll cases, however, yields little insight. Figure 4 illustrates that for a sampling of 500 numerical simulations, those runs which experienced moderate to large motions, as defined by exceeding 30 degrees of roll, within the first 500 seconds of simulation had a nearly identical distribution of maximum FTLEs as those which did not roll to angles in excess of 30 degrees.

Consider instead the finite-time Lyapunov exponent time series presented in Figures 5-6 for the first sample history of Figure 3. If, rather than focusing upon magnitude of the FTLE

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1As described in the ‘Analytical Model’ subsection, according to experimental data, the analytical model described leads to an overestimation of roll; therefore, the results presented for a simulated significant wave height of 2.644 m correspond to experimental cases nearer a significant wave height of approximately 3.7 m.
Figure 3. SAMPLE PARAMETRIC ROLL TIME HISTORIES FOR TR2 SUBJECT TO WAVES DEFINED BY BRETSCHNEIDER SPECTRUM WITH $\omega_m = 0.683$ AND $H_{1/3}/L_{pp} = 1/50$ FOR $Fn=0.04$.

Figure 4. HISTOGRAM OF MAXIMUM FTLES SEPARATED BY RUNS EXPERIENCING, AND NOT EXPERIENCING ROLL IN EXCESS OF 30 DEGREES AS SIMULATED OVER 500 SECONDS FOR $Fn=0.04$.
ing much the same behavior as in Figure 3 for $F_n = 0.04$. Again examining in detail the behavior of the first time series from Figure 7, Figures 8-9 illustrate again that the FTLE time series changes character in the vicinity of increased motions. Looking closely at the parametric roll event occurring shortly around 1650 seconds, it is clear that the behavior of the FTLE time series alter behavior beginning at approximately 1600 seconds. Additionally, during the event, from 1650 to 1675 seconds, the phase and shape of FTLE oscillations change dramatically before returning to the ‘steady state’ behavior after 1700 seconds.

**Relationship to roll damping**

From the analyzed examples, it can be seen that a strong correlation exists between the behavior of the SFTLEs and the amplitude of motion. In fact, it is quite likely that what the SFTLE time series is detecting is the onset of large motions via
changes in damping because, for nonlinear damping, large motions imply large velocities and thus more than linear damping. Recall, as previously stated, the sum of the FTLEs is measuring the rate of contraction or expansion of a unit ball of initial conditions around the fiducial trajectory. When the sum is negative the volume, or for this two state-space variable scenario, the area shrinks (dissipative condition), conversely the area expands when the sum is positive. Large negative SFTLEs corresponds to a large dissipation rate, this correlating, approximately speaking, to large damping. A general form of the equations of motion can be written as

\[ \ddot{\phi} + G(\phi, \dot{\phi}, t) = 0 \]  

(6)

which, as stated previously can be written in state space form as \( \dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}) \). First order perturbations of this trajectory are then written as \( \delta \dot{\mathbf{x}} = (\partial \mathbf{u}/\partial \mathbf{x}_i) \delta \mathbf{x}_i \) where the terms \( \partial \mathbf{u}/\partial \mathbf{x}_i \) comprise the Jacobian matrix for the system. Returning to the general form of our oscillator, namely Equation 6, we can write the Jacobian of the system at any time \( t+T \) (referring to the language of Equation 5) as follows, where \( (\phi_o, \dot{\phi}_o) \) represents the solution of Equation 6 for the fiducial trajectory at time \( t = t + T \).

\[
\mathbf{J}(t+T) = \begin{pmatrix} 0 & 1 \\ -\frac{\partial G}{\partial \mathbf{x}}|_{(\phi_o, \dot{\phi}_o, t+T)} & -\frac{\partial G}{\partial \dot{\mathbf{x}}}|_{(\phi_o, \dot{\phi}_o, t+T)} \end{pmatrix}
\]  

(7)

Recalling that unlike Lyapunov exponents, FTLEs are defined in terms of finite time and thus are local with respect to state space and time. If we assume \( T \) to be small, we can consider the matrix \( \mathbf{J}(t+T) \) to be almost constant during the numerical determination of the FTLEs over the time interval \([t, t+T]\). Starting from these considerations, one can derive the relation between the SFTLEs and the function \( G \). Assuming the smallness of \( T \), the equation of the perturbation can be written as:

\[
\delta \dot{\mathbf{x}} \approx \begin{pmatrix} 0 \\ -\frac{\partial G}{\partial \mathbf{x}}|_{(\phi_o, \dot{\phi}_o, t)} - \frac{\partial G}{\partial \dot{\mathbf{x}}}|_{(\phi_o, \dot{\phi}_o, t)} \end{pmatrix} \cdot \delta \mathbf{x} = \mathbf{J}_t \cdot \delta \mathbf{x}
\]  

(8)

where the subscript \( t \) recalls that the matrix is assumed con-
stant when dealing with the determination of FTLEs, although it is actually a time dependent matrix. We can say that the characteristic time scale of $J_t$ is much longer than the calculation time $T$, i.e. $J_t$ is a slowly varying matrix.

Equation 8 is now a linear differential equation with constant coefficients and the solution is known to be of the form:

$$\delta x(t + T) \approx A v_1 \exp(\lambda_1(t) \cdot T) + B v_2 \exp(\lambda_2(t) \cdot T)$$  \hspace{1cm} (9)

where $\lambda_j$ are the eigenvalues of $J_t$ (and they are thus slowly varying functions of the time $t$) and $v_j$ are the corresponding
to the numerical orientation of the orthonormal principal axes of eigenvectors. A and B are constant depending on the initial conditions. Therefore Equation 9 is saying that, considering a unit ball in state space in the vicinity of \((\phi_o(t), \dot{\phi}_o(t))\), it deforms into an ellipsoid and the principal axes of the ellipsoid stretch in time with rates \(\exp(\lambda_1 T)\) and \(\exp(\lambda_2 T)\). Because the area of the ellipse is proportional to the product of its principal axes, it can be said that locally the area \(V(t)\) changes according to [36]:

\[
V(T) = \exp((\lambda_1 + \lambda_2)T) \tag{10}
\]

The SFTLEs is actually measuring the exponential rate of change of the area, i.e.:

\[
SFTLE(t) \approx \frac{\log(V(T))}{T} = \lambda_{F1}(x(t)) + \lambda_{F2}(x(t)) \tag{11}
\]

Equation 11 states that the SFTLEs at each time instant can be determined by summing the eigenvalues of the matrix \(J_e\). Because the sum of the eigenvalues of \(J_e\) is equal to the trace of the matrix, this meaning that:

\[
SFTLE(t) \approx -\frac{\partial G}{\partial \phi} |_{(\phi_o(t), \dot{\phi}_o(t), t)} \tag{12}
\]

In the particular case of the modelling used herein, we have:

\[
\frac{\partial G}{\partial \phi} |_{(\phi_o(t), \dot{\phi}_o(t), t)} = 2 \cdot \mu + 2 \beta \cdot \dot{\phi}_o(t) \cdot \text{sign}(\dot{\phi}_o(t)) + 3 \cdot \delta \cdot \dot{\phi}_o^2(t) \tag{13}
\]

It is important to note that, in the field of 1-DOF models for roll in naval architecture, the function \(G\) contains damping, restoring, and forcing. Usually forcing and restoring are not dependent on the roll velocity, and the only term of \(G\) having an influence on the partial derivative with respect to \(\phi\) is the damping term. Because the linear plus quadratic plus cubic damping in velocity is mostly used, the result in 13 should apply to several models (beam sea, longitudinal sea, etc.). While the derivation above is not rigorous from a mathematical perspective, its results are quite accurate. As an example of the strong relation between the numerically determined relation \(SFTLE(\phi(t))\) and the curve determined using 13 is reported in Figure 10. It is important to emphasize that, although the reported idea allows one to understand part of the behavior of the FTLEs, other aspects, such as the odd behavior in time/frequency domains, are still undescribed mathematically.

This approach will result in a maximum limit for the SFTLE time series of \(-2\mu\), yet it can be seen that the SFTLEs at times, near large amplitude motions, exceed this \(-2\mu\) value. This is due to the numerical orientation of the orthonormal principal axes of the ellipsoid surrounding the fiducial trajectory. In the analytical approach described above, the basis vectors remain in their original orientation over the interval \([t, t + T]\), while numerically the first Lyapunov exponent will direct itself over the interval defined by \(T\) towards the direction of largest expansion with the following exponents arranged sequentially along the next most rapidly growing directions normal to the first exponent. Therefore, in this manner, at particularly rapid growth areas, the \(-2\mu\) limit is exceeded.

**Capsize**

A study of FTLE behavior when the vessel is subjected to very large amplitude parametrically induced roll, and capsize, was conducted using the analytical model in Equation 1 at forward speed of 1.414 m/s and waves defined by the Bretschneider spectrum, Equation 3 with modal period of 9.2 s and \(H_{1/3}/L_{pp} = 1/25\). As with the non-capsize runs studied previously, phase and other qualitative idiosyncratic behavior was noted prior to increases in roll response and/or capsize. Examine Figure 11 in which a sample roll time history leading to capsizex is presented along with the FTLE time series and the sum of the FTLE time series. In Figure 12, areas of particular interest are magnified. Note that ‘bumps’ in the FTLE time series are apparent prior to increases in roll angle and the largely divergent behavior near capsize. These phenomena are simply exaggerations of the effect detected for the more moderate roll motions discussed in the previous section.

Under parametric excitation the righting arm of the vessel is time dependent, therefore it becomes difficult to define a fixed ‘angle of vanishing stability.’ However, clearly it is important to ensure that any oddities in FTLE behavior are noted before the motions leave the initial stable attractor. Reflecting upon a
Figure 11. ROLL TIME HISTORY (TOP), FINITE-TIME LYAPUNOV EXPONENT TIME HISTORY (MIDDLE), SUM OF FTLE TIME HISTORY (BOTTOM) FOR CASE LEADING TO CAPSIZE FOR $F_n=0.04$.

Figure 12. MAGNIFIED PORTIONS OF FTLE TIME SERIES FROM FIGURE 11 FOR CASE LEADING TO CAPSIZE FOR $F_n=0.04$. 
According to the analytical model in 1, it can be seen that the restoring term is a function of the effective wave amplitude. Because the effective wave amplitude $\eta_{\text{eff}}$ is a function of time, the angle of static vanishing stability is also a function of time. From the analytical modelling of $GZ(\phi, \eta_{\text{eff}})$ it is possible to define a deterministic relation $\phi_v(\eta_{\text{eff}})$, where $\phi_v$ is the angle of vanishing static stability. Such relation is reported in Figure 14, where it can be seen the detrimental effect of the wave crest amidship. Although not reported in the figure, it should be noted that the static restoring curve shows a negative derivative at the origin when the effective wave amplitude exceeds about 2\text{m}, this leading to a loll angle of the order of 15 to 20 degrees in the range $\eta_{\text{eff}} \in [2\text{m}, 5\text{m}]$.

It is then possible, in principle, to trace the time evolution of $\phi_v(\eta_{\text{eff}}(t)) = \phi_v(t)$, and to check whether the instantaneous absolute value of the roll angle $\phi(t)$ (the system is symmetric) crosses the critical level given by $\phi_v(t)$. Equivalently, relying upon the definition of the angle of vanishing stability being a crossing from positive to negative righting arm, a time-dependent angle of vanishing stability can be calculated. E.g., capsize was defined as a change in sign of the righting arm value from positive to negative for positive roll angles beyond the angle of loll (or negative to positive for negative roll angles). Thus the GZ curve at each instant in time is used to update the angle of vanishing stability for the vessel in the conditions it encounters. Although such crossing do not necessitate that the ship capsizes, it is assumed that the probability that the ship is pushed back into the region of stability by the parametric excitation is sufficiently small to be neglected. For this reason the capsize instant is defined as the aforementioned crossing.

Five hundred separate simulations, each 1000s in length, with identical roll/roll velocity initial conditions and differing random wave trains were conducted to investigate the relationship between peak FTLE and capsize. The time-dependent angle of vanishing stability was used to define capsize. For the 500 runs studied, the ‘angle of vanishing stability’ ranged from 54.9 to 63.6 degrees. Note that this is related to the probability density function of $\eta_{\text{eff}}(t)$. If the standard deviation of $\eta_{\text{eff}}$ is small, you will have only small deviations in this quantity. As shown in Figure 15, those runs resulting in capsize encountered larger peak FTLE values than non-capsize cases. Additionally, Figure 16 displays the time from the occurrence of the peak FTLE to the capsize event. While peaks in the FTLE time series occur, oftentimes they occur too close to the capsize event for initiation of corrective measures or too far from the capsize event to lend credence to the idea that it is, in fact, the capsize instability detected. However, for some cases, peaks in the FTLE time series are discernable prior to capsize with sufficient time in which to issue an operator warning. A system based upon detection then of peaks along with qualitative changes in the FTLE time series, e.g Figure 12 could provide the basis of, or supplement to, a real-time motion monitoring system.

**CONCLUSIONS**

This work demonstrates that FTLE time series gives indicators of the onset of parametric resonance based upon simulation of a 1.5 DOF analytical model. While the warning signs of increasing motions are largely qualitative, ongoing research is devoted to improving upon and quantifying phase and/or shape irregularities in the FTLE time series along with the clearly quantifiable measures associated with capsize, i.e. peaks in the FTLE.
time series. The question could be posed, then, of why bother with FTLEs, could not the same information be gleaned from solely a roll time series? There are a number of benefits to incorporating FTLE calculations into real-time stability detection software. Specifically,

1. While a roll time series indicates that large roll motions have occurred, FTLEs are of use for indicating the existence of dangerous conditions, e.g. nearing the instantaneous separatrix from stable to unstable motions, which may not be obvious from a roll time history prior to a catastrophic result. The roll time history gives only an indication of what has occurred, with no insight as to if that behavior is abnormal or a threat. As the ultimate goal is not simply to detect large amplitude motions, but to warn of hazardous conditions, FTLEs yield greater information into the physical behavior of the system.

2. FTLEs capture rates of growth in all degrees of freedom. Rather than attempting to track deviations in multiple parameters, monitoring the FTLE time series provides a representation of divergent behavior in multiple dimensions.

3. Tracking trends in FTLE peaks, troughs, phase, and shape irregularities have been demonstrated in concept to indicate the onset of capsize [21,22], quiescence [37], and parametric roll as described herein. They can thus serve as a general tool which can be versatile for use in an on-board real-time motion monitoring system.

4. FTLEs provide another data point which can be used in any variety of tools for motion predictions as part of an intelligent system design.

Areas of on-going and future study include quantification of the idiosyncratic FTLE behavior and development of appropriate tools to monitor FTLEs in real time from realistic, e.g. noisy, experimental data.

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