2. The Two Body Problem

Preliminaries

There are two basic assumptions that we will make right up front. These are
1. Newton's Laws of motion apply
2. Newton's Law of gravitation applies.

Newton's Law of gravitation can be stated briefly as:

The gravitational force between two masses is proportional to the product of the masses and inversely proportional to the square of the distance between them .

It can be represented in equation form as:

$$|\vec{F}| = G\frac{m_1 m_2}{r^2} \tag{1}$$

where: G is the constant of proportionality known as the universal gravitational constant,
m_i is the mass of body i
r is the distance between the two bodies

General Problem

The general space mechanics problem that we would like to solve can be stated in the following manner:

Given n bodies in space with only Newtonian gravitational force interactions (inverse square gravitational force), and given their initial positions and velocities at some time, t_0 (epoch), that is $\vec{r}_0$, $\vec{V}_0$, at t_0, describe the motion of the n bodies in time.

As you might suspect, this problem is difficult and in fact is ***not possible to solve*** analytically. However, we can reduce the problem to one that is possible to solve by making certain assumptions based on Newton's law of gravitation. From Eq. (1) we can propose the first two of the following simplifications:
1. Neglect the effect of bodies far away (r is large making $|\vec{F}|$ small)
2. Neglect bodies that have a small amount of mass compared to the one of interest (m small)
3. In addition, we will neglect all non-Newtonian forces such as aerodynamic, magnetic, solar radiation, etc

Under these assumptions, we can reduce most problems to the so-called two-body problem. For example, the motion of an artificial satellite about the Earth, we can neglect the effect of the Sun (far away), the moon (far away and small compared to the Earth) , and we can neglect the effects of all the remaining planets and their moons as being either far away, small or both. The resulting problem will consider the Earth and the artificial satellite, the two-body problem. If we want to study the Moon's motion about the Earth, we can justify ignoring all contributions other than the Earth and the Moon, another two-body problem. Earth-Sun is another example.

Governing Differential Equations of Motion for the Two-Body Problem

The two-body problem can be loosely stated as follows:

> *Determine the motion of two spherically homogenous bodies interacting only through a Newtonian inverse square gravitational force.*

Practically, however, we are more interested in the motion of one body with respect to the other and can restate the two-body problem as:

> *Determine the motion of one spherically homogeneous body with respect to another assuming a Newtonian inverse square gravitational force..*
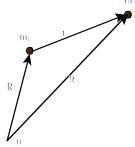
By determining the motion we mean determine the position and velocity as a function of time, that is, given the position and velocity at some epoch, t_0, find the position and velocity at some time, t. This statement of the problem is known as **The Kepler Problem**. We shall develop the key differential equations of motion here.

Consider two masses, m_1 and m_2 that are spherically symmetric. Although it won't be proved here, the gravitational field of a spherically symmetric mass is the same as if all the mass were concentrated at a point and may be treated as a particle (as long as the orbit doesn't fall below the surface of the mass). Consequently the forces of mass 1 on mass 2 and vice-versa are given by:

$$\vec{F}_{21} = -\frac{Gm_1m_2}{r^2}\frac{\vec{r}}{r} = -\frac{Gm_1m_2}{r^3}\vec{r}, \quad (2)$$

and

$$\vec{F}_{12} = \frac{Gm_1m_2}{r^3}\vec{r}, \quad (3)$$

where $\vec{F}_{ij}$ is the force on mass i due to mass j, and G is the universal gravitational constant. Note that Eqs. (2) and (3) are just statements of Newton's law of gravitation with the force assigned a direction (inverse square attractive force). We can now apply Newton's second law of motion,

$\vec{F} = m\vec{a}$, to each mass. In order to be able to write the second law, the reference frame must be inertial. Point 0 in the figure represents the origin of an inertial system. Hence we can write:

$$\vec{F}_{21} = -\frac{Gm_1 m_2}{r^3}\vec{r} = m_2\ddot{\vec{R}}_2,$$

$$\vec{F}_{12} = \frac{Gm_1 m_2}{r^3}\vec{r} = m_1\ddot{\vec{R}}_1.$$

(4)

If we add the two equations in Eq.(4) we get the result,

$$m_1\ddot{\vec{R}}_1 + m_2\ddot{\vec{R}}_2 = 0.$$

(5)

If we recall the definition of the center-of-mass, $(m_1 + m_2)\vec{R}_{cm} \equiv m_1\vec{R}_1 + m_2\vec{R}_2$, we can integrate Eq. (5) twice to get the result (obvious, if you think about it):

$$\vec{R}_{cm} = \vec{V}_{cm_0} t + \vec{R}_{cm_0},$$

(6)

that is, the center-of-mass of the two-body system moves at constant velocity in a straight line. We could have guessed this result since Newton's laws applied to system of particles says that the sum of the external forces equals the mass time the acceleration of the center-of-mass. Here the net external force on the system is zero, so the acceleration of the center-of-mass is zero, and the velocity is constant as is indicated by Eq. (6).

*At this point, let us take a side trip and discuss the differential equations in (4). If we consider both equations in (4), we can note that the two equation system is 12th order, that is, we have two second order ordinary differential equations (ODEs), each one with three components (2nd order x 3 components x 2 sets of equations = 12th order). Therefore if we integrate these equations to get the solution (to be discusses later), we expect to find 12 constants of integration! A complete solution will contain these 12 constants. If we look at Eq. (6) we see that it contains 6 constants, the three components each of $\vec{V}_{cm_0}$ and $\vec{R}_{cm_0}$, the initial values of the center-of-mass velocity and position. To complete the solution we must find 6 more **independent** constants. These are not so easy to determine.*

Let us now return to the problem at hand. In reality, we don't really care what the trajectories of the two masses are in inertial space, what we really want to know is the relative position of mass 2 with respect to mass 1 (satellite with respect to the Earth for example). Consequently it makes sense to take the differences or the two equation in (4). First, however, isolate $\ddot{\vec{R}}_1$ and $\ddot{\vec{R}}_2$ in each equation by dividing through by m_1 and m_2 respectively. Then we note that $\vec{r} = \vec{R}_2 - \vec{R}_1$ and therefore $\ddot{\vec{r}} = \ddot{\vec{R}}_2 - \ddot{\vec{R}}_1$. (Note that you can keep the direction of the

difference vector correct by thinking the difference vector of two vectors points in the direction of head - tail, the vector at the head of the difference vector minus the vector to the tail of the difference vector). Doing the algebra we find that:

$$\ddot{\vec{r}} = -\frac{G(m_1 + m_2)}{r^3}\vec{r}.$$ (7)

where we have used the fact that $\hat{e}_r = \frac{\vec{r}}{r}$. Generally we define the quantity, $\mu = G(m_1 + m_2)$.

However for the usual case of interest, $m_1 \gg m_2$, and $\mu \approx Gm_1$ The former definition is "exact" the latter an approximation. The latter value of μ is tabulated for the various planets and other bodies in our solar system. The resulting differential equation-of-motion for the relative motion in the two body problem is thus given as:

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0.$$ (8)

Equation (8) is the governing equation for the relative motion (body 2 with respect to body 1). It is a second order, ordinary differential equation, with three components. Consequently the system is 6th order, and a complete solution will contain 6 constants of integration (as expected).

In rectangular (Cartesian) coordinates, the three scalar components of Eq. (8) can be written as:

$$\ddot{x} + \frac{\mu}{r^3}x = 0,$$

$$\ddot{y} + \frac{\mu}{r^3}y = 0,$$ (9)

$$\ddot{z} + \frac{\mu}{r^3}z = 0,$$

where $r^3 = (x^2 + y^2 + z^2)^{3/2}$.

In cylindrical coordinates Eq. (8) takes the form:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2},$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0,$$ (10)

$$\ddot{z} = 0.$$

Later on it will be shown that the last equation in Eq. (10) can be dropped with proper selection of reference plane.

General Result

We can consider the case of a non-Newtonian attractive force. If we designate the attractive force as F_r, we can rewrite Eq (4) as:

$$m_2 \ddot{\vec{R}}_2 = -F_r \hat{e}_r$$
$$m_1 \ddot{\vec{R}}_1 = +F_r \hat{e}_r \tag{11}$$

Rearranging and subtracting as we did previously gives:

$$\ddot{\vec{R}}_2 - \ddot{\vec{R}}_1 = \ddot{\vec{r}} = -\left(\frac{F_r}{m_2} + \frac{F_r}{m_1} \right) = -F_r \left(\frac{m_1 + m_2}{m_1 m_2} \right) \hat{e}_r \tag{12}$$

If we rearrange this equation we will have an interesting result:

$$\vec{F} = -F_r \hat{e}_r = \left(\frac{m_1 m_2}{m_1 + m_2} \right) \ddot{\vec{r}} \tag{13}$$

where $\vec{F}$ is the attractive force $-F_r \hat{e}_r$. We observe that $\vec{F} \neq m\vec{a}$! (why?) The quantity in the parentheses is called the ***equivalent mass*** for a two-body system. Note that Eq. (13) is in the same form as if the primary body (mass 1) were inertially fixed and we were writing the equations-of-motion for a second mass with respect to it, if mass the second mass had a mass equal to the equivalent mass.

If we return to Eq. (12), we can use it to develop Eq. (8).

$$\ddot{\vec{r}} = -F_r \left(\frac{m_1 + m_2}{m_1 m_2} \right) \hat{e}_r = -G \frac{m_1 m_2}{r^2} \left(\frac{m_1 + m_2}{m_1 m_2} \right) \hat{e}_r = -\frac{G(m_1 + m_2)}{r^2} \hat{e}_r = -\frac{\mu}{r^3} \vec{r} \tag{14}$$