Astromechanics
The Two Body Problem (continued)

8.0 Recovering the Time

The differential equation of motion for the two body problem is given by:

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = \vec{0} \tag{1}$$

This equation is a second order, vector, ordinary differential equation for which the solution contains six constants of integration. Up to now we have extracted the following constants:

1) The angular momentum vector constant, $\vec{h} = \text{constant}$ is equivalent to three scalar constants. From our previous discussion, we found that the angular momentum constant establishes the orbit lies in a plane fixed in space. It turns out that two constants are sufficient to establish the plane of the orbit. The third constant is the magnitude of the angular momentum.

2) The energy of the orbit remains constant. $En = \frac{V^2}{2} - \frac{\mu}{r} = \text{constant}$

We took advantage of the fact that the motion was in a plane (yet to be determined by two of the constants associated with angular momentum), by writing Eq. (1) in plane polar coordinates:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{\mu}{r^2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0 \end{aligned} \tag{2}$$

We used two of the know constants to establish the plane so that the differential equations of the orbit in that plane, Eq. (2) are two second order ordinary differential equations that need four constants of integration to establish a solution. Two of these constants are known, the magnitude of the angular momentum, h, and the energy En. A third constant was extracted by removing time as the independent variable and obtaining the equation of the orbit:

$$r = \frac{\frac{h^2}{\mu}}{1 + e\cos(\theta - \omega)} \tag{3}$$

where ω is the third constant. The two constants h, and En were combined to form the constant called the eccentricity which is not a new independent constant. Hence we have one constant left to extract. This constant is associated with the time and will allow us to determine the position of the satellite in the orbit at any given time.

This final constant of motion is associated with integrating the transverse equation of motion to extract the time relationship. The transverse differential equation of motion can be written as

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \tag{4}$$

that gives us the magnitude of the angular momentum as

$$h = r^2\dot{\theta} \tag{5}$$

Since we used this equation to eliminate time to enable us to solve for the orbit, it is this same equation that we use to recover the time into the problem. We can recover time by rearranging Eq. (5):

$$dt = \frac{r^2}{h}d\theta \tag{6}$$

If we define the true anomaly as we have previously, $\nu = \theta - \omega$, and note that $d\nu = d\theta$, and also that at the point of closest approach (periapsis passage), $\nu = 0$, and we can define the time at periapsis passage as, $t = \tau$, (τ = time of periapsis passage), then Eq. (6) can be put in the form of a quadrature (integral with the integrand in terms of only the independent variable, in this case ν):

$$t - \tau = \frac{h^3}{\mu^2}\int_0^{\nu}\frac{d\nu}{\left(1 + e\cos\nu\right)^2} \tag{7}$$

The key thing to note here is that ν is the independent variable and the solution gives $t = f(\nu)$. In addition, τ is our ***sixth*** (fourth for in plane motion) and final constant of integration. In principle, the problem has been solved! However we will find that Eq. (7) is not the best for solving the original problem of interest, given the time, find the position and velocity. We will introduce an alternate from for the time equation when needed.

Time in a Parabolic Orbit **(e = 1)**

We can evaluate Eq. (7) relatively easy for a parabolic orbit, since the eccentricity equals one. Using some trigonometry half angle identities, we have:

$$\frac{\mu^2}{h^3}(t - \tau) = \int_{\nu_0 = 0}^{\nu}\frac{d\nu}{1 + \cos^2\frac{\nu}{2} - \sin^2\frac{\nu}{2}} = \int_{\nu_0 = 0}^{\nu}\frac{d\nu}{1 + \cos^2\frac{\nu}{2} - \left(1 - \cos^2\frac{\nu}{2}\right)} \tag{8}$$

$$\frac{\mu^2}{h^3}(t - \tau) = \int_{\nu_0 = 0}^{\nu} \frac{d\nu}{\left(2\cos^2\frac{\nu}{2}\right)} = \int_{\nu_0 = 0}^{\nu} \frac{1}{2}\sec^4\frac{\nu}{2}\, d\left(\frac{\nu}{2}\right)$$
$$= \frac{1}{2}\int_{\nu_0 = 0}^{\nu} \sec^2\frac{\nu}{2}\left(1 + \tan^2\frac{\nu}{2}\right) d\left(\frac{\nu}{2}\right) \tag{9}$$

The integration is easily carried out to give:

$$\frac{\mu^2}{h^3}(t - \tau) = \frac{1}{2}\left[\tan\frac{\nu}{2} + \frac{1}{3}\tan^3\frac{\nu}{2}\right] \tag{10}$$

We can put this equation in its final form by noting that the angular momentum can be written in terms of the orbit parameter, p as $h = \sqrt{\mu p}$. We can also define the parameter $\bar{n}$, that satisfies the following equation:

$$\bar{n}^2 p^3 = \mu \tag{11}$$

Then Eq. (10) can be rewritten in the form:

Time Equation for a Parabola (Barker's Equation)

$$2\bar{n}(t - \tau) = \tan\frac{\nu}{2} + \frac{1}{3}\tan^3\frac{\nu}{2} \tag{13}$$

Equation (13) is the time equation for a parabolic orbit and is known as Barker's equation.

Example

Consider a parabolic transfer orbit from Earth's orbit to Mars' orbit. We will assume that the perihelion radius of the transfer orbit is the same as the Earth's orbit radius so that the parabolic transfer orbit is tangent to the Earth's orbit at perihelion. We would like to find the ΔVrequired for the transfer, and the time of flight for the transfer. Further, compare the results with a Hohmann transfer.

We need to determine the properties at Earth orbit and at Mars orbit. The velocity anywhere in a parabolic orbit is escape velocity.

At Earth orbit: Velocities are parallel at perihelion

$$\Delta V_1 = V_{esc} - V_{cir} = \sqrt{\frac{2\mu}{r}} = \sqrt{\frac{\mu}{r}} = (\sqrt{2} - 1)\sqrt{\frac{\mu}{2}} = 0.4142\sqrt{\frac{1}{1}} = 0.4142 \text{ AU/TU}$$

At Mars' orbit:

$$r = \frac{p}{1 + \cos\nu} = \frac{2 r_p}{1 + \cos\nu} = 1.524 = \frac{2}{1 + \cos\nu_2}$$

$$\cos\nu_2 = 0.312 \qquad \Rightarrow \qquad \nu_2 = 71.80 \text{ deg}$$

The flight path angle at Mars' orbit is

$$\tan\phi_2 = \frac{\sin\nu}{1 + \cos\nu} = \frac{\sin 71.80}{1 + \cos 71.80} = 0.7239 \qquad \Rightarrow \qquad \phi_2 = 35.90 \text{ deg} \quad (= \frac{\nu_2}{2})$$

The orbit speed at Mars' orbit is just the escape speed and Mars orbit speed is just the circular speed:

$$V_2 = \sqrt{\frac{2\mu}{r}} = \frac{\sqrt{1\,(1)}}{1.524} = 1.1456 \text{ AU/TU} \qquad V_{c_2} = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{1}{1.524}} = 0.8100 \text{ AU/TU}$$

The ΔV_2 is obtained from the law of cosines:

$$\Delta V_2^2 = V_2^2 + V_{c_2}^2 - 2 V_2 V_{c_2} \cos\phi_2 = (1.1456)^2 + (0.8100)^2 - 2\,(1.1456)\,(0.8100)\cos 35.90 = 0.4652$$

$$\Delta V_2 = 0.6820 \text{ AU/TU}$$

Calculating the time of flight

The parabolic transfer orbit parameter $p = 2 r_p = 2\,(1) = 2 \text{ AU}$

$$\bar{n}^2 p^3 = \mu \qquad \Rightarrow \qquad \bar{n} = \sqrt{\frac{\mu}{p^3}} = \sqrt{\frac{1}{2^3}} = \frac{1}{2\sqrt{2}} \qquad 2\bar{n} = \frac{1}{\sqrt{2}}$$

Barker's equation is:

$$2\bar{n}(t - \tau) = \tan\frac{\nu}{2} + \frac{1}{3}\tan^3\frac{\nu}{2} = \frac{1}{\sqrt{2}}(t - \tau) = \tan\frac{71.80}{2} + \frac{1}{3}\tan^3\frac{71.80}{2} = 0.8503$$

$t - \tau = 1.2025 \ TU_{Sun} = 69.9 \ \text{days}$

We can now compare these results with a Hohmann transfer calculated previously

Earth/Mars Transfer	Hohmann	Parabola
ΔV_1 AU/TU	0.0989	0.4142
ΔV_2	0.0890	0.6820
ΔV_{total}	0.1879 AU/TU 5.5960 km/s	1.0962 AU/TU 32.65 km/s
TOF	4.4539 TU 258.92 days	1.2025 TU 69.90 days

The Kepler Problem (Parabolic Orbit)

Unfortunately, the original problem that we set out to solve was to find the position (and velocity) in orbit, given the time. Such a problem is called the Kepler problem. In other words, given the time in Barker's equation, find the true anomaly. Barker's equation is given by

$$2\bar{n}(t - \tau) = \tan\frac{\nu}{2} + \frac{1}{3}\tan^3\frac{\nu}{2} \tag{14}$$

which can be rearranged as

$$\tan^3\frac{\nu}{2} + 3\tan\frac{\nu}{2} - 6\bar{n}(t - \tau) = 0$$
$$x^3 + 3x - b = 0 \tag{15}$$

where we see Barker's equation is a (kind of special) cubic equation in $\tan^3\frac{\nu}{2}$. We can examine the expected roots that we will get using Descarte's rule of signs for polynomials of the form: f(x) = 0.

1) The number of sign changes equals the maximum number of positive real roots
2) The number of sign changes in f(-x) equals the maximum number of negative real roots.

Using (1) we have 1 sign change so we have at most one positive real root (note b>0)
Using (2) $(-x)^3 + 3(-x)$~-b = 0, we have zero sign changes so no negative real roots.

Our conclusion is that Eq. (15) has one real root (positive) and two complex conjugate roots. We only need to solve for the real root. For a cubic in this form, we can write the solution directly.

Let

$$R = 3\bar{n}(t - \tau) = \frac{b}{2} \tag{16}$$

then the solution is:

$$\tan\frac{\nu}{2} = \sqrt[3]{R + \sqrt{R^2 + 1}} + \sqrt[3]{R - \sqrt{R^2 + 1}} \tag{17}$$

Example

Consider $(t - \tau) = 1.2025\ \text{TU}$ in a parabolic orbit where p = 2 AU. Find the true anomaly, ν, and radial distance, r. First calculate $\bar{n}$ and subsequently, R.

$$\bar{n} = \sqrt{\frac{\mu}{p^3}} = \sqrt{\frac{1}{2^3}} = \frac{1}{2\sqrt{2}} \qquad\qquad R = 3\bar{n}(t - \tau) = 3\left(\frac{1}{2\sqrt{2}}\right)(1.2025) = 1.2754$$

then

$$\begin{aligned}\tan\frac{\nu}{2} &= \sqrt[3]{1.2754 + \sqrt{1.2754^2 + 1}} + \sqrt[3]{1.2754 - \sqrt{1.2754^2 + 1}} \\ &= \sqrt[3]{2.8961} + \sqrt[3]{-0.3453} \\ &= 1.4254 - 0.7016 \\ &= 0.7238\end{aligned}$$

$$\frac{\nu}{2} = 38.90\ \text{deg} \qquad \Rightarrow \qquad \nu = 71.80\ \text{deg}$$

The time given, was the travel time to Mars along a parabolic orbit, and the true anomaly calculated was that to arrive at Mars. The radial distance is calculated from:

$$r = \frac{p}{1 + \cos\nu} = \frac{2}{1 + \cos 71.80} = 1.524\ \text{AU}$$

Time in an Elliptic Orbit (e < 1)

We can extract the time equation for an elliptic orbit in the same manner as we did for the parabolic orbit. However, since $e \neq 1$, the results are not so nice. We have from the original time expression:

$$(t - \tau) = \frac{h^3}{\mu^2} \int_{\nu=0}^{\nu} \frac{d\nu}{(1 + e\cos\nu)^2} \tag{18}$$

For an elliptic orbit, $\frac{h^2}{\mu} = a(1 - e^2) \quad \Rightarrow \quad h = \sqrt{\mu a}\sqrt{a(1 - e^2)}$. If we substitute in for the angular momentum, and bring the terms containing μ and a to the left hand side we arrive at:

$$\sqrt{\frac{\mu}{a^3}}(t - \tau) = n(t - \tau) = (1 - e^2)^{3/2} \int_0^{\nu} \frac{d\nu}{(1 + e\cos\nu)^2} \tag{19}$$

where we have incorporated the relation developed previously:

$$n^2 a^3 = \mu \tag{20}$$

The integral can be evaluated for the case where e < 1. The result is

$$n(t - \tau) = \left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\right) - \frac{e\sqrt{1-e^2}\sin\nu}{1 + e\cos\nu}\right] \tag{21}$$

Or

$$M = \left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\right) - \frac{e\sqrt{1-e^2}\sin\nu}{1 + e\cos\nu}\right] \tag{22}$$

where we have introduce a definition of the ***mean anomaly***:

Mean Anomaly

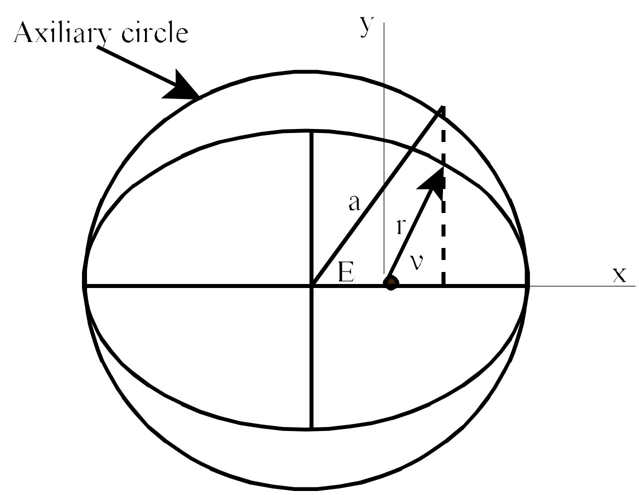
$$M \equiv n(t - \tau) \tag{23}$$

The mean anomaly is not a physical angle that we can designate on a drawing. It is just the angle that would be travel through by a satellite moving at its constant mean angular rate, n, in time (t - τ). It varies uniformly in time and is often used instead of time as the independent variable.

Equation (22) is a messy equation. However it can be used to solve for the time (or mean anomaly) given the true anomaly. On the other hand, solving the Kepler problem, given the time, find the true anomaly, is much more complicated. Consequently an alternative approach is available which ends up still solving a non-linear equation for the angle, but it is not nearly as messy.

The Eccentric Anomaly

In order to simply the time equation, we can introduce a new variable called the ***eccentric anomaly.*** It is defined with the help of an auxiliary circle. The auxiliary circle is centered at the center of the ellipse (not the focus) and has a radius equal to the semi-major axis, a. As a result it is tangent to both the periapsis and the apoapsis. At the point the radius vector touches the ellipse, we construct a perpendicular to the semi-major axis and extende it outward until it touches our auxiliary circle. A line is then drawn from this point to the center of the ellipse. The angle this line makes with respect to the major axis line is designated as *E* and is called the ***eccentric anomaly***. It turns out that using this variable as the angle variable leads to a rather nice form of the time equation. In order demonstrate this, we will use a coordinate system centered at the focus of the ellipse with the x axis in the direction of the periapsis, and the y axis perpendicular to it and pointing in the direction of the semi-latus rectum. Such a coordinate system is called the ***perifocal coordinate system***. We could use a coordinate system centered at the center of the ellipse, but this one will save us some algebra, believe it or not. We first want to get an expression for the radius as a function of *E*.



Consider determining the x and y positions on the ellipse in terms of *E*. From the geometry we can write:

$$a \cos E = ae + x \qquad \Rightarrow \qquad x = a(\cos E - e) \tag{24}$$

The equation of an ellipse with the center located at the point (x_0, y_0) is given by

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1 \tag{25}$$

In our case the center of the ellipse is locate at $(x_0, y_0) = (-ae, 0)$. In addition we can recall that $b = a\sqrt{1 - e^2}$, so that Eq. (25) becomes:

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \tag{26}$$

or

$$y^2 = a^2(1 - e^2)(1 - \cos^2 E) \tag{27}$$

The desired result for now is

$$y = a\sqrt{1 - e^2}\sin E = b\sin E \tag{28}$$

We now have expressions for *x* and *y* in terms of the eccentric anomaly, *E* so we can find *r* as a function of *E*:

$$\begin{aligned} r^2 = x^2 = y^2 &= a^2(\cos E - e)^2 + a^2(1 - e^2)\sin^2 E \\ &= a^2[\cos^2 E - 2e\cos E + e^2 + \sin^2 E - e^2\sin^2 E] \\ &= a^2[1 - 2e\cos E + e^2\cos^2 E] \end{aligned} \tag{29}$$

Or, taking the square root:

r in Terms of the Eccentric Anomaly

$$r = a(1 - e\cos E) \tag{30}$$

In order to put this result in perspective we can note:

$$r = a(1 - e\cos E) = \frac{a(1 - e^2)}{1 + e\cos\nu} \tag{31}$$

Hence we have an expression for the radius in terms of two different variables, the true anomaly and the eccentric anomaly. They are equivalent, and can e interchanged. Equation (31) can be used to determine $E = f(\nu)$ or $\nu = f(E)$ which we will do later. Now we would like to proceed to the time equation.

We can recover the time equation in a similar manner that we did previously. However here, all we will need to do is evaluate the angular momentum. We will do this in the perifocal coordinate system. We have the following components of the position and velocity:

$$x = a(\cos E - e) \qquad y = a\sqrt{1 - e^2}\sin E$$
$$\dot{x} = -a\sin E\dot{E} \qquad \dot{y} = a\sqrt{1 - e^2}\cos E\dot{E} \tag{32}$$

The angular momentum is given by:

$$\vec{h} = \vec{r}\ x\ \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = \begin{Bmatrix} 0 \\ 0 \\ x\dot{y} - y\dot{x} \end{Bmatrix} \tag{33}$$

or the magnitude of $\vec{h}$is $h = x\dot{y} - y\dot{x}$.If we substitute in our expressions from Eq. (32) we have:

$$\begin{aligned} h &= a(\cos E - e)(a\sqrt{1 - e^2}\cos E\dot{E}) - a\sqrt{1 - e^2}\sin E(-a\sin E\dot{E}) \\ &= a^2\sqrt{1 - e^2}\,\dot{E}\left(\cos^2 E - e\cos E + \sin^2 E\right) \\ &= a^2\sqrt{1 - e^2}\,(1 - e\cos E)\dot{E} \end{aligned} \tag{34}$$

We also can recall that $h = \sqrt{\mu a}\sqrt{1 - e^2}$. Substituting the angular momentum expression in Eq. (34), and breaking our the differentials in $\dot{E}$leads to the equation:

$$\sqrt{\frac{\mu}{a^3}}\,dt = (1 - e\cos E)\,dE \tag{35}$$

We can integrate the above equation from the time of periapsis passage to the current time and note that at periapsis both the true anomaly and eccentric anomaly are zero. So, integrating Eq. (35) and putting in the limits gives:

Kepler's Equation

$$\begin{aligned} n(t - \tau) &= E - e\sin E \\ M &= E - e\sin E \end{aligned} \tag{36}$$

Equation (36) is known as Kepler's equation and is generally the time equation related to elliptic orbits. If we are given the position or eccentric anomaly, we can solve for the time and more importantly, if we are given the time, we can solve for the eccentric anomaly, iteratively, a lot easier than we could for true anomaly.

We can complete this section on elliptic orbit time by comparing Eq. (36) with Eq. (22).

These equations must be the same. We can recognize the first term in each is an angle, and the second term in each is multiplied by an e. By comparison we can find:

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\nu}{2} \tag{37}$$

and

$$\sin E = \frac{\sqrt{1-e^2}\sin\nu}{1+e\cos\nu} \tag{38}$$

In addition, from the relations: $x = r\cos\nu = a(\cos E - e)$, and Eq. (31), we can get the results:

$$\cos E = \frac{e+\cos\nu}{1+\cos\nu} \tag{39}$$

and

$$\tan E = \frac{\sqrt{1-e^2}\sin\nu}{e+\cos\nu} \tag{40}$$

Finally, these results can be inverted to give:

$$\begin{aligned} \tan\frac{\nu}{2} &= \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \\ \sin\nu &= \frac{\sqrt{1-e^2}\sin E}{1-e\cos E} \\ \cos\nu &= \frac{\cos E - e}{1-e\cos E} \end{aligned} \tag{41}$$

Note that these relationships can be developed directly from geometry.

In all these equations it is important to note the quadrant in which the angles occur. We should note from the figure above that:

$$\begin{array}{llll} \text{if} & 0 \le \nu \le \pi & \text{then} & 0 \le E \le \pi \\ \text{if} & \pi \le \nu \le 2\pi & \text{then} & \pi \le E \le 2\pi \end{array} \qquad (42)$$

Note that this doesn't mean of E is in the first quadrant that ν is in the first quadrant, ν could be in the first or second quadrant, and vice versa.

Time in a Hyperbolic Orbit (e > 1)

We can extract the time equation for an hyperbolic orbit in the same manner as we did for the elliptic orbit.

$$(t - \tau) = \frac{h^3}{\mu^2} \int_{\nu = 0}^{\nu} \frac{d\nu}{(1 + e\cos\nu)^2} \qquad (43)$$

For a hyperbolic orbit, $\frac{h^2}{\mu} = a(e^2 - 1) \quad \Rightarrow \quad h = \sqrt{\mu a}\sqrt{a(e^2 - 1)}$. If we substitute in for the angular momentum, and bring the terms containing μ and a to the left hand side we arrive at:

$$\sqrt{\frac{\mu}{a^3}}(t - \tau) = n(t - \tau) = (e^2 - 1)^{3/2} \int_0^{\nu} \frac{d\nu}{(1 + e\cos\nu)^2} \qquad (44)$$

where we again have incorporated the relation developed previously: $\;^2 a^3 = \mu$

The integral can be evaluated for the case where e > 1. The result is

$$n(t - \tau) = \left[\frac{e\sqrt{e^2 - 1}\,\sin\nu}{1 + e\cos\nu} - \ln\left(\frac{\sqrt{e+1} + \sqrt{e-1}\,\tan\frac{\nu}{2}}{\sqrt{e+1} - \sqrt{e-1}\,\tan\frac{\nu}{2}}\right)\right]$$
$$= \left[\frac{e\sqrt{e^2 - 1}\,\sin\nu}{1 + e\cos\nu} - 2\tanh^{-1}\left(\sqrt{\frac{e-1}{e+1}}\tan\frac{\nu}{2}\right)\right] \qquad (45)$$

In addition we will introduce a new variable called the ***hyperbolic anomaly*** that is analogous to the eccentric anomaly. It leads to a simplified time equation similar to Keplers's equation that we developed for elliptic orbits. In addition there are relations similar to those for elliptic orbits for relating the hyperbolic anomaly to the true anomaly. These relations follow.

Time Equation for Hyperbolic Orbit

$$n(t - \tau) = e\sinh F - F \qquad (46)$$

where F is the hyperbolic anomaly.

Many hyperbolic equations are the same as elliptic equations with a replace with $-a$. However in the following equations we will define $a > 0$, or $a = |a| > 0$.

Radial Distance

$$r = a(e\cosh F - 1) = \frac{a(e^2 - 1)}{1 + e\cos\nu} \qquad (47)$$

$$\cosh F = \frac{e + \cos\nu}{1 + e\cos\nu}$$
$$\sinh F = \frac{\sqrt{e^2 - 1}\,\sin\nu}{1 + e\cos\nu} \qquad (48)$$
$$\tan\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\nu}{2}$$

The inverse relations:

$$\cos\nu = \frac{e - \cosh F}{e\cosh F - 1}$$

$$\sin\nu = \frac{\sqrt{e^2 - 1}\,\sinh F}{e\cosh F - 1} \tag{49}$$

Additional Relations of Interest

Since dealing with hyperbolic functions is sometimes uncomfortable for students, alternative representations can be used. We have:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \sinh^{-1} y = \ln[y + \sqrt{y^2 + 1}]$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \cosh^{-1} y = \ln[y + \sqrt{y^2 - 1}] \tag{50}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + x^{-x}} \qquad \tanh^{-1} = \frac{1}{2}\ln\left[\frac{1 + y}{1 - y}\right]$$

Time of Flight Problems

It is assumed that the properties of the orbit are known, the semi-major axis, a, the eccentricity, e, (or their equivalents, angular momentum and time), and the time of periapsis passage, τ, Then there are two types of time of flight problems:

1) Given the true anomaly, find the time from periapsis ,
2) Given the time past periapsis, find the true anomaly. (Kepler problem)

The first type of problem is easily solved since we have been using the true anomaly as the independent variable and we have a time equation with time as a function of true anomaly. The second problem requires us to “invert” the time equation that is nonlinear. Consequently most schemes for solving this problem require iteration.

Another aspect of either time problem is the fact that the times determined using the time equations developed are times measured from periapsis passage. Consequently, if we want to find the time of flight between two arbitrary points in the orbit, we must find the times from periapsis to those points and then take the difference.

Methods of solving time problems are best illustrated by examples. Several examples of both types of time problems are now presented.

Example

We are in the Earth's orbit about the Sun and apply a tangent Delta-V = 0.20 AU/TU. Find the time it takes to reach Mars' orbit, r = 1.524 AU.

As in all problems, we will find the properties of the original orbit assumed to be a circular orbit at 1 AU radius from the Sun.

$$V_c = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{1}{1}} = 1 \text{ AU/TU} \qquad En = \frac{V^2}{2} - \frac{\mu}{r} = \frac{1^2}{2} - \frac{1}{1} = -\frac{1}{2}\,\frac{\text{AU}^2}{\text{DU}^2}$$

$$h = r V \cos\phi = r V = 1\,(1) = 1 \text{ AU}^2\text{/TU}$$

Next we find the properties of the new orbit. Here we note the radius is unchanged and the velocity increment is tangent to the original velocity so that the new flight path angle is still zero, so that the point in the new orbit is at the periapsis of the new orbit. We have:

$V = V_c + \Delta V = 1 + 0.20 = 1.2 \text{ AU?TU}$ Then energy and angular momentum is given by:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{1.2^2}{2} - \frac{1}{1} = -0.280 \text{ AU}^2\text{/TU}^2\,, \quad h - r V \cos\phi = 1\,(1.2)\cos 0 = 1{,}2 \text{ AU}^2\text{/TU}$$

$$a = -\frac{\mu}{2\,En} = -\frac{1}{2\,(-0.2800} = 1.75857\ AU$$

Semi-major axis

$$e = \sqrt{1 + \frac{2\,h^2\,En}{\mu^2}} = \sqrt{1 + \frac{2\,(1.2^2)\,(-0.280)}{1^2}} = 0.4400$$

We need to determine the eccentric anomaly when the vehicle reaches Mars' orbit

$$r = a(1 - e\cos E) = 1.524 = 1.7857\,(1 - 0.4400\cos E) \quad \Rightarrow \quad \cos E = 0.3331$$

$E = 70.544 \text{ deg} = 1.2321 \text{ rad}$

Note that we must be careful we have the correct quadrant In this case it could be 1st or 4th. Since it is going out, it must be 1st.

We can find the mean angular rate or in this case its reciprocal is more convenient:

$$n^2 a^3 = \mu \quad \Rightarrow \quad \frac{1}{n} = \sqrt{\frac{a^3}{\mu}} = \sqrt{\frac{1.7857^3}{1}} = 2.3863 \quad (n = 0.4191)$$

Then the time of flight from perihelion to Mars' orbit is given by Kepler's equation:

$$t - \tau = \frac{1}{n}[E - e\sin E] = 2.3863\,[\,1.2312 - 0.4400\sin(1.2321)\,] = 1.9481\ \text{TU} = 113.25\ \text{days}$$

Alternatively we could have solved for the true anomaly and then either used it directly in Eq, (22) or then convert to eccentric anomaly:

$$r = \frac{a(1 - e^2)}{1 + e\cos\nu} = 1.524 = \frac{1.7857\,(1 - 0.4400^2)}{1 + 0{,}4400\cos\nu} \quad \Rightarrow \quad \cos\nu = -0.1253 \quad \Rightarrow \quad \nu = 97.1972\ \text{deg}$$

and

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\nu}{2} = \sqrt{\frac{1 - 0.4400}{1 + 0.4400}}\tan\frac{97.1972}{2} = 0.7073$$

$\tan\frac{E}{2} = 35.2722 \quad \Rightarrow \quad E = 70.544\ \text{deg}$ (As before - we can now use Kepler's equation

<u>Note that here the eccentric anomaly is in the first quadrant, and the true anomaly is in the second.</u>

Example

We will now get to Mars using a hyperbolic orbit of eccentricity 2. We will thrust tangentially from Earth's orbit. Find the time of flight to reach Mars' orbit.

Since we are again launching tangent to the original orbit, we know that we will be at the perihelion of the new orbit. Hence we have:

$$r_p = 1 = a(e - 1) = a(2 - 1) \quad \Rightarrow \quad a = 1\ \text{AU}$$

From the energy equation:

$$\frac{V_p^2}{2} - \frac{\mu}{2} = -\frac{\mu}{2a} = \frac{V_p^2}{2} - \frac{1}{1} = -\frac{1}{2(1)} \quad \Rightarrow \quad V_p = \sqrt{3} = 1.7320\ \text{AU/TU}$$

$$\Delta V_p = 1.7320 - 1 = 0.7320\ \text{AU/TU}$$

We need to calculate the hyperbolic anomaly when we reach Mars' orbit:

$$r = a(e\cosh F - 1) = 1.524 = (2\cosh F - 1) \quad \Rightarrow \quad \cosh F = 1.2620$$

$$F = 0.7089$$

We can now use "Kepler's" equation:

$$t - \tau = \frac{1}{n}(e \sinh F - F) = \sqrt{\frac{a^3}{\mu}}(e \sinh F - 1) = \sqrt{\frac{1^3}{1}}(2 \sinh(0.7089) - 0.7089)$$

$$= 0.8307 \text{ TU} = 48.29 \text{ days}$$

We can now summarize the results of the last few examples:

Earth-Mars	ΔV (AU/TU)	TOF (days)
Hohmann	0.0989	258.3
Ellipse	0.2000	113.2
Parabola	0.4142	69.9
hyperbola	0.7089	48.4

Example

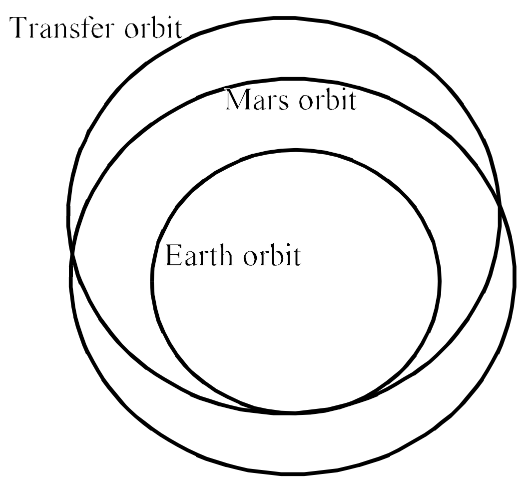
We will now consider a transfer orbit from Earth to Mars that is designed to return to the Earth if it does not intercept Mars. That is it is an orbit with a two year period so that if a vehicle is launched in this orbit from a perihelion at Earth orbit radius, it will return to that point in two years. Hence if launched from Earth, it will return to Earth in two years. This orbit will intersect Mars' orbit in two places. We are interested in the time from Earth to Mars. We are also interested in other times relating to this orbit.

The period of this orbit is twice the period of Earth in its orbit. In time units, the period of the reference orbit is

$$T_p - 2\pi\sqrt{a^3/\mu} = 2\pi\sqrt{1^3/1} = 2\pi \text{ TU}$$

Hence the period of this "free return" orbit is 2(2π). Then we have:

$$2(2\pi) = 2\pi\sqrt{a^3/1} \Rightarrow a = 1.5874 \text{ AU}$$



Since we know the perihelion distance, we have

$$r_p = a(1 - e) = 1 = 1.5874(1 - e)$$

or $e = 0.3700$.

WE can compute the eccentric anomaly at Mars orbit from :

$$r = a(1 - e\cos E) = 1.524 = 1.5874(1 - 0.3700\cos E) \Rightarrow \cos E = 0.1079$$

$E = 83.8032$ deg (First quadrant). The time of flight is obtained from Kepler's equation:

$$t - \tau = \frac{1}{n}[E - e\sin E] = \sqrt{\frac{a^3}{\mu}}[E - e\sin E] = \sqrt{\frac{1.5874^3}{1}}[83.8032\frac{\pi}{180} - 0.3700\cos(83.8932)]$$
$$t - \tau = 2.1896 \text{ TU} = 127.29 \text{ days}$$

We now ask the question, how long does it take to get from Mars' orbit on the way out, to Mars' orbit on the way in? We calculate this time of flight by calculating the time of flight from perihelion to point 1, and from perihelion to point 2, and taking the difference to get the time from point 1 to point 2.

The time of flight from perihelion to point 1 is the time calculated above. In order to calculate the time from perihelion to point 2, we need to determine the eccentric anomaly at point 2. We do this from the radius equation:

$$r_2 = r_1 = 1.524 - a(1 - e\cos E) = 1.5874(1 - 0.3700\cos E) \quad \Rightarrow \quad \cos E = 0.1079$$

Exactly the same value as before. Here, however, we will be in the 4th quadrant since we are returning. (Positive cosine in 1st or 4th , so it must be 4th). Hence angle of interest is 360 - 83.8032 = 276.1968 deg. We can now calculate the time from perihelion to point 2.

$$t - \tau = \frac{1}{n}[E - e\sin E] = \sqrt{\frac{1.5874^3}{1}}\left[276.1968\frac{\pi}{180} - 0.3700\sin(276.1968)\right] = 10.3768 \text{ TU}$$

The time of flight from point 1 to point 2 is then

$$TOF = (t_2 - \tau) - (t_1 - \tau) = (t_2 - t_1) = 10.3768 - 2.1896 = 8.1872\ TU = 475.94 \text{ days}$$

The time to go from Earth's orbit to the second crossing of Mars' orbit is just what we calculated above, 10.3768 TU .

Although the proper way to calculate time of flights is a shown previously, sometime one can take advantage of orbit symmetry about the semi-major axis to simplify calculations. For this example we could calculate the time from Mars' to Mars' orbit (over the top) by calculation the time from Mars' orbit to the aphelion, and then double it. We already know the time from perihelion to Mars orbit (point 1) and we know the time from perihelion to aphelion is half the period, $T_p/2 = 2\pi$. Hence the time from point 1 to aphelion is $TOF_a = 2\pi - 2.1896$
Then $TOF_{2-1} = 2\,TOF_a = 2(2\pi - 2.1896) = 8.1872$ TU as before.

Solving the Kepler Problem

A version of the Kepler problem can be stated as follows: Given the orbit properties, the initial position of the vehicle on the orbit at some time t_0, find the position on the orbit after some time of flight, *TOF*. A more general statement of the Kepler problem will be given later. In order to solve this problem we must use Kepler's equation to solve for the proper eccentric anomaly at the end of the time of flight. The procedure is as follows:

It is assumed we know the energy and angular momentum of the orbit and hence the semi-major axis and the eccentricity of the orbit. At epoch, time t_0, we know the true anomaly, ν_0, and the corresponding radius, r_0. We now have to find the time past periapsis to the initial time, t_0. We do this using the procedures in the above examples. Since we know the radius, (or true anomaly), we can solve for the eccentric anomaly at epoch:

$$r_0 = a(1 - e\cos E_0) \tag{51}$$

Once we know the eccentric anomaly, we can use Kepler's equation to solve for time past periapsis:

$$t_0 - \tau = \frac{1}{n}\left(E_0 - e\sin E_0\right) \tag{52}$$

where $n^2 a^3 = \mu$. We never have to solve for τ, just for $t_0 - \tau$. Then, after the time of flight, the new time past periapsis will be:

$$(t - \tau) = (t_0 - \tau) + TOF \tag{53}$$

Kepler's equation at the end of the time of flight becomes:

$$n(t - \tau) = M = E - e\sin E \tag{54}$$

where M is the mean anomaly at the new position, and is known. Hence we know M and must solve for E. There are several ways to solve this equation for E, one of which is to use the nonlinear equation solver on your calculator or computer if it has one. This equation is well behaved, especially if the eccentricity is small and these algorithms should have no trouble solving for the eccentric anomaly. If such a tool is not available, the equation can typically be solved using one of two methods: fixed point iteration or Newton's method.

Fixed-Point Iteration

If the eccentricity is small, then this method converges fast and is easily implemented. The first thing we need to do is to rearrange Kepler's equation in the form:

$$E = M + e\sin E \tag{55}$$

Then if the eccentricity were zero, the solution would be $E = M$. Hence we will start with this as our first guess. We then simply put that value back into Eq. (55) and solve for a new value of E. The algorithm is as follows:

$$
\begin{aligned}
E_0 &= M \\
E_1 &= M + e \sin E_0 \\
E_2 &= N + e \sin E_1 \\
\cdot &= \cdot + \cdot \\
\cdot &= \cdot + \cdot \\
E_{k+1} &= M + e \sin E_k
\end{aligned}
\tag{56}
$$

The algorithm is repeated until $E_{k+1} - E_k = \Delta E_k$ is as small as one desires. As the eccentricity increases, the number or iterations needed for the result to converge becomes large.

A method that converges faster, but requires more algebraic calculations is Newton's method, likely the method used in many calculators. In general, Newton's method is looking for the solution to the problem: find *x* given *f(x) = 0*, where *f(x)* is some arbitrary function of *x*. the idea is that we can expand the function in a Taylor's series two first order and set that equal to zero

$$
f(x) = f(x_0) + f'(x_0)(x - x_0) + h.o.t = 0 \tag{57}
$$

where x_0 is some initial ***guess***, and $f'(x_0)$ is the derivative of $f(x)$ evaluated at $x = x_0$. We can solve Eq.(57) for a new estimate of x to get:

$$
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{58}
$$

For our problem, the function in which we are interested is Kepler's equation. Hence we have:

$$
\begin{aligned}
f(E) &= E - e \sin E - M \\
f'(E) &= 1 - e \cos E
\end{aligned}
\tag{59}
$$

Then the Newton algorithm becomes:

$$
\begin{aligned}
E_{k+1} &= E_k - \frac{E_k - e\sin E_k - M}{1 - e\cos E_k} \\
&= E_k + \frac{M - E_k + e\sin E_k}{1 - e\cos E_k}
\end{aligned}
\tag{60}
$$

A modified Newton's method can be developed by keeping the second order terms in the Taylor series expansion. A little algebra will lead to the result:

$$
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\left(1 + \frac{1}{2}\frac{f(x_k)f''(x_k)}{\left(f'(x_k)\right)^2}\right) \tag{61}
$$

or for the Kepler problem:

$$
E_{k+1} = e_k + \frac{(M - M_k)}{1 - e\cos E_k} - \frac{1}{2}\frac{e\sin E_k}{1 - e\cos E_k}\left(\frac{M - M_k}{1 - e\cos E_k}\right)^2 \tag{62}
$$

where $M_k = E_k - e\sin E_k$.

Example

Here we are given the orbital properties, En = -0.28 AU^2/TU^2 and h = 1.2 AU^2/TU. We would like to find the position of the satellite 1.9481 TU into the flight assuming the satellite was launched at perihelion.

We can compute the semi-major axis and eccentricity from:

$$
a = -\frac{\mu}{2\,En} = -\frac{1}{2\,(-0.28)} = 1.7857 \ \text{TU} \qquad e = \sqrt{1 + \frac{2\,h^2\,En}{\mu^2}} = \sqrt{1 + \frac{2\,(1.2^2)\,(-0.28)}{1^2}} = 0.4400
$$

The mean angular rate and the mean anomaly is determined from:

$$
n = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{1}{(1.7857)^3}} = 0.4191 \ \text{rad/TU} \qquad M = n\,(t - \tau) = 0.4191\,(1.9481) = 0.8164 \ \text{rad}
$$

Kepler's equation then becomes:

$$
M = E - e\sin E = 0.8164 = E - 0.4400\sin E
$$

that must be solved for E.

Fixed Point Iteration Solution $\quad E_{k+1} = M + e\sin E_k \ , \quad E_0 = M$

The results are:

$$E_0 = 0.8164$$
$$E_1 = 0.8164 + 0.4400 \sin 0.8164 = 1.1370$$
$$E_2 = 1.2156$$
$$E_3 = 1.2289$$
$$E_4 = 1.2309$$
$$E_5 = 1.2312$$
$$E_6 = 1.2313$$
$$E_7 = 1.23128 = 70.5473 \text{ deg}$$

The radial distance is determined from:

$$r = a(1 - e\cos E) = 1.7857\left[1 - 0.4400\cos(1.23128)\right] = 1.524 \text{ AU}$$

This problem is just the trip to Mars problem that we solved for the time of flight to Mars previously. Here we put in the time and determined the distance.

The true anomaly can be determined in several ways:

$$\cos\nu = \frac{\cos E - e}{1 - \cos E}$$
$$= \frac{0.3330 - 0.4400}{1 - 0.4400\,(0.3330)} = -0.1254$$
$$\nu = 1.6965 = 97.200 \text{ deg}$$

$$\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$
$$= \sqrt{\frac{1 + 0.4400}{1 - 0.4400}}\tan\frac{70.5473}{2}$$
$$= 1.1343$$
$$\frac{\nu}{2} = 0.8482 = 48.6000 \text{ deg}$$
$$\nu = 97.200 \text{ deg}$$

Newton's Method

$$E_{k+1} = E_k + \frac{M - E_k + e\sin E_k}{1 - e\cos E_k}, \qquad E_0 = M$$

$$E_0 = M = 0.8164$$

$$E_1 = 0.8164 + \frac{0.8164 - (0.8164 + (0.4400)\sin(0.8164))}{1 - (o.4400)\cos(0.8164)} = 1.27531$$

$$E_2 = 1.23175$$

$$E_3 = 1.23128$$

Note that it takes only 3 iterations here.

Modified Newton's Method

$$E_{k+1} = e_k + \frac{(M - M_k)}{1 - e\cos E_k} - \frac{1}{2}\frac{e\sin E_k}{1 - e\cos E_k}\left(\frac{M - M_k}{1 - e\cos E_k}\right)^2$$

where $M_k = E_k - e\sin E_k$.
Here, M = 0.8164, $M_0 = 0.8164 - 0.4400\sin(0.8164) = 0.49578$

$$E_0 = 0.8164$$

$$E_1 = 1.22698$$

$$E_2 = 1.23128$$

Example

Given: a = 2.0 DU, and e = 0.2. The initial position is at r = 1.7 DU and the flight path is greater than zero. Find the position after 10.1365 TU.

We must find the time past periapsis at the initial position (position 1). We can find the eccentric anomaly at position 1 from:

$$r_1 = a(1 - e\cos E_1) = 1.7 = 2(1 - 0.2\cos E_1)$$

$$\cos E_1 = 0.7500$$

$$E_1 = 41.4096 \text{ deg}$$

We can find the mean anomaly at point 1 from:

$$M_1 = n(t - \tau) = E_1 - e\sin E_1 = \frac{(41.4096)\pi}{180} - 0.2\sin 41.4096° = 0.5904$$

We can find the mean anomaly at the new location from the following:

$$M_2 = n(t_2 - \tau) = n(t_2 - t_1) + n(t_1 - \tau) = \Delta M + M_1$$

or (Recall, $n^2 a^3 = \mu$)

$$M_2 = \sqrt{\frac{1}{2^3}}(10.1356) + 0.5904 = 4.17424 \text{ rad}$$

We can now solve Kepler's equation for E_2.

$$M_2 = E_2 - e\sin E_2 = 4.17424 = E_2 - 0.2\sin E_2 \quad \Rightarrow \quad E_2 = 4.02026 \text{ rad} = 230.3439 \text{ deg}$$

Then the radial distance is:

$$r_2 = a(1 - e\cos E) = 2[1 - 0.2\cos(4.02026)] = 2.25525 \text{ DU}$$

The true anomaly is found from:

$$\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E_2}{2} = \sqrt{\frac{1+0.2}{1-0.2}}\tan\frac{230.3439}{2} = -2.6060$$

$\frac{\nu}{2}$ Has to be in the first or second quadrant. Since the tangent is negative, we know it is in the second quadrant. A calculator would give $\frac{\nu_p}{2} = -69.0069 \text{ deg}$, the principal angle. To convert to

the angle of interest we have $\frac{\nu}{2} = 180 + \frac{\nu_p}{2} = 180 - 69.0069 = 110.9931 \text{ deg}$

or

$$\nu = 221.9862 \text{ deg}$$

Alternatively: $\tan\nu = \frac{\sqrt{1-e^2}\sin E_2}{\cos E_2 - e} = \frac{-0.75439}{-0.8382} = 0.8999$ (third quadrant)

The calculator will give: $\nu_p = 41.9828 \text{ deg}$, so that $\nu = 180 + \nu_p = 221.98 \text{ deg}$

Application to Hyperbolic Orbits

The Kepler problem for the hyperbolic orbit is carried out in exactly the same manner as for the elliptic orbit. The main difference is that the orbit is not periodic and be don't have to be concerned with multiple revolutions. Generally we use Newton's method for solving the hyperbolic form of Kepler's equation. An example will best illustrate how to treat time in the hyperbolic orbit.

Example

Given |a| = 2 DU, and e = 1.2. The initial radius is r = 1 AU with the flight path angle grater than zero. Find the radial distance and true anomaly after a flight time of 0.4238 TU.

Again, we must find the conditions at the initial flight time, position 1. First some preliminaries from $n^2 a^3 = \mu$.

$$n = \sqrt{\frac{\mu}{|a|^3}} = \sqrt{\frac{1}{2^3}} = 0.35355 \text{ rad/TU} \quad \text{and} \quad \frac{1}{n} = 2.8284 \text{ TU/rad}$$

The initial hyperbolic anomaly is obtained from:

$$r = |a|\,(e \cosh F - 1) = 1 = 2\,(1.2 \cosh F - 1)$$

$$\cosh F = 1.2500 \qquad \Rightarrow \qquad F = 0.6931$$

Since the flight path angle is greater than zero, the vehicle is on the outward bound leg of the hyperbola, so we must calculate the time past periapsis passage to the current point 1. From Kepler's equation:

$$M = n\,(t_1 - \tau) = e \sinh F - F = 1.2 \sinh(0.6931) - 0.6931\,) = 0.2068$$

The mean anomaly at the new position is given by:

$$M_2 = M_1 + n\,(TOF) = 0.2068 + 0.3536\,(0.4238) = 0.3566 \text{ rad}$$

Kepler's Equation now looks like:

$$n\,(t_2 - \tau) = M_2 = e \sinh F - F = 0.3566 = 1.2 \sinh F - F$$

Newton's method becomes:

$$F_{k+1} = F_k - \frac{e \sinh F - F - M}{e \cosh F - 1} \qquad F_0 = M$$

For this problem we have:

$$F_0 = 0.3566$$

$$F_1 = 0.3566 - \frac{1.2\cosh(0.3566) - 0.3566 - 0.3566}{1.2\cosh(0.3566) - 1} = 1.3531$$

$$F_2 = 1.04373$$

$$F_3 = 0.94292$$

$$F_4 = 0.93353$$

$$F_5 = 0.93345$$

$$F_6 = 0.93346$$

Then:

$$\boldsymbol{r} = |a|\,(e\cosh F - 1) = 2\,(1.2\cosh(0.93346) - 1) = 1.524 \ \text{AU}$$

The true anomaly is found from:

$$\tan\frac{\nu}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} = \sqrt{\frac{1.2+1}{1.2-1}}\tanh\frac{0.93346}{2} = 1.4440 \qquad \text{(first quadrant)}$$

$$\frac{\nu}{2} = 55.307 \ \text{deg} \qquad \Rightarrow \qquad \nu = 110.614 \ \text{deg}$$
