Astromechanics

1. Preliminaries

Since we will be using vectors in this course, it is of interest to review some ideas relating to them. Of particular interest is the distinction between a vector and it representation. Once the difference between the two is identified, we will be interested in the representation of position, velocity, and acceleration in various coordinate systems.

Vectors and Their Representations

A vector is an abstract symbol introduced by mathematicians and engineers to represent a quantity which has magnitude and direction. Typically in this course we are interested in three dimensional vectors representing position, velocity, and acceleration. A vector is generally represented in its abstract form by an arrow whose length is proportional to the magnitude of the vector quantity. We can deal directly with the vector symbol when doing certain operations such as addition, subtraction, vector, and scalar multiplication by using graphical techniques or by just indicating them, such as: $\vec{A} + \vec{B} = \vec{C}$. However, to perform calculations it is generally necessary to use a ***representation*** of the vector. These representations are not unique but are related to some coordinate system. In general the same vector can be represented in many different ways, depending upon the coordinate system selected. We can pick ***different types*** of coordinate systems, e.g. rectangular (Cartesian), spherical, cylindrical, and in two dimensions, plane rectangular, and plane polar, and we can pick ***different orientations*** of the same type of system. In either case, the ***representation*** of the ***same*** vector will appear quite ***different***. Although in general we are usually interested in different orientations of the same type of coordinate system (to be dealt with later), at the present time we are interested in different types, in this case rectangular and cylindrical (in 2D, plane polar), and spherical coordinate systems.

We generally define a coordinate system by a set of mutually orthogonal unit vectors, called ***basis vectors***. These vectors are of unit length and are perpendicular to each other forming a unit triad. Typically they are designated by the symbol $\hat{e}_i$, where, i, indicates a direction. For example the following is an equivalent representation of the generic vector $\vec{A}$ in a ***rectangular*** coordinate system:

$$\begin{aligned} \vec{A} &= A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \\ &= A_x \hat{e}_x + A_y \hat{e}_y + A_z e_{\hat{z}} \\ &= A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \end{aligned} \tag{1}$$

or in the last case we could write:

$$\vec{A} = \sum_{i=1}^{3} A_i \hat{e}_i \tag{2}$$

Here, A_x, A_y, A_z or the A_i terms are called ***components*** of the vector, and are ***scalars***, and the $(\hat{i}, \hat{j}, \hat{k})$, $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$, or $(\hat{e}_i,\ i = 1,2,3)$ are the ***basis vectors*** for this coordinate system.

The position vector in a rectangular coordinate system is generally represented as

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \tag{3}$$

with $\hat{i}, \hat{j}, \hat{k}$ being the mutually orthogonal unit vectors along the x, y, and z axes respectively. The values x, y, and z are the ***scalar*** components of the position vector $\vec{r}$.

All coordinate systems have two items in common, ***a reference plane***, and a ***reference direction*** in that reference plane. For rectangular coordinates one can think of the reference plane as the x-y plane and the reference direction as the x direction or $\hat{i}$. Further, the coordinate z is measured perpendicular to the reference plane, (along $\hat{k}$) giving us the coordinates (x, y, z). If we consider cylindrical (or plane-polar) coordinates, the reference plane is the one in which the radial component is measured, (r), and the reference direction, the one from which the angle to the radial component is measured (θ). In addition, in cylindrical coordinates, the coordinate z is measured perpendicular to the reference plane, giving us the coordinates (r, θ, z). In spherical coordinates we can think of some equatorial-like plane as the reference plane. The magnitude of the position vector (r) is one coordinate. The reference direction is that direction from which the angle to the projection of the position vector on the reference plane is measured (θ), and the elevation of the position vector with respect to the reference plane is the third coordinate (ϕ), giving us the coordinates (r, θ, ϕ). (Note that sometimes the third angle is measured from the normal to the reference plane (z axis) instead of from the reference plane).

Here, for reasons to become clear later, we are interested in plane polar (or cylindrical) coordinates and spherical coordinates. Cylindrical coordinates have mutually orthogonal unit vectors in the ***radial*** (parallel to the radius vector), ***transverse*** (perpendicular to the radius vector in the plane of interest) and ***normal*** (perpendicular to the plane of interest). They are designated as $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$ respectively. A generic vector $\vec{A}$ would be represented as:

$$\vec{A} = A_r\,\hat{e}_r + A_\theta\,\hat{e}_\theta + A_z\,\hat{e}_z, \tag{4}$$

where A_r, A_θ, A_z are the ***scalar*** radial, transverse, and normal (z) components of the vector $\vec{A}$.

Spherical coordinates also have mutually orthogonal unit vectors in the radial (parallel to the position vector), the longitudinal (parallel to the reference plane and perpendicular to the position vector), and the elevation or latitude (along a constant longitude line and perpendicular to the position and longitudinal unit vectors). A generic vector $\vec{A}$ would be represented as:

$$\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_\phi \hat{e}_\phi, \tag{5}$$

where A_r, A_θ, A_ϕ are the ***scalar*** radial, longitudinal, and latitudinal components of the vector $\vec{A}$. It should be clear that the scalar components of the representation of the vector $\vec{A}$ in plane polar or spherical coordinates are not the same as those in rectangular coordinates. Hence the ***same vector*** has a different representation in different types of coordinate systems. Also it should be clear that the same vector will have a different representation in two rectangular coordinate systems oriented with different reference directions and reference planes.

Although the two representations are different in the two systems, they are related to each other. If we consider the same vector represented in a rectangular coordinate system, in a cylindrical, or in a spherical coordinate system, we have the following relations between the two representations:

$$\begin{aligned} \vec{A} &= A_x \hat{i} + A_y \hat{j} + A_z \hat{k}, \\ &= A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_z \hat{e}_z, \\ &= A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_\phi \hat{e}_\phi, \end{aligned} \tag{6}$$

where even the components A_r and A_θ are different in the two different representations.

The relationship amongst the various components is called a transformation. We can write the transformation matrix relating the cylindrical and spherical components of the vector to the rectangular components. The results are, for rectangular to cylindrical:

$$\begin{Bmatrix} A_r \\ A_\theta \\ A_z \end{Bmatrix}^{cyl} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}^{rect}, \tag{7}$$

and for spherical:

$$\begin{Bmatrix} A_r \\ A_\theta \\ A_\phi \end{Bmatrix}^{sphere} = \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \\ -\sin\theta & \cos\theta & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \end{bmatrix} \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}^{rect} \tag{8}$$

Hence, although the representations of the same vector are different in different coordinate systems, these representations are related to each other. Note that these transformations can be represented in matrix form, and these matrices are called transformation matrices.

Furthermore we can note that we can represent vectors as column matrices if we can specify in what coordinate system they are represented. For example if we have a position vector in a rectangular coordinate system, we can write:

$$\vec{r} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}^{system1} \tag{9}$$

When transforming the representations of the vectors from one system with an orthogonal unit triad of basis vectors to another system with a different orthogonal unit triad of basis vectors, the transformation matrix takes on a very special form. If both basis triads are right handed (the unit vectors obey the right hand rule for taking a vector product, that is: $\hat{i}\ x\ \hat{j} = \hat{k},\ \ \hat{j}\ x\ \hat{k} = \hat{i},\ \ \hat{k}\ x\ \hat{i} = \hat{j}$, or $\hat{e}_r\ x\ \hat{e}_\theta = \hat{e}_\phi,\ \ \hat{e}_\theta\ x\ \hat{e}_\phi = \hat{e}_r,\ \ \hat{e}_\phi\ x\ \hat{e}_r = \hat{e}_\theta$), the transformation matrix is an ortho-normal matrix. That is, its determinant is 1, and its inverse can be obtained by just taking the matrix transpose. Additional properties include: each row or column can be thought of has a vector (with three components) that has a magnitude of 1. Further, each row (vector) is orthogonal to every other row and each column (vector) is orthogonal to every other column. In short the scalar (dot) product of any column (row) with any other column (row) is zero, while the scalar (dot) product with itself is 1. See definitions of vector and scalar products below.

Vector Algebra

Addition and Subtraction

Generally we can manipulate vector equations using the generic vector itself. However, it is useful to know how to do basic vector algebra using the representations of the vector. For addition and subtraction we have for the generic operation,

$$\vec{A} + \vec{B} = \vec{C} \tag{10}$$

In addition, if we could draw each vector on a surface, we could add the two vectors graphically by drawing the vector $\vec{B}$ with the tail of $\vec{B}$ attached to the head of $\vec{A}$. The vector $\vec{C}$ is the vector drawn from the tail of $\vec{A}$ to the head of $\vec{B}$. Once we get more than two vectors, they may not all be in the same plane, and such a construction could become unwieldy and certainly not accurate. Consequently we would like to be able to perform these operations with precision and with ease.

We can achieve both goals since we can perform this operation in terms of the vector representations, if we have each vector represented in the ***same coordinate system***. Then the addition and subtraction operation is just the addition and subtraction of the vector components:

$$C_i = A_i + B_i \tag{11}$$

The Scalar (or Dot) Product

The scalar (dot) product is generically given as $\vec{A} \cdot \vec{B} = S$ where S is a scalar (independent of coordinate system in which $\vec{A}$ and $\vec{B}$ are represented). Furthermore, the definition is
$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}|\,\cos\angle(\vec{A},\vec{B})$. In terms of the vector representations, one can use the definition of the scalar product to show it can be calculated simply as the sum of the products of like components. Here we have

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i\,, \tag{12}$$

or
and the result is the ***same scalar***, regardless of which representation is used.

The magnitude of a vector is given by:

$$|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}} \tag{14}$$

$$\begin{aligned} \vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y + A_z B_z, \\ &= A_r B_r + A_\theta B_\theta + A_z B_z, \\ &= A_r B_r + A_\theta B_\theta + A_\phi B_\phi, \\ &= S \end{aligned} \tag{15}$$

The Vector (or Cross) Product

The vector (cross) product is generically given as $\vec{A}\, x\, \vec{B} = |\vec{A}|\,|\vec{B}|\,\sin\angle(\vec{A},\vec{B})\hat{e}_\perp$, where $\hat{e}_\perp$ is a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ and directed in accordance with the right hand rule. If you rotate $\vec{A}$ into $\vec{B}$ then the vector $\hat{e}_\perp$ points in the direction in which a right handed screw would advance. Alternatively, take your right hand, point you fingers in the direction of $\vec{A}$ and curl them towards $\vec{B}$, your thumb will be pointing in the direction of $\hat{e}_\perp$. If we write each vector in terms of its components and basis vectors, and apply the definition of the vector product to the unit vectors when we perform the vector multiplication, we can arrive at a convenient way to calculate this vector multiplication operation. It can be performed using the representations of the vector in the following manner:

$$\vec{A}\ x\ \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{Bmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{Bmatrix}, \tag{16}$$

or

$$\vec{A}\ x\ \vec{B} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ A_R & A_\theta & A_z \\ B_r & B_\theta & B_z \end{vmatrix} = \begin{Bmatrix} A_\theta B_z - A_z B_\theta \\ A_z B_r - A_r B_z \\ A_r B_\theta - A_\theta B_r \end{Bmatrix}, \tag{17}$$

or

$$\vec{A}\ x\ \vec{B} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_\phi \\ A_R & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix} = \begin{Bmatrix} A_\theta B_\phi - A_\phi B_\theta \\ A_\phi B_r - A_r B_\phi \\ A_r B_\theta - A_\theta B_r \end{Bmatrix}. \tag{18}$$

Vector Calculus

We generally have to deal with derivatives of the above vectors. Or particular interest here is the representation in plane-polar coordinates. It should be noted that the basis vectors in the (inertial) rectangular coordinate system do not change in magnitude or direction, and hence are constant. In plane-polar coordinates, the basis vectors are constant in magnitude, but are changing direction. Hence their derivatives are not zero. We can note the following development:

$$\dot{\hat{e}}_r = \lim_{\Delta t \Rightarrow 0} \frac{\Delta \hat{e}_r}{\Delta t} = \lim_{\Delta t \Rightarrow 0} \frac{\Delta \theta\, \hat{e}_\theta}{\Delta t} = \dot{\theta}\, \hat{e}_\theta . \tag{19}$$

Similarly,

$$\dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r. \tag{20}$$

Then, since $\vec{r} = r\,\hat{e}_r$,

$$\frac{d\vec{r}}{dt} = \vec{V} = \dot{r}\,\hat{e}_r + r\,\dot{\hat{e}}_r = \dot{r}\,\hat{e}_r + r\,\dot{\theta}\,\hat{e}_\theta = V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta. \tag{21}$$

In a similar manner we can represent the acceleration.

$$\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{V}}{dt} = \ddot{r}\,\hat{e}_r + r\,\dot{\hat{e}}_r + \dot{r}\,\dot{\theta}\,\hat{e}_\theta + r\,\ddot{\theta}\,\hat{e}_\theta + r\,\dot{\theta}\,\dot{\hat{e}}_\theta,$$

or

$$\begin{aligned} \vec{a} = \frac{d\vec{V}}{dt} &= \ddot{r}\,\hat{e}_r + \dot{r}\,\dot{\theta}\,\hat{e}_\theta + \dot{r}\,\dot{\theta}\,\hat{e}_\theta + r\,\ddot{\theta}\,\hat{e}_\theta - r\,\dot{\theta}^2\,\hat{e}_r \\ &= \left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{e}_r + \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\hat{e}_r \end{aligned} \tag{22}$$

Similar, but much more complicated, calculations can be carried out for spherical coordinates. The resulting unit vector rates can be determined to be:

$$\begin{aligned} \dot{\hat{e}}_r &= \dot{\phi}\,\hat{e}_\phi + \dot{\theta}\cos\phi\,\hat{e}_\theta \\ \dot{\hat{e}}_\theta &= \dot{\theta}\sin\phi\,\hat{e}_\phi - \dot{\theta}\cos\phi\,\hat{e}_r \\ \dot{\hat{e}}_\phi &= -\dot{\phi}\,\hat{e}_r + \dot{\theta}\sin\phi\,\hat{e}_\theta \end{aligned} \tag{23}$$

Summary

The position, velocity, and acceleration for each coordinate system are given next.

Rectangular Coordinates

$$\begin{aligned} \vec{r} &= x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \\ \vec{V} &= \dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k} \\ \vec{a} &= \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k} \end{aligned}$$

Polar coordinates (in-plane components only)

$$\begin{aligned} \vec{r} &= r\,\hat{e}_r \\ \vec{V} &= \dot{r}\,\hat{e}_r + r\,\dot{\theta}\,\hat{e}_\theta \\ \vec{a} &= (\ddot{r} - r\,\dot{\theta}^2)\,\hat{e}_r + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\hat{e}_\theta \end{aligned}$$

(24)

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$
$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$
$$\nabla(\cdot) = \frac{\partial(\cdot)}{\partial x}\hat{i} + \frac{\partial(\cdot)}{\partial y}\hat{j} + \frac{\partial(\cdot)}{\partial z}\hat{k}$$
$$\nabla\Phi\cdot d\vec{r} = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$$

$$\vec{F} = F_e\,\hat{e}_r + F_\theta\,\hat{e}_\theta$$
$$d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta$$
$$\nabla(\cdot) = \frac{\partial(\cdot)}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial(\cdot)}{\partial\theta}\hat{e}_\theta$$
$$\nabla\Phi\cdot d\vec{r} = \frac{\partial\Phi}{\partial r}dr + \frac{\partial\Phi}{\partial\theta}d\theta$$
$$x = r\cos\phi, \qquad y = r\sin\phi. \tag{25}$$

Spherical Coordinates:

$$\vec{r} = r\,e_r$$

$$\vec{V} = V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta + V_\phi\,\hat{e}_\phi$$
$$= \dot{r}\,\hat{e}_r + r\dot{\theta}\cos\phi\,\hat{e}_\theta + r\dot{\phi}\,\hat{e}_\phi \tag{26}$$

$$\vec{a} = a_r\,\hat{e}_r + a_\theta\,\hat{e}_\theta + a_\phi\,\hat{e}_\phi$$
$$= (\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\cos^2\phi)\hat{e}_r + (r\ddot{\theta}\cos\phi + 2\dot{r}\dot{\theta}\cos\phi - 2r\dot{\phi}\dot{\theta}\sin\phi)\,\hat{e}_\theta$$
$$+ (r\ddot{\phi} + 2\dot{r}\dot{\phi} + r\dot{\theta}^2\sin\phi\cos\phi)\,\hat{e}_\phi$$

$$\vec{F} = F_r\,\hat{e}_r + F_\theta\,\hat{e}_\theta + F_\phi\,\hat{e}_\phi$$
$$d\vec{r} = dr\,\hat{e}_r + r\cos\phi\,d\theta\,\hat{e}_\theta + r\,d\phi\,\hat{e}_\phi$$
$$\nabla(\cdot) = \frac{\partial(\cdot)}{\partial r}\hat{e}_r + \frac{1}{r\cos\phi}\frac{\partial(\cdot)}{\partial\theta}\hat{e}_\theta + \frac{1}{r}\frac{\partial(\cdot)}{\partial\phi}\hat{e}_\phi$$
$$\nabla\Phi\cdot d\vec{r} = \frac{\partial\Phi}{\partial r}dr + \frac{\partial\Phi}{\partial\theta}d\theta + \frac{\partial\Phi}{\partial\phi}d\phi$$

Finally we note:

$$\boxed{\vec{A}\cdot\dot{\vec{A}} = A\dot{A}} \quad \text{and} \quad \boxed{\vec{r}\cdot\dot{\vec{r}} = r\dot{r}} \tag{27}$$

Also you can verify from above that

$|\vec{V}| = |\dot{\vec{r}}| \neq \dot{r}$ E.g. In plane polar coordinates: $|\vec{V}| = \left[\dot{r}^2 + r^2\dot{\theta}^2\right]^{1/2}$