Astromechanics

10. The Kepler Problem

One of the fundamental problems in astromechanics is the Kepler problem The Kepler problem is stated as follows: Given the current position a velocity vectors and a time of flight, find the new position and velocity vectors, or given $\vec{r}_0$, $\vec{V}_0$, and TOF, find $\vec{r}$ and $\vec{V}$. There are two methods that we can use to solve this problem. The first uses material that we have already developed, and the second introduces a new idea that is extremely useful.

Method 1

The first method builds on material already developed. Consequently we will just outline the procedure in a algorithmic format.

1. Find the orbital elements and other orbit characteristics. In particular we can find the orbital elements and parameters: a, e, i, Ω, ω, and ν_0 using the orbit determination scheme developed previously. In addition, we can compute the mean angular rate and the eccentric anomaly (or hyperbolic anomaly since from the value of a or e we can determine if elliptic or hyperbolic). Mean angular rate:

$$n = \sqrt{\frac{\mu}{a^3}} \tag{1}$$

Eccentric (hyperbolic) anomaly:

$$\cos E = \frac{e + \cos\nu_0}{1 + e\cos\nu_0} = \cosh F_0 \tag{2}$$

or

$$r_0 = a\left(1 - e\cos E_0\right) \qquad \text{Or} \qquad r_0 = a\left(e\cosh F_0 - 1\right)$$

2. Determine time past periapsis to the current position (position at epoch) or mean anomaly:

$$M_0 = E_0 - e\sin E_0 \qquad \text{Or} \qquad M_0 = e\sinh F_0 - F_0 \tag{3}$$

3. Determine te mean anomaly after the time of flight:

$$M = M_0 + n\,(TOF) \tag{4}$$

where n is the mean angular rate given in Eq. (1)

4. Solve Kepler's equation for the new eccentric (hyperbolic) anomaly:

$$M = E - e\sin E \qquad \text{Or} \qquad M = e\sinh F - F \tag{5}$$

5. Determine the new radius and true anomaly once we know E (F):

$$r = a(1 - e\cos E) \qquad \text{Or} \qquad r = |a|(e\cosh F - 1) \tag{5}$$

$$\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \qquad \text{Or} \qquad \tan\frac{\nu}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} \tag{6}$$

or from the orbit equation:

$$r = \frac{h^2/\mu}{1 + e\cos\nu} \tag{7}$$

6. Find the orbit-plane components of the position and velocity vectors. We can do this in the "local" coordinate system or in the perifocal coordinate system:

Local

$$\vec{r}^{\ell} = r\,\hat{i}^{\ell}$$

$$\vec{V}^{\ell} = \frac{\mu}{h}e\sin\nu\,\hat{i}^{\ell} + \frac{\mu}{h}(1 + e\cos\nu)\hat{j}^{\ell}$$

Perifocal

$$\vec{r}^{p} = r\cos\nu\,\hat{i}^{p} = r\sin\nu\,\hat{j}^{p}$$

$$\vec{V}^{p} = -\frac{\mu}{h}\sin\nu\,\hat{i}^{p} + \frac{\mu}{h}(e + \cos\nu)\hat{j}^{p} \tag{8}$$

7. Transform from local or from perifocal coordinate system to the classical inertial system

$$\vec{r}^{I} = T_{\ell 2I}\vec{r}^{\ell} \qquad \vec{r}^{I} = T_{p2I}\vec{r}^{p}$$

$$\vec{V}^{I} = T_{\ell 2I}\vec{V}^{\ell} \qquad \vec{V}^{I} = T_{p2I}\vec{V}^{p} \tag{9}$$

where $T_{\ell 2I}$ and T_{p2I} are the local and perifocal to inertial coordinate transformation matrices developed previously.

Method 2

The second method of solving Kepler's equation requires the introduction of a new concept. Recall that the orbit always lies in a plane. Then any vector in that plane can be represented by the some of two constant vectors multiplied by the appropriate constants. In particular we will select the original position and velocity vectors as our constant vectors so that we can represent any other position and velocity vector in the form:

$$\begin{aligned} \vec{r}(t) &= f(t)\,\vec{r}_0 + g(t)\,\vec{V}_0 \\ \vec{V}(t) &= \dot{f}(t)\,\vec{r}_0 + \dot{g}(t)\,\vec{V}_0 \end{aligned} \tag{10}$$

where $f,\ g,\ \dot{f},$ and $\dot{g}$ are to be determined. As might be expected, rather than determining these variables in terms of time, we will eventually determine them in terms of change in true anomaly or eccentric anomaly, and possibly time. To help us find these functions, we can consider the following relations:

$$\begin{aligned} \vec{r}\ x\ \vec{V} &= \left(f\vec{r}_0 + g\,\vec{V}_0\right) x \left(\dot{f}\vec{r}_0 + \dot{g}\,\vec{V}_0\right) \\ &= f\dot{f}\left(\vec{r}_0\, x\, \vec{r}_0\right) + f\,\dot{g}\left(\vec{r}_0\, x\, \vec{V}_0\right) + g\,\dot{g}\left(\vec{V}_0\, x\, \vec{V}_0\right) + g\dot{f}\left(\vec{V}\vec{r}_0\, x\, \vec{r}_0\right) \\ \vec{h} &= \left(f\dot{g} - \dot{f}g\right)\vec{h} \end{aligned}$$

or

$$1 = f\dot{g} - \dot{f}g \tag{11}$$

Hence if we can determine any three, we can obtain the fourth.

In order to get the functions $f,\ g,\ \dot{f},$ and $\dot{g}$, we will use the perifocal coordinate system and represent the initial position and velocity vector as well as the current position and velocity vector in that system using true or eccentric anomaly formulation developed previously. For ow we will just need the component representation of these vectors as follows:

$$\begin{aligned} \vec{r} &= x^p\,\hat{i}^p + y^p\,\hat{j}^p & \qquad \vec{r} &= x_0^p\,\hat{i}^p + y_0^p\,\hat{j}^p \\ \vec{V} &= \dot{x}^p\,\hat{i}^p + \dot{y}^p\,\hat{j}^p & \qquad \vec{V} &= \dot{x}_0^p\,\hat{i}^p + \dot{y}_0^p\,\hat{j}^p \end{aligned} \tag{12}$$

Now, by performing certain operations, we can isolate the four functions, $f,\ g,\ \dot{f},$ and $\dot{g}$ in terms of known quantities in Eq. (12). An example will be given in detail for the function f.

Isolating f

Consider the vector product $\vec{r}\ x\ \vec{V}_0$. Then we have:

$$\vec{r}\ x\ \vec{V}_0 = \left(f\vec{r}_0 + g\vec{V}_0\right) x\ V_0 = f\left(\vec{r}_0\ x\ \vec{V}_0\right) + g\left(\vec{V}_0\ x\ \vec{V}_0\right) = f\vec{h} = fh\hat{k}^p \tag{13}$$

But Eq. (13) also gives us (using Eq. (12)):

$$\vec{r}\ x\ V_0 = \begin{vmatrix} \hat{i}^p & \hat{j}^p & \hat{k}^p \\ x^p & y^p & z^p \\ x_0^p & y_0^p & z_0^p \end{vmatrix} = 0\,\hat{i}^p + 0\hat{j}^p \left(x^p\dot{y}_0^p - y^p\dot{x}_0^p\right)\hat{k}^p \tag{14}$$

Equating Eqs. (13) and (14) leads to

$$fh = x^p\dot{y}_0^p - y^p\dot{x}_0^p$$

or to the desired result:

$$f = \frac{x^p\dot{y}_0^p - y^p\dot{x}_0^p}{h} \tag{15}$$

Isolating $\dot{f}$

In a similar manner if we look at $\vec{V}\ x\ \vec{V}_0$ we can obtain the result:

$$\dot{f} = \frac{\dot{x}^p\dot{y}_0^p - \dot{x}_0^p\dot{y}^p}{h} \tag{16}$$

Isolating g

Here we will look at $\vec{r}_0\ x\ \vec{r}$ and obtain:

$$g = \frac{x_0^p y^p - x^p y_0^p}{h} \tag{17}$$

and finally,

Isolating $\dot{g}$

Here we will look at $\vec{r}_0 \ x \ \vec{V}$and obtain:

$$\dot{g} = \frac{x_0^p \dot{y}^p - \dot{x}^p y_0^p}{h} \tag{18}$$

We can now evaluate these functions in terms of the true anomaly or the eccentric anomaly. The following true anomaly relations have been developed previously:

$$\begin{array}{llll} x^p = r\cos\nu & x_0^p = r_0\cos\nu_0 & y^p = r\sin\nu & y_0^p = r_0\sin\nu_0 \\ \dot{x}^p = -\frac{\mu}{h}\sin\nu & \dot{x}_0^p = -\frac{\mu}{h}\sin\nu_0 & \dot{y}^p = \frac{\mu}{h}\left(e + \cos\nu\right) & \dot{y}_0^p = \frac{\mu}{h}\left(e + \cos\nu_0\right) \end{array} \tag{19}$$

We can substitute these expressions in the Eqs (15 - 18) and apply suitable trig identities to get our ***true anomaly f & g expressions***:

$$\begin{aligned} f &= 1 - \frac{\mu r}{h^2}\left[1 - \cos(\nu - \nu_0)\right] \\ g &= \frac{r r_0}{h}\sin(\nu - \nu_0) \\ \dot{f} &= \frac{\mu}{h}\tan\frac{(\nu - \nu_0)}{2}\left[\frac{1 - \cos(\nu - \nu_0)}{h^2/\mu} - \frac{1}{r} - \frac{1}{r_0}\right] \\ &= \frac{\sqrt{\mu}}{r_0 h^2/\mu}\left[\frac{\vec{r}_0 \cdot \vec{V}_0}{\sqrt{\mu}}\left(1 - \cos(\nu - \nu_0)\right) - \sqrt{h^2/\mu}\sin(\nu - \nu_0)\right] \\ \dot{g} &= 1 - \frac{\mu r_0}{h^2}\left[1 - \cos(\nu - \nu_0)\right] \end{aligned} \tag{20}$$

In order to develop the eccentric anomaly relations we need to determine the derivaitve of the eccentric anomaly. We have from Kepler's equation:

$$\frac{d}{dt}\left[n(t - \tau)\right] = n = (1 - e\cos E)\dot{E} = \frac{r}{a}\dot{E} \tag{21}$$

Hence we can write the derivative of the eccentric anomaly as:

$$\dot{E} = \frac{a\,n}{r} = \frac{1}{r}\sqrt{\frac{\mu}{a}} \tag{22}$$

From previous work we have:

$$\begin{aligned} x^p &= a(\cos E - e) & x_0^p &= a(\cos E_0 - e) \\ \dot{x}^p &= -\frac{1}{r}\sqrt{\mu\, a}\,\sin E & \dot{x}_0^p &= -\frac{1}{r_0}\sqrt{\mu\, a}\,\sin E_0 \end{aligned} \tag{23}$$

and

$$\begin{aligned} y^p &= a\sqrt{1-e^2}\,\sin E & y_0^p &= a\sqrt{1-e^2}\,\sin E_0 \\ \dot{y}^p &= \frac{h}{r}\cos E & \dot{y}_0^p &= \frac{h}{r_0}\cos E_0 \end{aligned} \tag{24}$$

If we substitute these expression in to the ***f & g*** expressions we get them in terms of the eccentric anomaly and time:

$$\begin{aligned} f &= 1 - \frac{a}{r_0}\big(1 - \cos(E - E_0)\big) \\ g &= (t - t_0) - \sqrt{\frac{a^3}{\mu}}\big((E - E_0) - \sin(E - E_0)\big) \\ \dot{f} &= -\frac{\sqrt{\mu\, a}}{r\, r_0}\sin(E - E_0) \\ \dot{g} &= 1 - \frac{a}{r}\big(1 - \cos(E - E_0)\big) \end{aligned} \tag{25}$$

There are similar expressions for hyperbolic orbits.

$$
\begin{aligned}
f &= 1 + \frac{|a|}{r_0}\Big(1 - \cosh(F - F_0)\Big) \\
g &= (t - t_0) - \sqrt{\frac{|a|^3}{\mu}}\Big((F - F_0) - \sinh(F - F_0)\Big) \\
\dot{f} &= -\frac{\sqrt{\mu\,|a|}}{r\,r_0}\sinh(F - F_0) \\
\dot{g} &= 1 + \frac{|a|}{r}\Big(1 - \cosh(F - F_0)\Big)
\end{aligned}
\tag{26}
$$

We can now summarize an algorithm to solve Kepler’s problem using method 2:

Given: $\vec{r}_0,\ \vec{V}_0,\ \text{and TOF}$, Find: $\vec{r},\ \text{and}\ \vec{V}$.

1. Find the magnitudes of the initial position and velocity vectors:

$$
r_0 = \sqrt{\vec{r}_0 \cdot \vec{r}_0} \qquad\qquad V_0 = \sqrt{\vec{V}_0 \cdot \vec{V}_0} \tag{27}
$$

2. Find the angular momentum vector and its magnitude:

$$
\vec{h} = \vec{r}_0\ x\ \vec{V}_0 \qquad\qquad h = \sqrt{\vec{h} \cdot \vec{h}} \tag{28}
$$

3. Find the Energy of the orbit and the semi-major axis:

$$
En = \frac{V_0^2}{2} - \frac{\mu}{r_0} = -\frac{\mu}{2\,a} \qquad \Rightarrow \qquad a \tag{29}
$$

4. Find the eccentricity:

$$
e^2 = 1 + \frac{2\,h^2\,En}{\mu^2} \qquad \Rightarrow \qquad e \tag{30}
$$

5. (Optional) Find the flight path angle:

$$
h = r_0\,V_0\cos\phi_0 \qquad \Rightarrow \qquad \phi_0 \qquad (\text{Check } \vec{r}_0 \cdot \vec{V}_0 > 0,\ \phi_0 > 0;\quad \vec{r}_0 \cdot \vec{V}_0 < 0,\ \phi_0 < 0) \tag{31}
$$

6. Determine the true anomaly at epoch:

$$\tan\nu_0 = \frac{\dfrac{r_0 V_0^2}{\mu}\sin\phi_0\cos\phi_0}{\dfrac{r_0 V_0^2}{\mu}\cos^2\phi_0 - 1} \qquad \text{or} \qquad \cos\nu_0 = \frac{1}{e}\left[\frac{h^2}{\mu r_0} - 1\right] \tag{32}$$

The first form will give the proper quadrant if the two argument tangent function is used. The proper quadrant for the true anomaly can be obtained for the second form from the result of the scalar product $\vec{r}_0 \cdot \vec{V}_0$ being greater (first half) or less (second half) than zero.

7. Compute the eccentric anomaly at epoch:

$$r = a\left(1 - e\cos E_0\right) \quad \Rightarrow \quad \cos E_0 = \frac{1}{e}\left[1 - \frac{r_0}{a}\right] \quad \Rightarrow \quad E_0 \tag{33}$$

8. Compute Mean anomaly at epoch and at current value:

$$M_0 = E_0 - e\sin E_0 \qquad\qquad n = \sqrt{\frac{\mu}{a^3}} \tag{34}$$

$$M = M_0 + n\,(TOF) \tag{35}$$

9. Solve Kepler's equation for the current eccentric anomaly:

$$M = E - e\sin E \quad \Rightarrow \quad E_{k+1} = E_k - \frac{E_k - e\sin E_k - M}{1 - e\sin E_k} \quad \Rightarrow \quad E \tag{36}$$

10. Determine the radius distance and (optional) true anomaly

$$r = a\left(1 - e\cos E\right) \qquad\qquad \cos\nu = \frac{\cos E - e}{1 - e\cos E} \tag{37}$$

11. Determine the position and velocity vectors using f & g functions:

$$\vec{r} = f\vec{r}_0 + g\vec{V}_0 \qquad\qquad \vec{V} = \dot{f}\vec{r}_0 + \dot{g}\vec{V}_0 \tag{38}$$

where:

$$f = 1 - \frac{a}{r_0}\left(1 - \cos(E - E_0)\right) \qquad \dot{f} = -\frac{\sqrt{\mu a}}{r r_0}\sin(E - E_0)$$

$$g = TOF - \sqrt{\frac{a3}{\mu}}\left[(E - E_0) - \sin(E - E_0)\right] \qquad \dot{g} = 1 - \frac{a}{r}\left[1 - \cos(E - E_0)\right]$$

Or the true anomaly form of the f & g functions could be used.

Example:

Given: $\vec{r}_0 = 1\hat{i} + 1\hat{j}$ DU $\qquad \vec{V}_0 = 2\hat{k}$ DU/TU $\qquad \Delta\nu = 60$ deg

Find: $\vec{r}, \vec{V}$

1. Find magnitudes of initial vectors: $r_0 = \sqrt{2}$ DU $\qquad V_0 = 2$ DU/TU

2. Find $\vec{h} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = \begin{Bmatrix} 2 \\ -2 \\ 0 \end{Bmatrix}$ $\qquad |\vec{h}| = 2\sqrt{2}$ DU^2/TU

3. Find Energy,

$$En = \frac{V_0^2}{2} - \frac{\mu}{r_0} = \frac{4}{2} - \frac{1}{\sqrt{2}} = 1.2929 \text{ DU}^2/\text{TU}^2 = -\frac{1}{2a} \qquad a = -0.3867 \text{ (hyperbola)}$$

or

$|a| = 0.3867$

4. Determine the eccentricity: $e = \sqrt{1 + \frac{2h^2 En}{\mu^2}} = \sqrt{1 + \frac{(2)(8)^2(1.2929)}{1^2}} = 4.6568$

5. Determine initial flight path angle: $h = r_0 V_0 \cos\phi_0 = 2\sqrt{2} = 2\sqrt{2}\cos\phi_0 \quad \Rightarrow \quad \phi_0 = 0$

6&7. Determine true anomaly at epoch, Since at periapsis, $\nu_0 = M_0 = F_0 = 0$

8. Determine true anomaly: $\nu = \nu_0 + \Delta\nu = 0 + 60 = 60$ deg

9. Not necessary here, will repeat problem using hyperbolic anomaly later

10. Determine radial distance: $r = \dfrac{h^2/\mu}{1 = e\cos\nu} = \dfrac{8}{1 + 4.6568\cos 60} = 2.4035 \text{ DU}$

11, Determine $\vec{r}$ and $\vec{V}$ using f and g functions:

$$f = 1 - \frac{\mu r}{h^2}\left[1 - \cos\Delta\nu\right] = 1 - \frac{(1)(2.4035)}{8}\left[1 - \cos 60\right] = 0.8498$$

$$g = \frac{r r_0}{h}\sin\Delta\nu = \frac{(2.4035)(\sqrt{2})}{2\sqrt{2}}\sin 60 = 1.0407$$

Then

$$\vec{r} = f\vec{r}_0 + g\vec{V}_0 = 0.8498\begin{Bmatrix}1\\1\\0\end{Bmatrix} + 1.0407\begin{Bmatrix}0\\0\\2\end{Bmatrix} = \begin{Bmatrix}0.8498\\0.8498\\2.081\end{Bmatrix}$$

and $|\vec{r}| = 2.4034 \text{ DU}$

The velocity:

$$\dot{f} = \frac{\mu}{h}\tan\Delta\frac{\nu}{2}\left[\frac{1 - \cos\Delta\nu}{h^2/\mu} - \frac{1}{r} - \frac{1}{r_0}\right]$$

$$= \frac{1}{2\sqrt{2}}\tan 30\left[\frac{1 - \cos 60}{8} - \frac{1}{2}.4035 - \frac{1}{\sqrt{2}}\right] = -0.2165$$

$$\dot{g} = 1 - \frac{\mu r_0}{h^2}\left(1 - \cos\Delta\nu\right) = 1 - \frac{(1)(\sqrt{2}}{8}\left(1 - \cos 60\right) = 0.9116$$

and finally:

$$\vec{V} = \dot{f}\vec{r}_0 + \dot{g}\vec{V}_0 = -0.2165\begin{Bmatrix}1\\1\\0\end{Bmatrix} + 0.9116\begin{Bmatrix}0\\0\\2\end{Bmatrix} = \begin{Bmatrix}-0.2165\\-0.2165\\1.8232\end{Bmatrix}$$

and $|\vec{V}| = 1.8487 \text{ DU/TU}$

We can check the energy equation:

$$En = \frac{V^2}{2} - \frac{\mu}{r} = \frac{1.8487^2}{2} - \frac{1}{2.4034} = 1.2929 \ \mathrm{DU}^2/\mathrm{TU}^2$$ As before!

Example

Here we will use the same example as above but use the hyperbolic anomaly. Since the epoch time is at periapsis, many calculations are not needed and some that are needed were done in the previous problem.

We need to determine the radial distance and the hyperbolic anomaly after the time of flight. Generally, we would need to solve the Kepler equation for the hyperbolic anomaly. However here we can take advantage of the solution obtained in the previous problem. The radial distance we found to be: $r = 2.4034 \ \mathrm{DU}$. We can use this to determine the hyperbolic anomaly:

$$r = |a|(e\cosh F - 1) = 2.4034 = 0.3867(4.6568\cosh F - 1) \quad \Rightarrow \quad \cosh F = 1.5494$$

$$F = 1.0054$$

Since we are at periapsis at epoch, we have $F_0 = 0, \text{ and } F - F_0 = \Delta F = 1.0054$

Now we can determine the position and velocity vectors using the hyperbolic anomaly form of the f and g functions:

$$f = 1 + \frac{|a|}{r_0}(1 - \cosh\Delta F) = 1 + \frac{0.3867}{\sqrt{2}}(1 - \cosh 1.0054) = 0.8498$$

$$\dot{f} = -\frac{\sqrt{\mu a}}{r r_0}\sinh\Delta F = -\frac{\sqrt{(1)(0.3867}}{(2.4034)(\sqrt{2})}\sinh 1.0054 = -0.2165$$

$$\dot{g} = 1 + \frac{|a|}{r}[1 - \cosh\Delta F] = 1 + \frac{0.3867}{2.4034}[1 - \cosh 1.0054] = 0.9116$$

Before we can find g, we need to find the time of flight. In this case since it is time past periapsis we can use Kepler's equation:

$$t - \tau = TOF = \frac{1}{n}[e\sinh F - F[= \sqrt{\frac{(0.3867^3)}{1}}[4.6568\sinh 1.0054 - 1.0054] = 1.0826 \ \mathrm{TU}$$

$$g = TOF - \sqrt{\frac{|a|^3}{\mu}}(\sinh\Delta F - \Delta F) = 1.0836 - \sqrt{\frac{0.3867^3}{1}}(\sinh 1.0054 - 1.0054) = 1.0407$$

These number are the same as determined previously!