Astromechanics
Two-Body Problem (Cont)

5. Orbit Characteristics

We have shown that the in the two-body problem, the orbit of the satellite about the primary (or vice-versa) is a conic section, with the primary located at the focus of the conic section. Hence the orbit is either an ellipse, parabola, or a hyperbola, depending on the orbit energy and hence eccentricity. For conic sections we have the following classifications:

$e < 1$	ellipse	
$e = 1$	parabola	(An exception this relation is that all rectilinear orbits have e = 1, and angular momentum = 0)
$e > 1$	hyperbola	

Of main interest for Earth centered satellites (Geocentric satellites) and Sun centered satellites (Heliocentric satellites) are elliptic orbits. However when we go from one regime to another such as leaving the Earth and entering into an interplanetary orbit then we must deal with hyperbolic orbits. In a similar manner, if we approach a planet from a heliocentric orbit, hyperbolic orbits are of interest. Parabolic orbits, on the other hand are more theoretical than practical and simply define the boundary between those orbits which are periodic and “hang around,” (elliptic orbits), and those orbits which allow one to escape from the system (hyperbolic orbits). So one might say that the parabolic orbit is the minimum energy orbit that allows escape. In the following, we will determine the properties of each of these types of orbits and write some equations that are applicable only to the type of orbit of interest.

PROPERTIES OF THE ORBITS

<u>Parabolic Orbit (e = 1, En = 0)</u>

The parabolic orbit serves as a boundary between the elliptic (periodic) orbits and the hyperbolic (escape) orbits. It is the orbit of least energy that allows escape. The orbit equation becomes,

$$r(\nu) = \frac{h^2/\mu}{1 + \cos\nu} = \frac{p}{1 + \cos\nu} \qquad (1)$$

Further, the periapsis distance = $r(0) = p/2 = h^2 / 2\mu$.

The energy equation becomes:

Escape Velocity

$$\frac{V^2}{2} - \frac{\mu}{r} = 0 \quad \Rightarrow \quad V_{escape} = \sqrt{\frac{2\,\mu}{r}} \qquad (2)$$

Equation (2) defines the ***escape velocity***, which is the minimum velocity to escape the two body system at the given radius r. Note that the speed required for escape is independent of its direction!

The flight path angle in a ***parabolic orbit*** is given by:

$$\tan\phi = \frac{\sin\nu}{1 + \cos\nu} = \frac{2\sin\nu/2\,\cos\nu/2}{1 + \cos^2\nu/2 - \sin^2\nu/2} = \tan\nu/2$$

or

$$\phi = \frac{\nu}{2} \qquad \textbf{(3)}$$

Also it is easy to show (use V cos ϕ = r $\dot{\nu}$) that $\cos\phi = \sqrt{\frac{h^2}{2\,\mu\,r}} = \sqrt{\frac{r_p}{r}}$.

Example

A satellite is in a circular orbit pf 100 n mi. Find the escape speed and the minimum ΔV needed to insert the satellite in and escape orbit.

Convert to canonic units: r = R_e + 100 n mi~=~ $1 + \frac{100}{3443.9181} = 1.0290$ DU

$$V_{escape} = \sqrt{\frac{2\,\mu}{r}} = \sqrt{\frac{2\,(1)}{1.0290}} = 1.3941 \text{ DU/TU}$$

$$V_{escape} = 1.3941 \cdot 7.9054 \frac{\text{km/s}}{\text{DU/TU}} = 11.021 \text{ km/s}$$

If we assume a launch tangent to the orbit all we have to do is add the increment in velocity (ΔV) to the existing circular velocity:

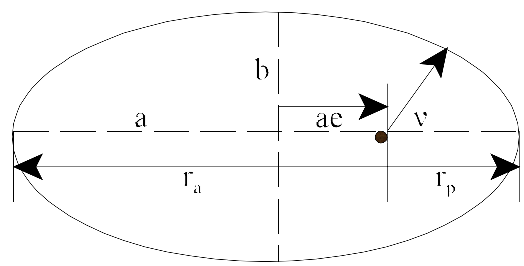
$$\Delta V = V_{es} - V_{cir} = \sqrt{\frac{2\mu}{r}} - \sqrt{\frac{\mu}{r}} = (\sqrt{2} - 1)\sqrt{\frac{\mu}{r}}$$

$$= 0.4142\sqrt{\frac{1}{1.0290}} = 0.4083 \text{ DU/TU} \cdot 7.9054 \frac{\text{km/s}}{\text{DU/TU}}$$

$$= 3.228 \text{ km/s} = 10589.77 \text{ ft/sec}$$

Elliptic Orbits (e < 1, En < 0)

Recall that the radius position is measured from the focus of the ellipse (not the center). By evaluating the orbit equation at values of the true anomaly of 0 and π, we can determine the closest approach (periapsis distance, r_p) and the furthest distance (apoapsis distance, r_a). The sum of these two comprise the ***major axis*** of the ellipse. Of interest to us is half the distance or the ***semi-major axis***, usually designated by the symbol ***a***. Hence ***a*** is the distance from the center of the ellipse to the periapsis or apoapsis. We can set ν = 0 and π to obtain r_p and r_a respectively,



$$r(0) = r_{min} = r_p = \frac{h^2/\mu}{1+e}, \qquad r(\pi) = r_{max} = r_a = \frac{h^2/\mu}{1-e} \tag{4}$$

Then the major axis, $\boldsymbol{2a = r_a + r_p}$, and we have,

$$2a = h^2/\mu\left[\frac{1}{1+e} + \frac{1}{1-e}\right] = h^2/\mu\left[\frac{2}{1-e^2}\right] \tag{5}$$

or

$$h^2/\mu = a(1-e^2) \tag{6}$$

.

Hence the orbit equation for the ellipse becomes;

$$r(\nu) = \frac{a(1-e^2)}{1 + e\cos\nu} \tag{7}$$

Also it follows:

Periapsis and apoapsis distances

$$r_p = a(1 - e)$$
$$r_a = a(1 + e) \tag{8}$$

and

$$e = \frac{r_a - r_p}{r_a + r_p} \tag{9}$$

From the geometry:

$$a\,e = a - r_p = r_a - a \equiv c \tag{10}$$

Define the semi-minor axis:

$$b = a\sqrt{1 - e^2} \tag{11}$$

Energy in an Elliptic Orbit

We can recall the expression for the eccentricity in terms of the energy and angular momentum:

$$e^2 = 1 + \frac{2h^2 En}{\mu^2} \tag{12}$$

We can eliminate the eccentricity from Eq. (6) using Eq. (12):

$$h^2/\mu = a(1-e^2) = a\left[1 - \left(1 + \frac{2h^2 En}{\mu^2}\right)\right] = -\frac{2ah^2 En}{\mu^2} \tag{13}$$

or

Energy in an elliptic orbit

$$En = -\frac{\mu}{2a} \tag{14}$$

Hence the energy equation takes the form (for an elliptic orbit):

Energy equation, elliptic orbit

$$\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \tag{15}$$

Solving for the velocity we have

$$V^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right] \tag{15a}$$

Additionally we have the flight path angle relation

$$\tan\phi = \frac{e\sin\nu}{1 + e\cos\nu} \tag{16}$$

and in terms of r,

$$\cos\phi = \frac{r\dot{\nu}}{V} = \sqrt{\frac{a(1-e^2)}{r\left[2 - \frac{r}{a}\right]}} \tag{17}$$

An interesting set of relations happen at the point on the orbit at the end of the semi-minor **axis**. At that point, the following relations can be found: (can you prove these relations?)

True only at the end of the semi-minor axis:

r = a, cos ν = -e, $\tan\phi = \frac{e}{\sqrt{1-e^2}}$, sin ϕ = e, $b = a\sqrt{1-e^2}$

Orbit Properties in Terms of Apoapsis and Periapsis Properties (r_p, r_a, V_p, V_a)

We know that the major axis of an elliptic orbit is equal to the sum of the periapsis distance and the apoapsis distance $2a = r_p + r_a$. Consequently, the energy in an elliptic orbit is given by:

$$En = -\frac{\mu}{r_p + r_a} \tag{18}$$

Starting with this equation we can determine an expression for the velocity at periapsis and apoapsis:

$$\frac{V_p^2}{2} - \frac{\mu}{r_p} = -\frac{\mu}{r_a + r_p} \qquad \frac{V_a^2}{2} - \frac{\mu}{r_a} = -\frac{\mu}{r_a + r_p}$$

These equations can be rearranged to give the velocities at each apse:

$$V_p = \sqrt{\frac{\mu}{r_p}}\sqrt{\frac{2\frac{r_a}{r_p}}{1 + \frac{r_a}{r_p}}} \qquad V_a = \sqrt{\frac{\mu}{r_a}}\sqrt{\frac{2}{1 + \frac{r_a}{r_p}}} \tag{19}$$

Angular momentum in terms of r_a and r_p

The radial velocity component is zero at both the apoapsis and periapsis so the total velocity is the transverse velocity. Consequently we have the nice relation:

$$h = r_p V_p = r_a V_a \tag{20}$$

Substituting in for either V_p or V_a from Eq. (19) in the above equation provides the angular momentum in terms the distances only. Using either one it is easily shown that

$$h = \sqrt{\frac{2\mu r_a r_p}{r_a + r_p}} \tag{21}$$

Angular Momentum in terms of V_a and V_p

We can use Eq. (20) to get the relation $\frac{r_a}{r_p} = \frac{V_p}{V_a}$ to replace the ratios in Eq. (19), and get the result:

$$V_p^2 = \frac{\mu}{r_p}\frac{2\,V_p}{V_a + V_p} \tag{21}$$

or by rearranging:

$$h = r_p V_p = \frac{2\,\mu}{V_a + V_p} \tag{22}$$

Orbit Energy in terms of V_a and V_p

If we rearrange Eq. (21) we can obtain an expression for r_p in terms of the velocities at the apses:

$$r_p = \frac{\mu}{V_p}\frac{2}{r_a + r_p} \tag{23}$$

Then the energy can be obtained from:

$$En = \frac{V_p^2}{2} - \frac{\mu}{r_p} = -\frac{V_a V_p}{2} \tag{24}$$

Period of an Elliptic Orbit

An elliptic orbit is the only type of orbit that has a period. We can determine the period by recalling the angular momentum equation and noting that half the angular momentum is the aerial rate,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\nu} = \frac{h}{2} \tag{25}$$

Then integrating both sides over one period or once around the orbit we have

$$\int_0^A dA = \int_0^{T_p} \frac{h}{2} dt = \frac{h}{2} T_p = \pi\, a\, b \quad \text{(area of ellipse)} \tag{26}$$

But $h = \sqrt{\mu a (1 - e^2)}$, and $b = a\sqrt{(1 - e^2)}$, and the (1 - e^2) terms cancel leading to the period given as

$$T_p = 2\pi\sqrt{\frac{a^3}{\mu}} \qquad (27)$$

We can see that both the ***energy*** and the ***period*** of the orbit ***depend only on the size*** of the orbit and not on the shape (e).

It is useful to define the ***mean angular rate,*** $n = 2\pi / T_p$. With this definition we can write a form of ***Kepler's third law, the square of the period of an orbit is proportional to the cube of the orbit size.***

$$n^2 a^3 = \mu \qquad (28)$$

<u>Circular Orbit - As a Special Case of Elliptic Orbit</u> (e = 0, $En = -\frac{\mu}{2 r_c}$)

With e = 0, the orbit equation gives,

$$r = p = h^2/\mu = a = const = r_c \qquad (29)$$

From the energy equation,

$$\frac{V_c^2}{2} - \frac{\mu}{r_c} = -\frac{\mu}{2 r_c} \quad \Rightarrow \quad V_c = \sqrt{\frac{\mu}{r_c}} \qquad (30)$$

V_c is called the circular speed and is ***defined*** at every radius r as $V_c \equiv \sqrt{\frac{\mu}{r}}$ regardless of the orbit.

Example

We will launch from the Earth's surface with a velocity of 3 km/s at a flight path angle of 30 degrees. Find the properties of the orbit. The first thing that we do is to calculate the energy and angular momentum!

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{3^2}{2} - \frac{3.9860x10^5}{6378.1363} = -57.9947 \ \text{km}^2/\text{s}^2$$

and

$$h = r\,V\cos\phi = (6378.1363)\,(3)\cos 30 = 16570.8842 \ \text{km}^2/\text{s}$$

From these properties we can calculate the eccentricity and semi-major axis of the orbit:

$$e = \sqrt{1 + \frac{2\,h^2\,En}{\mu^2}} = \sqrt{1 + \frac{2\,(16570.8842)^2\,(-57.9947)}{(3.9860x10^5)^2}} = 0.8942$$

$$En = -\frac{\mu}{2\,a} = -57.9947 = -\frac{3.9860x10^5}{2\,a} \qquad \Rightarrow \qquad a = 3436.5209 \ \text{km}$$

Now we can calculate the maximum and minimum radii of the orbit.

$$r_{\max} = a\,(1 + e) = (3436.5209)\,(1 + 0.8942) = 6509.4579 \ \text{km}$$

$$r_{\min} = a\,(1 - e) = 3436.5209\,(1 - 0.8942) = 363.5839 \ \text{km}$$

Note that part of the “orbit” lies interior to the Earth. The perigee is only 363.6 km from the center of the Earth. On the other hand, the maximum radius is well outside the earth and the satellite (missile?) achieves an altitude of

$$h_{alt} = r_{\max} - R_e = 6509.4579 - 6378.1363 = 131.3216 \ \text{km}$$

We could have done this problem in canonic units in the following way:

$$r = 1 \ \text{DU} \qquad V = \frac{3}{7.9054} = 0.3794 \ \text{DU/TU}$$

Then the energy and angular momentum become:

$$En = \frac{0.3794^2}{2} - \frac{1}{1} - -0.9280 \ \text{DU}^2/\text{TU}^2, \ \ h = (1)\,(0.3794)\cos 30 = 0.3286 \ \text{DU}^2/\text{TU}$$

The eccentricity is obtained from:

$$e = \sqrt{1 + \frac{2 h^2 En}{\mu^2}} = \sqrt{1 + \frac{2\,(0.3286^2)\,(-0.9280)}{1^2}} = 0.8942$$

$$a = -\frac{\mu}{2\,En} = -\frac{1}{2\,(-0.9280)} = 0.5388 \text{ DU}$$

$$r_{\mathbf{max}} = a\,(1 + e) = 0.5388\,(1 + 0.8942) = 1.0206 \text{ DU} \cdot 6378.1363 = 6509.5 \text{ km}$$

<u>Hyperbolic Orbits (e > 1, En > 0)</u>

From the orbit equation, $r = \dfrac{p}{1 + e\cos\nu}$, it is clear that the radial distance will go to infinity when the denominator term goes to zero.

$$r \mapsto \infty \quad \text{when } (1 + e\cos\nu) = 0$$

Then

$$\nu_{\substack{\mathbf{min}\\ \mathbf{max}}} = \cos^{-1}\left(-\frac{1}{e}\right) \tag{31}$$

Equation (31) puts limits on the true anomaly of a hyperbolic orbit.

Just as in an elliptic orbit, we can calculate the closest approach at $\nu = 0$. We can also calculate the “apoapsis” distance by letting $\nu = \pi$. However the result is negative and represents the “closest approach” of the other branch of the hyperbola, one that has no meaning in our orbit. However, formally we can then note that (remember, e > 1):

$$r_p = \frac{h^2/\mu}{1 + e}, \qquad r_\pi = \frac{h^2/\mu}{1 - e} = -(r_p + 2a) \tag{32}$$

Substituting for r_p in the above equation yields the hyperbolic orbit result,

$$h^2/\mu = a\,(e^2 - 1) \tag{33}$$

and the corresponding hyperbolic orbit equation

Hyperbolic Orbit Equation

$$r(\nu) = \frac{a(e^2 - 1)}{1 + e\cos\nu} \qquad (34)$$

If we substitute for $e^2 = 1 + \frac{2h^2 En}{\mu^2}$ in Eq. (33) we find the hyperbolic orbit energy,

$$En = \frac{\mu}{2a} \qquad (35)$$

The corresponding energy equation is

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a} \qquad (36)$$

or

$$V^2 = \mu\left[\frac{2}{r} + \frac{1}{a}\right] \qquad (37)$$

One of the consequences of Eq. (36) (or 37) is that at infinity, the velocity is no longer zero and is given by

$$V_\infty = \sqrt{\frac{\mu}{a}} \qquad (38)$$

and is defined as the ***hyperbolic excess velocity.***

We can also determine the flight path angle in terms of ν or in terms of r in a similar manner as we did for elliptic orbits. The results are

$$\tan\phi = \frac{e\sin\nu}{1 + e\cos\nu} \qquad (39)$$

and

$$\cos\phi = \frac{h}{r\,V} = \frac{a\,(e^2 - 1)}{r\left[2 + \dfrac{r}{a}\right]} \tag{40}$$

Finally, the hyperbolic orbit has a property that no other orbit has. A vehicle traveling the length of the orbit will arrive coming from some point at infinity, and then fly by through the closest approach point (periapsis), and then leave, going to some point at infinity. The approach direction comes in from a direction of $-\nu_\infty$ and the departure direction in the direction of $+\nu_\infty$. The angle through which the vehicle turns is called the ***turning angle,*** δ. This turning angle can be determined from the properties of the hyperbola.

$$\delta = 2\,\nu_\infty - \pi \qquad \Rightarrow \qquad \frac{\delta}{2} = \nu_\infty - \frac{\pi}{2} \tag{41}$$

Then $\sin\frac{\delta}{2} = \sin\nu_\infty \cos\frac{\pi}{2} - \cos\nu_\infty \sin\frac{\pi}{2} = -\cos\nu_\infty = \frac{1}{e}$. Consequently the turning angle is given by

$$\sin\frac{\delta}{2} = \frac{1}{e} \tag{42}$$

The properties characteristic of all orbits were presented in this section. Equations which apply to all orbits are given and those which are applicable to specific orbits are presented. Be sure you don’t apply equations that apply to a specific case to a general case.