Astromechanics
The Two Body Problem (Continued)

4. General Solution to Differential Equations of Motion

The vector differential equation of motion which describes the relative motion of a satellite with respect to a "primary" body is

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} \qquad (1)$$

We have shown that starting with this equation, that the angular momentum of the system is constant and that the energy of the system is constant. ***A consequence of the angular momentum being a constant is that the motion takes place in a plane.*** As a result, we can deal with the two-dimensional problem for now and represent the above equation using plane polar coordinates.
We will write the radial acceleration equals the radial force per unit mass, and the transverse acceleration equals the transverse force per unit mass. From previous work we can write directly,

$$\begin{aligned} \hat{e}_r: &\quad \ddot{r} - r\dot{\theta} = -\frac{\mu}{r^2} \\ \hat{e}_\theta: &\quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{aligned} \qquad (2)$$

These equations are two second order, ordinary differential equations in the dependent variables, r and θ, with the independent variable, t. A solution consists of determining r(t) and θ(t). Determining such a solution requires determining 4 constants of integration. Such a solution ***will not be possible***. However we will try and extract as much information as we can. Remember, two of the constants that we already have (the magnitude of the angular momentum, and the energy equation) must be contained in Eq. (2).

If we look at the transverse (θ) equation, we can note that it can be rewritten as

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \qquad (3)$$

Assuming $r \neq 0$, Eq. (3) gives us the angular momentum constant, in particular, the magnitude of the specific angular momentum is constant.

$$r^2\dot{\theta} = h = const \quad \text{(magnitude of angular momentum)} \qquad (4)$$

The resulting equations to be solved are:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$
$$r^2\dot{\theta} = h \tag{5}$$

Here we have a second order and a first order ordinary differential equation, whose solution is r(t) and θ(t), which we are unable to obtain. Our strategy then is to ***change the independent variable*** from time, t, to the angle, θ. However, by making this transformation, we will lose some information (and a constant) so that we don't get the true solution that we are looking for. We will get how *r* changes with θ or $r(\theta)$ that is an equation of the orbit! We have lost the time dependence.

The procedure for eliminating time is accomplished by noting that we would like to generate the derivative dr/dθ which implies that $r = f(\theta)$. We can obtain this derivative by the following method,

$$\frac{d(\cdot)}{d\theta} = \frac{\frac{d(\cdot)}{dt}}{\frac{d\theta}{dt}} = \frac{1}{\dot{\theta}}\frac{d(\cdot)}{dt} = \frac{r^2}{h}\frac{d(\cdot)}{dt} \tag{6}$$

Therefore we can replace any derivative with respect to time with,

$$\frac{d(\cdot)}{dt} = \frac{h}{r^2}\frac{d(\cdot)}{d\theta} \tag{7}$$

We can apply Eq. (7) to the radial component of the acceleration found in Eq. (5), and use the transverse equation in Eq. (5) to remove the $\dot{\theta}$ term. The result is,

$$\frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right) - \frac{h^2}{r^3} = -\frac{\mu}{r^2} \tag{8}$$

This equation seems worse than the one we started with! However, we will now seek to look at a ***change of dependent variable*** to see if we can simplify Eq. (8). The clue we get is from the term inside ( ). We can note that if we let u = 1/r, then

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\theta} \tag{9}$$

Noting that h is a constant and replacing r with 1/u, and $\dot{\theta}$ with h/r^2, and dividing through by h^2u^2 Eq. (8) becomes,

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \qquad (10)}$$

that has a well known solution. The solution is that of a simple harmonic oscillator with a constant forcing function. It can be written in several ways, all equivalent. Here we will introduce two constants, but only one of which is a new independent constant. The solution to Eq. (10) is given by,

$$\begin{aligned} u(\theta) &= A\cos\theta + B\sin\theta + \frac{\mu}{h^2} \\ &= C\cos(\theta-\omega)+\frac{\mu}{h^2} \\ &= \frac{\mu}{h^2}[1 + e\cos(\theta-\omega)] \end{aligned} \tag{11}$$

where e and ω are the two constants. However only ω is a new one. (3 down, 1 to go). We will show later that e is related to the magnitude of the angular momentum and to energy. We can now note that since u = 1/r we can just take the reciprocal of Eq. (11) to get a general expression for r(θ),

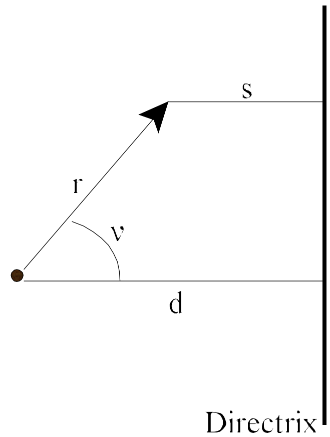
$$\boxed{\begin{aligned} r(\theta) &= \frac{h^2/\mu}{1 + e\cos(\theta - \omega)} \\ &= \frac{h^2/\mu}{1 + e\cos\nu} \end{aligned} \qquad (12)}$$

where θ - ω has been replace by ν.

An Aside (Conic sections)

Recall that one definition of a conic section is that it is the locus of points whose ratio of their distances from a fixed point (focus) that from a straight line (directrix) is a constant (eccentricity).

From the definition and figure, the eccentricity is given by:



$$e = \frac{r}{s}$$

$$= \frac{r}{d - r\cos\nu}$$

$$r = \frac{e\,d}{1 + e\cos\nu} = \frac{p}{1 + e\cos\nu}$$

If we compare the last equation above to Eq. (12), we see that Eq. (12) is the equation of a ***Conic Section*** where r is measured from the focus. The quantity ν is called the ***True Anomaly*** and is the angle measured from the radius vector defined by the point of the conic section closest to the origin (focus), and the current position (radius) vector. The quantity e is called the eccentricity of the conic section.

A quantity called the orbit parameter p is defined such that

$$\boxed{r = \frac{p}{1 + e\cos\nu} \qquad (13)}$$

The orbit parameter $p = h^2/\mu$. It is also called the ***semi-latus rectum***, and is the radius distance when the true anomaly ν= 90 deg (π/2 rad). Eqs. (12) and (13) are the solution to the equations of motion when time is eliminated. They are called the ***equation of the orbit*** and describe how the radial distance varies with the change in the central angle. The particular angle ν is called the ***true anomaly*** and is measured from the point of closest approach, i.e. $r = r_{min}$ when ν = 0. Further, the solution is the equation of a conic section with the radius vector measured from a ***focus*** of that conic section. Consequently from the properties of conic sections, we know that,

e < 1, conic section is an ellipse
e = 1, conic section is a parabola
e >1, conic section is an hyperbola

In both parabolic and hyperbolic orbits the radius vector becomes infinitely long and they represent orbits on which the satellite escapes the two-body system.

It should be pointed out that Eq. (10) is sometimes called the ***differential equation of the orbit***. If an arbitrary central force law was used, (of which an inverse square law is a special case) then the differential equation of the orbit would take on the general form:

$$\frac{d^2 u}{d\theta^2} + u = -\frac{F_r(1/u)}{m' h^2 u^2}, \tag{14}$$

where $F_r(r)$ is an arbitrary central force, and m' is the equivalent mass of system (for two body problem $m' = \frac{m_1 m_2}{m + m_2}$). Note that for any given orbit, Eq. (14) can be used to determine the central force required to generate that orbit. Given r(θ), set u = 1/r(θ), differentiate twice and substitute into Eq. (14) and solve for $F_r(1/u)$ or $F_r(r)$. (Easier said then done!)

Verifying r_{min}

The equation for the radial distance r is given by:

$$r = \frac{p}{1 + e\cos\nu} \tag{15}$$

We can determine the minimum by taking the derivative of *r* with respect to ν, setting it equal to zero, and solving for ν. Substituting that value of ν back into Eq. (15) will give us the minimum (or maximum) value of *r*. The derivative of Eq. (15) with respect to ν gives:

$$\frac{dr}{d\nu} = -\frac{p(-e\sin\nu)}{(1 + e\cos\nu)^2} = 0 \qquad \Rightarrow \qquad \nu = 0,\ \pi \tag{16}$$

It is easily verified that the minimum occurs when $\nu = 0$. Hence the true anomaly is measured from the position of closest approach, $r_{min} = r(\nu)_{\nu = 0} = r(0)$. If we substitute $\nu = 0$ into Eq. (15) we find,

$$\boxed{r_{min} = \frac{p}{1 + e}} \tag{17}$$

Likewise, for an ***elliptic orbit only***, the furthest distance achieved in an elliptic orbit is

$$\boxed{r_{max} = \frac{p}{1 - e}} \tag{18}$$

Clearly, if $e \geq 1$, the denominator can go to zero and $r_{max} = \infty$

These positions in the orbit have special names. The point of closest approach is known as the ***periapsis***, and for an elliptic orbit, the furthest point is called the ***apoapsis***. Generally, if the orbit is about a planet or the sun, the name of the planet or sun is substituted for ..apsis, For example if we are discussing orbits about the Earth (Geoid), we reference the two points of closest and furthest approach as ***perigee*** and ***apogee*** respectively. If we discuss orbits about the Sun (Helios), we reference the two points as ***perihelion*** and ***aphelion***. Sometime the words periapse, or pericenter are used to describe the periapsis, with similar words, apoapse or apocenter to describe the apoapsis.

Relating the Energy and Angular Momentum Constants to the Eccentricity

We have stated several times previously that energy and angular momentum are of primary importance to the orbit problem. In fact, knowing both of these constants will completely describe the size and shape of the orbit. Here we will discuss how they determine the shape (eccentricity) of the orbit. In order to determine this relationship we will calculate the energy in an orbit. In order to make the calculations easier, we will take advantage of the fact that the energy is constant throughout the orbit and evaluate the energy equation at the periapsis. The orbit equation is given by:

$$r = \frac{\frac{h^2}{\mu}}{1 + e\cos\nu} \qquad h = r^2\dot{\theta} \tag{19}$$

and the energy equation:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{\dot{r}^2 + r^2\dot{\theta}^2}{2} - \frac{\mu}{r} = En \tag{20}$$

Since the energy is a constant, we can evaluate it anywhere in the orbit. Let us pick the periapsis where $r = r_{\min}$, and $\dot{r} = 0$. Using these two conditions, we see that

$$r = r_{\min} = r(\nu)\big|_{\nu = 0} = \frac{\frac{h^2}{\mu}}{1 + e} \tag{21}$$

The energy equation becomes:

$$\frac{r^2\dot{\theta}^2}{2} - \frac{\mu}{r} = \frac{h^2}{2r^2} - \frac{\mu}{r} = \frac{h^2(1+e)^2}{2\left(\frac{h^2}{\mu}\right)^2} - \frac{\mu(1+e)}{\frac{h^2}{\mu}} = En$$

or

$$\frac{\mu^2}{h^2}\left[\frac{1}{2}(1 + 2e + e^2) - (1 + e)\right] = \frac{\mu^2}{h^2}\left[-\frac{1}{2}(1 - e^2)\right]$$

and finally:

Eccentricity in terms of energy and angular momentum

$$e^2 = 1 + \frac{2 h^2 En}{\mu^2} \tag{22}$$

Equation (22) is valid for all orbits. Consequently we note the following:

En < 0	e < 1	orbit = ellipse	$\lvert KE \rvert < \lvert PE \rvert$
En = 0	e = 1	orbit = parabola	$\lvert KE \rvert = \lvert PE \rvert$
En > 0	e > 1	orbit = hyperbola	$\lvert KE \rvert > \lvert PE \rvert$

Summary of Results that are Valid for All Orbits

The orbit equation:

$$r = \frac{\dfrac{h^2}{\mu}}{1 + e\cos\nu} \tag{23}$$

where r = distance from focus (center of attracting body)
ν = true anomaly (measured from periapsis)

The orbit parameter (semi latus rectum)

$$p = \frac{h^2}{\mu} = r(\nu)_{\nu = \frac{\pi}{2}} \tag{24}$$

Energy Equation:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{V_r^2 + V_\theta^2}{2} - \frac{\mu}{r} = En \tag{25}$$

Kinematics:

$$V_r = \dot{r} = V \sin\phi$$
$$V_\theta = r\dot{\nu} = V \cos\phi \qquad (26)$$

Eccentricity as a function of energy and angular momentum:

$$e^2 = 1 + \frac{2\,h^2\,En}{\mu^2} \qquad (27)$$

Flight path angle

Before we can write this equation, we need some additional kinematic relations that we can extract from previous work.

$$r = \frac{p}{1 + e\cos\phi} \qquad \dot{r} = \frac{dr}{d\nu}\left(\frac{d\nu}{dt}\right) = \frac{p\,e\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)^2} = \frac{r\,\dot{\nu}\,e\sin\nu}{1 + e\cos\nu} \qquad (28)$$

Then we can write the tangent of the flight path angle from Eq. (26)

Flight path angle

$$\tan\phi = \frac{V_r}{V_\theta} = \frac{\dot{r}}{r\,\dot{\nu}} = \frac{e\sin\nu}{1 + e\cos\nu} \qquad (29)$$

Radial Velocity in Terms of True Anomaly

The radial velocity is just $\dot{r}$, given by Eq, (28).

$$\dot{r} = \frac{p\,e\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)^2} = \frac{r\,\dot{\nu}\,e\sin\nu}{1 + e\cos\nu}$$

By multiplying and dividing by $p = h^2/\mu$ we can rewrite Eq. (28) as,

$$\dot{r} = \frac{1}{p} r^2 \dot{\nu}\, e \sin\nu = \sqrt{\frac{\mu}{p}}\, e \sin\nu = \frac{\mu}{h}\, e \sin\nu \tag{30}$$

Orbit Symmetry

If we define the line of apsides as the line joining the focus and the periapsis, or for an ellipse, the apoapsis and the periapsis, we can note that the orbit is symmetric about that line. Form the orbit equation, we can see that the function, $\cos\nu$, is an even function and consequently the radius is the same for $\pm\ \nu$.