

**Astromechanics**  
**The Two Body Problem (Continued)**

**3. Constants of Motion for the Two Body Problem**

The vector differential equation of motion which describes the relative motion of a satellite with respect to a “primary” body is

$$\ddot{\vec{r}} = -\frac{\mu}{r^3} \vec{r} \quad (1)$$

Equation (1) is a second order ordinary vector differential equation. Once we pick a coordinate system, we can write the representation of Eq. (1) in scalar form. For Cartesian coordinates, for example, we can write Eq. (1) in scalar form:

$$\begin{aligned} \ddot{x} &= -\frac{\mu}{r^3} x \\ \ddot{y} &= -\frac{\mu}{r^3} y \\ \ddot{z} &= -\frac{\mu}{r^3} z \end{aligned} \quad (2)$$

where  $r^3 = (x^2 + y^2 + z^2)^{3/2}$ . We could also pick cylindrical or spherical coordinates in which to represent Eq. (1). In any case, no matter how we represent it, Eq. (1) represents a sixth order dynamic system. We can obtain the solution to the system if we can determine 6 constants of integration! We can extract 4 of these constants by applying standard techniques in dynamic analysis to Eq. (1). We can save ourselves a lot of grief if we keep Eq. (1) in vector form. These standard techniques include looking at the angular motion or moment equation, and applying the work-energy relation. Both of these techniques can be applied to the Newton equation  $\vec{F} = m\vec{a}$ , which is essentially the form of Eq. (1). (Equation (1) can loosely be considered to be  $\vec{a} = \vec{F}/m$ ).

***Angular Motion (Angular Momentum Constant)***

The moment of force is defined to be  $\vec{r} \times \vec{F}$ , and the corresponding time derivative of the moment of momentum (angular momentum) is given as  $\vec{r} \times m\vec{a}$ . In this problem we have divided through by m, hence the results are associated with the specific angular momentum. Here we can take the cross product of the vector,  $\vec{r}$  with Eq. (1),

$$\vec{r} \times \ddot{\vec{r}} = \vec{r} \times \left( -\frac{\mu}{r^3} \vec{r} \right) = 0 \quad (3)$$

The last equality holds since  $\vec{r} \times \vec{r} = \mathbf{0}$ . (a radial force passes through the origin and hence cannot have a moment about the origin) The first term (the term on the left side of the equation) can be rewritten, and Eq. (3) becomes,

$$\frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = \frac{d}{dt}(\vec{r} \times \vec{V}) = 0 \quad (4)$$

Therefore we can write that the specific angular momentum (angular momentum per unit mass) is a constant which we shall label,  $\vec{h} = \text{const}$ .

$\vec{r} \times \vec{V} = \vec{r} \times \dot{\vec{r}} = \vec{h} = \text{const} \quad (5)$
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Note that  $\vec{h}$  is a vector constant so it is equivalent to 3 scalar constants (3 down, 3 to go). ***Any given orbit has constant angular momentum.***

**Consequences of Angular Momentum Constant**

The fact that the angular momentum is a constant simplifies the problem considerably. The angular momentum being constant means that it is fixed in (inertial) space. One can also note that the velocity vector and the position vector are always perpendicular to the angular momentum vector (by definition of the cross product). Furthermore the position vector goes through the center of attraction (Earth, Sun, or whatever is the primary mass). Note that  $\vec{r}$  and  $\vec{V}$  form a plane, and that plane is perpendicular to the angular momentum vector. We can now come to the following conclusions for the two-body problem:

1. Any given orbit lies in a plane fixed in space.
2. The fixed orbit plane must pass through the center of attraction.

***Reducing the Equations of Motion***

Since the orbit lies in a fixed plane, the equations of motion may be simplified by noting that we can now (for the time being) reduce the problem to two dimensions. Hence we can write Eq. (1) in plane coordinates, either x and y as in Eq. 2 (with z = 0), or we can use plane polar coordinates. It turns out more useful to reduce the problem by using plane polar coordinates. Formally, we can use the two of the three angular momentum constants to locate the plane in space (we will do that later). This leaves one of the constants associated with angular momentum that we can apply to our reduced equations of motion.

## Plane Polar Coordinates

Before examining Eq. (1) further, we will look in more detail at the representation of various vector quantities in a plane polar coordinate system. From previous work we can see that the position, velocity, and acceleration vectors can be represented as:

$$\begin{aligned}\vec{r} &= r \hat{e}_r \\ \vec{V} &= \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta = V_r \hat{e}_r + V_\theta \hat{e}_\theta \\ \vec{a} &= (\ddot{r} - r \dot{\theta}^2) \hat{e}_r + (r \ddot{\theta} + 2\dot{r} \dot{\theta}) \hat{e}_\theta = a_r \hat{e}_r + a_\theta \hat{e}_\theta\end{aligned}\quad (6)$$

The angular momentum vector (per unit mass) is given by  $\vec{h} = \vec{r} \times \vec{V}$ . We can implement this operation in the following way:

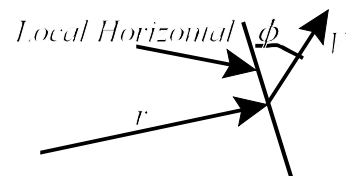
$$\vec{h} = \vec{r} \times \vec{V} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ r & 0 & 0 \\ \dot{r} & r \dot{\theta} & 0 \end{vmatrix} = \begin{Bmatrix} 0 \\ 0 \\ r^2 \dot{\theta} \end{Bmatrix} = r^2 \dot{\theta} \hat{e}_z = r V_\theta \hat{e}_z \quad (7)$$

Hence the magnitude of the angular momentum is given by:

$$|\vec{h}| = h = r^2 \dot{\theta} = r V_\theta \quad (8)$$

We can introduce a new “in-plane” variable, the flight path angle  $\phi$ , the angle between the velocity vector and the local horizontal. With this definition, we can write the angular momentum and the components of velocity in another way:

$$\begin{aligned}V_r &= V \sin \phi \\ V_\theta &= V \cos \phi \\ h &= r V_\theta = r V \cos \phi\end{aligned}\quad (9)$$



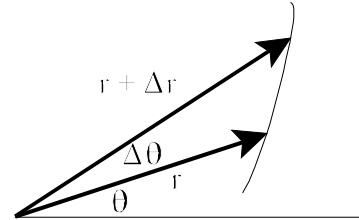
### Kepler's Second Law

We are now in a position to extract Kepler's second law: "Equal areas are swept out in equal times," or "the areal rate is a constant." This law can be established by determining the area swept out as the position vector moves from a location  $\theta$  to  $\theta + \Delta\theta$ . The area between the two vectors is approximately:

$$\Delta A = \frac{1}{2} r (r \Delta \theta) + \frac{1}{2} r \Delta r \Delta \theta + O(\Delta^3)$$

Then, we can divide by  $\Delta t$  and take the limit as  $\Delta t \rightarrow 0$  to get:

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h = \text{const} \quad (10)$$



or the areal rate is a constant ( equal areas are swept out in equal times). The consequence of this law is that when the radial vector is small, the angular rate is large and vice-versa. Hence a satellite in an orbit that varies in altitude will be moving the fastest when closest to the Earth and slower when further from the Earth.

### Specific Energy Constant

We will now extract the work-energy integral. We can extract this integral by noting that  $\dot{\vec{r}} dt = d\vec{r} = \vec{V} dt$ . If we take the dot product of this equation with Eq. (1), we have,

$$\ddot{\vec{r}} \cdot \dot{\vec{r}} dt = -\frac{\mu}{r^3} \vec{r} \cdot d\vec{r} \quad (11)$$

Recall,  $\vec{A} \cdot \dot{\vec{A}} = A \dot{A}$  or  $\vec{A} \cdot d\vec{A} = A dA$ . Then using these relations we can rewrite Eq. (11) in the following manner,

$$\frac{1}{2} d(\dot{\vec{r}} \cdot \dot{\vec{r}}) = -\frac{\mu}{r^2} dr = d\left(\frac{\mu}{r}\right) \quad (12)$$

Integrating both sides of Eq. (12) leads to

$$\frac{1}{2} \vec{v} \cdot \vec{v} - \frac{\mu}{r} = \frac{V^2}{2} - \frac{\mu}{r} = \text{constant} = E_n \quad (13)$$

Eq. (13) is the specific energy equation and says that the kinetic energy plus the potential energy (all per unit mass) is a constant. Typically we use the symbols  $T + U = E_n$ , where  $E_n$  is the total mechanical energy/unit mass. Therefore, the potential energy for an **inverse square** gravitational field is given by  $U = -\frac{\mu}{r}$ , where the datum is selected such that the potential energy is **zero at**

**infinity**. As a consequence of this reference point, the *Potential Energy of All Orbits is Negative!* The energy equation for all orbits is given by

$$\frac{V^2}{2} - \frac{\mu}{r} = E_n \quad (14)$$

$E_n$  in Eq. (14) represents the fourth constant of integration (4 down, 2 to go), and is the Energy constant. *Any given orbit has a constant energy.*

### *Equations-of-Motion in Plane Polar Coordinates*

The equations of motion can now be written in plane polar coordinates. We will write the radial acceleration equals the radial force per unit mass, and the transverse acceleration equals the transverse force per unit mass. From previous work we can write directly,

$$\begin{aligned} \hat{e}_r: \quad \ddot{r} - r\dot{\theta}^2 &= -\frac{\mu}{r^2} \\ \hat{e}_\theta: \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0 \end{aligned} \quad (15)$$

These equations are two second order, ordinary differential equations in the dependent variables,  $r$  and  $\theta$ , with the independent variable,  $t$ . A solution consists of determining  $r(t)$  and  $\theta(t)$ . Determining such a solution requires determining 4 constants of integration. Such a solution **will not be possible**. However we will try and extract as much information as we can. Remember, two of the constants that we already have are the constant magnitude of the angular momentum,  $h$ , and the specific energy  $E_n$ . Hence for a complete solution, we need two more constants of integration.

### *Using Angular Momentum and Energy Constants to Solve Problems*

The two constants, angular momentum and energy for a given orbit, fully describe the size

and shape of the orbit in the orbit plane. They also serve to describe the velocity magnitude and orientation at different points in the orbit. Here we will look at determining certain properties of the orbit, namely the speed and flight path angle of the satellite, and maximum and minimum distances from the attracting body. Note that we can find these properties without knowing any particular details about the orbit. These problems are presented to illustrate the importance of these two constants. There are better ways to calculate some of the properties in the problems than that given here. However, knowing the angular momentum and the energy of an orbit is always the *first step in solving virtually any problem!*

In the examples that follow we will use astrodynamic constants from the Joint Gravitational Model 2 (JGM-2). The constants of interest here are the Earth's gravitational constant,  $\mu$ , and the radius of the Earth,  $R_e$ .

$$\mu_e = 3.9860 \times 10^5 \text{ km}^3/\text{s}^2 = 9.563 \times 10^4 \text{ mi}^3/\text{sec}^2 = 6.275 \times 10^4 \text{ Nm}^3/\text{sec}^2 = 1.4076 \times 10^{16} \text{ ft}^3/\text{sec}^2$$

$$R_e = 6378.1363 \text{ km} = 3963.1902 \text{ mi} = 3443.9181 \text{ Nm} = 2.0926 \times 10^7 \text{ ft}$$

More accurate values appear on the JGM-2 constant sheet.

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### Example

Consider the launch from the surface of the Earth (assume no atmosphere) with a launch velocity of 3 km/s (in any direction!). Determine the speed at 300 km altitude (if it reaches that altitude). We can solve this problem using the energy equation.

At launch:

$$\frac{V^2}{2} - \frac{\mu}{r} = En = \frac{3^2}{2} - \frac{3.9860 \times 10^5}{6378.1363} = -57.9947 \text{ km}^2/\text{s}^2$$

Hence the energy of the orbit is negative! Remember that potential energy is zero at infinity. Here, the potential energy is more negative than the kinetic energy is positive so the sum is negative.

At 300 km:

$$\frac{V^2}{2} - \frac{\mu}{r} = En = \frac{V^2}{2} - \frac{3.9860 \times 10^5}{6378.1363} = -57.9947 \text{ km}^2/\text{s}^2$$

$$V = 1.8399 \text{ km/s}$$

If the satellite in this orbit reaches the altitude of 300 km, it will have a speed of 1.8399 km/s.

Note that the direction of launch is not important, and the direction of the velocity at 300 km is

not determined. In fact we do not know if the orbit will even go that high without specifying the direction of launch.

Example

If, in the previous problem, the satellite was launched at a flight path angle of 30 deg., what would be the maximum altitude reached?

In order to solve this problem, we will need to calculate the angular momentum for this orbit. However, prior to doing that, let us solve this generically (no numbers) and substitute the number is at the end. From the angular momentum we can determine the transverse component of velocity:

$$h = r V_{\theta} \quad \Rightarrow \quad V_{\theta} = \frac{h}{r}$$

From the energy equation we can write:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{V_r^2 + V_{\theta}^2}{2} - \frac{\mu}{r} = En \quad \Rightarrow \quad V_r^2 + \frac{h^2}{r^2} = 2En + 2\frac{\mu}{r}$$

However at the peak of the orbit,  $V_r = 0$  (  $V$  does not equal zero!), so that we can write:

$$2En r_{\max}^2 + 2\mu r_{\max} - h^2 = 0$$

$$r_{\max}^2 + \frac{\mu}{En} r_{\max} - \frac{h^2}{2En} = 0$$

The solution is given by:

$$r_{\max} = -\frac{\mu}{2En} \pm \sqrt{\left(\frac{\mu}{2En}\right)^2 + \frac{h^2}{2En}}$$

For our problem:

$$h = r V \cos\phi = (6378.1363) (3) \cos 30 = 16570.8842 \text{ km}^2/\text{s}; \quad En = -57.9947 \text{ km}^2/\text{s}^2$$

Then:

$$r_{\min}^2 + \frac{3.986 \times 10^5}{-57.9947} r_{\min} - \frac{16570.8842^2}{2(-57.9947)} = 0 \Rightarrow r_{\min}^2 - 6873.0418 r_{\min} + 23674 \times 10^6 = 0$$

$$r_{\max} = 6509.3483 \text{ km}, \quad r_{\min} = 363.6935 \text{ km}$$

The maximum altitude is determined by subtracting the radius of the earth from  $r_{\max}$

$$h_{alt_{\max}} = r_{\max} - R_e = 6509.3483 - 6378.1363 = 131.2120 \text{ km}$$

The minimum “altitude” is obtained the same way:

$$h_{alt_{\min}} = r_{\min} - R_e = 363.6935 - 6378.1363 = -6014.4428 \text{ km}$$

The minimum altitude is below the Earth’s surface. This result should not be surprising since we launched from the surface of the Earth, the lower part of the “orbit” should be inside the Earth. That is, in order to launch into an orbit from the surface of the Earth, the orbit must intersect the Earth.

What is the satellite orbit speed at  $r_{\max}$  and  $r_{\min}$ ? An easy way to calculate these speeds is to note that at these two points in the orbit,  $V_r = 0$ , and  $V = V_\theta$ . So we can write:

$$V = V_\theta = \frac{h}{r} \Rightarrow V = V_\theta = \frac{16570.8842}{6509.3483} = 2.5457 \text{ km/s}$$

$$\text{At the minimum radius} \quad V = V_\theta = \frac{16570.8842}{363.6935} = 45.5628 \text{ km/s}$$

Also, we could have used the energy equation. For Example at the maximum radius ( or height)

$$\frac{V^2}{2} - \frac{\mu}{r} = En = \frac{V^2}{2} - \frac{3.9860 \times 10^5}{6509.3483} = -57.9947 \text{ km}^2/\text{s}^2 \Rightarrow V = 2.5457 \text{ km/s}$$

We can note a few things here. One is that as the distance gets smaller, the speed gets faster. Here it is 20 times faster at the closest point than at the peak.

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Example

Consider a vertical launch with the same velocity. Find the maximum altitude that can be reached. Here the flight path angle is 90 deg.

$$h = r V \cos \phi = r V \cos 90 = 0 = r V_\theta \Rightarrow V_\theta = 0 \Rightarrow V = V_r$$



Then the maximum altitude would occur when  $V_r = V = 0$ . Then we can use the energy equation to determine the maximum altitude.

$$\frac{V^2}{2} - \frac{\mu}{r} = En = \frac{V_r^2 + V_\theta^2}{2} - \frac{\mu}{r} = \frac{0}{2} - \frac{3.986 \times 10^5}{r_{\max}} = -57.9947$$

$$r_{\max} = 6873.0418 \text{ km, or } h_{alt_{\max}} = 6873.0418 - 6378.1363 = 494.9055 \text{ km}$$

### Example

Suppose that we launch with a higher speed, say, 15 km/s (in any direction). How high will this vehicle be able to go?

For all problems dealing with speed, we use the energy equation. In this case we have:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{15^2}{2} - \frac{3.986 \times 10^5}{6378.1363} = 50.0052 \text{ km}^2/\text{s}^2 = En$$

If we substitute  $r = \infty$  into the energy equation we get  $\frac{V^2}{2} = En$ . So for our problem we can solve for the velocity when the spacecraft is at infinity,  $V_\infty$ .

$$V_\infty = \sqrt{2En} = \sqrt{2(50.0052)} = 10.00 \text{ km/s}$$

Hence for this orbit, the satellite still has speed at a large distance from the center of the Earth. This speed is called the *hyperbolic excess velocity*.

$$V_\infty = \sqrt{2En} = \text{hyperbolic excess velocity} \quad (16)$$

We can now ask the question if there is some launch speed that will just get us to infinity, but have no velocity when we get there. Or another way to think about it is if there is an energy level that will just let us get to infinity. Well we want  $V_\infty = 0$  at  $r = \infty$ . If we evaluate the energy equation at infinity with these values we get:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{0}{2} - \frac{\mu}{\infty} = 0$$

The result is that an orbit whose energy level is 0 will have just the right amount of energy to get to infinity with zero velocity. An orbit with negative energy cannot get to infinity, and one with

positive energy gets there with an excess of velocity. This energy level is the boundary below which the satellite cannot escape the Earth, and above which it can. Consequently we can calculate the velocity at any radius that will be the lower bound for escape:

Escape Velocity at any radius  $r$

$$\frac{V^2}{2} - \frac{\mu}{r} = 0 \quad \Rightarrow \quad V_{es} = \sqrt{\frac{2\mu}{r}} \quad (17)$$

At any given radius (distance from the center of the primary attracting body) we can calculate the minimum velocity (in any direction) required for escape). At the surface of the Earth we have:

Escape Velocity from Earth's Surface

$$V_{es} = \sqrt{\frac{2\mu}{r}} = \sqrt{\frac{2(3.986 \times 10^5)}{6378.1363}} = 11.1799 \text{ km/s} = \sqrt{\frac{2(9.5630 \times 10^4)}{3.963.1902}} = 6.7469 \text{ mi/sec}$$

### *Special Case - Circular Orbit*

We can return to the general differential equations of motion and examine them to see if we can extract a solution for the special case of a circular orbit. In plane polar coordinates, a circular orbit would be described by the fact that the radius of the orbit is constant,  $r = \text{constant}$ . Let us look at plane polar differential equations of motion and substitute in the fact the radius is a constant. The general equations are:

$$\begin{aligned} \hat{e}_r: \quad \ddot{r} - r\dot{\theta}^2 &= -\frac{\mu}{r^2} \\ \hat{e}_\theta: \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0 \end{aligned} \quad (18)$$

If we specify that  $r = \text{constant}$  we have  $r = \text{const} = r_c$ ,  $\dot{r} = 0$ ,  $\ddot{r} = 0$ , where we will designate the constant as  $r_c$  ( $r$  circular). Substituting these values into Eq. (18) gives:

$$-r_c \dot{\theta}^2 = -\frac{\mu}{r_c^2} \quad (19)$$

Equation (19) implies that  $\dot{\theta} = \text{constant} = n$  ( $n \equiv$  mean angular rate, in this case constant)

We can also note:  $\vec{V} = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta = V_r \hat{e}_r + V_\theta \hat{e}_\theta = V_\theta \hat{e}_\theta \Rightarrow V = V_\theta$ . That is that the circular orbit speed is same as the transverse component of velocity since  $V_r = \dot{r} = 0$ . We can rewrite Eq (19) by multiplying through by  $r_c$ :

$$r_c^2 \dot{\theta}^2 = V_c^2 = \frac{\mu}{r_c}$$

Then the circular speed is given by:

### Circular Speed at radius r

$$V_c = \sqrt{\frac{\mu}{r_c}} = \sqrt{\frac{\mu}{r}} \quad (20)$$

Equation (20) is the speed a satellite would have if it were in a circular orbit at radius r. However, it also serves to define the circular speed a radius r. For example we might be in an orbit that at some radius r, has a speed V that might be twice the circular speed. Hence we might describe the orbit speed as twice circular. Hence the circular speed often serves as a reference speed from which to define other speeds. Later, we will use such a definition to define canonical units.

### Energy in a Circular Orbit

We can compute the energy in a circular orbit by substituting the orbit speed in the energy equation:

### Energy in a Circular Orbit

$$\frac{V_c^2}{2} - \frac{\mu}{r} = En = \frac{1}{2} \frac{\mu}{r_c} - \frac{\mu}{r_c} = -\frac{\mu}{2r_c} = En_c \quad (21)$$

Here we see that the energy of a circular orbit is always negative and depends only on the size of the orbit, the bigger the radius the higher the energy.

### Angular Momentum in a Circular Orbit

The angular momentum of a circular orbit is easily found from:

### Angular Momentum in a Circular Orbit

$$h = r_c V_\theta = r_c^2 \dot{\theta} = r_c V = r_c \sqrt{\frac{\mu}{r_c}} = \sqrt{\mu r_c} \quad (22)$$

If we square both sides of Eq. (22) (using the third term and the last term), we can arrive at the result:

$$n^2 r_c^3 = \mu \quad (23)$$

where  $n$  is the (mean) angular rate defined earlier.

### Period of a Circular Orbit

Since a circular orbit closes on itself, we can ask the question about how long it takes to return to a given position, or what is the period of the orbit? This calculation is easy since we know the speed and the radius (and hence the circumference) we can simply divide the circumference by the speed:

### Period of Circular Orbit

$$T_p = \frac{\text{circumfrance}}{\text{speed}} = \frac{2 \pi r_c}{\sqrt{\frac{\mu}{r_c}}} = 2 \pi \sqrt{\frac{r_c^3}{\mu}} \quad (24)$$

Example: Find the properties of a 200 km circular orbit.

Position:

$$r_c = 6378.1863 + 200 = 6578.1363 \text{ km}$$

Orbit speed:

$$V_c = \sqrt{\frac{\mu}{r_c}} = \sqrt{\frac{3.986 \times 10^5}{6578.1363}} = 7.7842 \text{ km/s}$$

Angular rate:

$$\dot{\theta}_c = n = \sqrt{\frac{\mu}{r_c^3}} = \sqrt{\frac{3.986 \times 10^5}{6578.1363^3}} = 0.001183 \text{ rad/s} = 0.0678 \text{ deg/s}$$

Period:

$$T_p = \frac{2\pi}{n} = 2\pi \sqrt{\frac{r_c^3}{\mu}} = 2\pi \sqrt{\frac{6578.1363^3}{3.986 \times 10^5}} = 5309.6455 \text{ s} = 88.494 \text{ min}$$


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Example: What is the altitude of a synchronous, 24 hour orbit?

$$T_p = 2\pi \sqrt{\frac{r_c^3}{\mu}} = 2\pi \sqrt{\frac{r_c^3}{3.986012 \times 10^5}} = 24(60)(60)$$

$$r_c = 42241.1221 \text{ km} = 6.6228 R_e \quad \Rightarrow \quad h_{alt} = 35862.9 \text{ km}$$


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### Canonic Units

We hinted previously about using the circular speed as a reference speed and measuring other speeds relative to it. If in addition, select the corresponding radius as a reference length, and measure other lengths relative to it, the resulting units that we end up with are called canonic units. Consider satellites orbiting the Earth. We can define the Earth radius as a reference length,  $R_e = 6378.1363 \text{ km} = 1 \text{ DU}$ , where DU means “distant unit.” The reference speed is the speed of a circular orbit at the radius of the Earth (sometimes called the speed of a surface satellite). Since

$$V_c = \sqrt{\frac{\mu}{r_c}}, \text{ we have } V_c = \sqrt{\frac{3.986 \times 10^5}{6378.1363}} = 7.9054 \text{ km/s} = 1 \text{ SU} = 1 \text{ DU/TU}, \text{ where SU}$$

stands for “speed unit”, and TU stands for a “time unit”. With this definition, a satellite in a circular orbit of 1 DU has a circular speed of 1 DU/TU. If we apply the expression for the circular

$$\text{orbit speed we have: } 1 \text{ DU/TU} = \sqrt{\frac{\mu}{1 \text{ DU}}} \quad \Rightarrow \quad \mu = [1 \text{ DU}^3 / \text{TU}^2]. \text{ When we}$$

reference distances and speeds to the radius of the Earth (or any other planet) and its associated circular speed, we call these canonic units. In canonic units referred to a given planet, the value of  $\mu$  associated with that planet is always  $1 \text{ DU}^3/\text{TU}^2$ . For example in canonic units the energy

$$\text{equation is written as } \frac{V^2}{2} - \frac{1}{2} = En, \text{ where the units of energy are } \text{DU}^2 / \text{TU}^2.$$

$$\text{The period of the reference orbit is given by } T_p = 2\pi \sqrt{\frac{r_c^3}{\mu}} = 2\pi \sqrt{\frac{1^3}{1}} = 2\pi \text{ TU}.$$

Therefore the time unit, TU is the time it takes a satellite in the reference orbit to travel through 1 radian. For orbits about the Sun, the Earth’s orbit radius is the reference distance, and the Earth’s mean orbital speed (EMOS) is the reference speed. In order to convert from canonical units to scientific units we can use the following conversion factors (more accurate ones appear on the JGM-2 constant sheet).

### Canonic Units

	Earth	Sun
1 TU	13.4468 min	58.1328 days
1 DU	6378.1363 km	1.4960x10 <sup>8</sup>
1 SU (DU/TU)	7.9054 km/s	29.7848 km/s

When using canonic units, the orders of magnitude of the numbers in the equations is about 1.

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Example:

What is the energy of a synchronous (24 hour) satellite?  
A synchronous satellite has an orbit radius of 6.6228 Earth radii so  $r = 6.6228$  DU

The speed of a synchronous satellite is  $V_c = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{1}{6.6228}} = 0.3886$  DU/TU

The orbital energy of a synchronous satellite is given by:

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{0.3886^2}{2} - \frac{1}{6.6228} = -0.0755 \frac{\text{DU}^2}{\text{TU}^2}$$

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