Maneuvering Flight

The feature that maneuvering flight adds to the equations is the pitch rate. Two types of maneuvering flight that we will look at are a symmetric pull up, and a horizontal turn. In either case an added pitch rate is encountered that must be accounted for in the governing equations. Exactly what this pitch rate is we can obtain from the equations of motion.

Pitch-up

For a symmetric pitch up we will look at the vehicle as if it is moving in a large vertical circle. We will consider the vehicle when it is at the bottom of this vertical. If we sum the forces in the vertical direction at this instant we have

$$L - W = m\frac{V^2}{R} \tag{1}$$

where $\frac{V^2}{R}$ is the acceleration radially inward, in the direction of the lift. We can further note that the angular rate of the aircraft as it moves around this vertical circle is $\frac{V}{R} = q$, the same as the pitch rate. If in addition, we introduce the concept of a load factor, $n = \frac{L}{W}$, we can rewrite the force equation as:

$$(n - 1)W = \frac{W}{g}Vq \tag{2}$$

that can be rearranged to give:

$$q = \frac{g}{V}(n - 1) \tag{3}$$

q for pull up

Equation (3) is the pitch rate of an aircraft doing a pull-up (at the bottom of the circle).

Horizontal Turn

If an aircraft is banked at some angle ϕ, and is desired to maintain horizontal flight, then the vertical and horizontal force-balance equations must be satisfied in the following way. The vertical balance equation is

$$L\cos\phi = W = nW \quad \Rightarrow \quad \cos\phi = \frac{1}{n} \tag{4}$$

The horizontal force equation becomes the radial equation of the turn and takes the form::

$$L\sin\phi = m\frac{V^2}{R_h} \tag{5}$$

However the turn rate is given by $\Omega = \dfrac{V}{R_h}$ and is along the horizontal axes system's vertical z axis. Since the aircraft is banked at the angle ϕ, the component of the turn rate along the y axis, the pitch rate, is given by

$$q = \Omega \sin \phi = \frac{V}{R_h} \sin \phi \tag{6}$$

If we multiply the horizontal force equation, Eq.(5) by $\sin \phi$, the result is

$$L \sin^2 \phi = m V q = n W \sin^2 \phi \tag{7}$$

or, rearranging:

$$q = \frac{g}{V} n \sin^2 \phi = \frac{g}{V} n (1 - \cos^2 \phi) = \frac{g}{V} n (1 - \frac{1}{n^2}) \tag{8}$$

or in the final form:

$$q = \frac{g}{V} \left(\frac{n^2 - 1}{n} \right) \tag{9}$$

q for horizontal turn

Effect on pitch moment and lift equations

The pitch and lift coefficient equations now have an additional term due to the pitch rate that makes them appear as follows:

$$\begin{aligned} C_L &= C_{L_\alpha} \bar{\alpha} + C_{L_{\delta_e}} \delta_e + C_{L_q} \hat{q} \\ C_m &= C_{m_{0L}} + C_{m_\alpha} \bar{\alpha} + C_{m_{\delta_e}} \delta_e + C_{m_q} \hat{q} \end{aligned} \tag{10}$$

Here all the derivatives must be and are non-dimensional. Consequently we need to examine the "q" derivatives closer. Consider the lift equation and the term C_{L_q}. The non-dimensional derivative is defined as follows:

$$C_{L_q} \equiv \frac{\partial C_L}{\partial \left(\dfrac{q b}{2 V} \right)} = \frac{\partial C_L}{\partial \hat{q}} \tag{11}$$

where

$$\hat{q} \equiv \frac{q b}{2 V}$$

Non-dimensional pitch rate (11a)

Similarly for the pitch coefficient we have:

$$C_{m_q} = \frac{\partial C_m}{\partial \hat{q}} \tag{12}$$

From the definitions we can note that

$$\frac{\partial C_{(x)}}{\partial q} = \frac{\bar{c}}{2V} C_{(x)_q}$$
$$C_{(x)_q} = \frac{2V}{\bar{c}} \frac{\partial C_{(x)}}{\partial q} \tag{13}$$

where x = L, m.

Estimating C_{L_q}

The major contribution to lift due to pitch rate is from the horizontal surfaces at extremes ends of the fuselage such as the horizontal tail (or canard surface). This contribution comes from a additional increment in tail angle of attack due to the pitch rate. The horizontal tail has an additional velocity due to the pitch rate given by $w_{ht} = q\, l_{ht}$. The increment of tail angle of attack is then given by (assuming small angles)

$$\Delta \alpha_{ht} = \frac{q\, l_{ht}}{V_{ht}} \quad \Rightarrow \quad \Delta C_{L_{ht}} = a_{ht} \frac{q\, l_{ht}}{V_{ht}} \tag{14}$$

Then the aircraft increment in lift coefficient is given by

$$\Delta C_L = \Delta C_{L_{ht}} \eta_{ht} \frac{S_{ht}}{S} = a_{ht} \eta_{ht} \frac{S_{ht}}{S} \frac{l_{ht}}{V_{ht}} q \tag{15}$$

To get Eq (15) in terms of $\hat{q}$, it is necessary to multiply and divide by $\frac{\bar{c}}{2V}$, or take the derivative of Eq. (15) with respect to q and apply the second equation in Eq.(13). Doing the latter, we have

$$C_{L_q} = \frac{2V}{\bar{c}} \frac{\partial C_m}{\partial q} = \frac{2V}{\bar{c}} a_{ht} \eta_{ht} \frac{S_{ht}}{Sl_{ht}} \frac{l_{ht}}{V_{ht}}$$

or rearranging and noting $\eta_{ht} = V_{ht}^2 / V^2$, we have

$$C_{L_q} = 2\, a_{ht} \sqrt{\eta_{ht}} \left(\frac{S_{ht}\, l_{ht}}{S\, \bar{c}} \right) = 2\, a_{ht} \sqrt{\eta}\, \forall_{ht} \tag{16}$$

Similarly for the pitch moment:

$$\Delta C_m = -\Delta C_{ht} \eta_{ht} \frac{S_{ht} l_{ht}}{S \bar{c}} = -a_{ht} \eta_{ht} \frac{S_{ht} l_{ht}}{S \bar{c}} \frac{l_{ht} q}{V_{ht}} \tag{17}$$

then

$$C_{m_q} = \frac{2V}{\bar{c}} \frac{\partial C_m}{\partial q} = -2\, a_{ht} \sqrt{\eta_{ht}}\, \forall_{ht} \frac{l_{ht}}{\bar{c}} \tag{18}$$

However, unlike the lift coefficient, the wing and other parts of the aircraft contribute to the pitch moment due to pitch rate. We can estimate the contributions from the wing using techniques in Appendix B of Etkin and Reid or for preliminary calculations, add about 10 % to the value calculated in Eq. (18). Then the pitch moment due to pitch rate can be estimated using the following equation:

$$C_{m_q} = -2\, K_q\, a_{ht} \sqrt{\eta_{ht}}\, \forall_{ht} \frac{l_{ht}}{\bar{c}} \tag{19}$$

where K_q = 1.10.

Elevator Control Deflection During Pull up or Horizontal Turn

Return now to the governing equations for the pull-up (or horizontal turn). We are interested in determining the additional elevator deflection required for the maneuver over and above that required for level flight at the same speed. Our strategy then is to determine the elevator angle required for the maneuver, and subtract from it the elevator angle required for level flight. The result will be displayed in the form of elevator angle / g.

First we need the equations for maneuvering flight:

$$\begin{aligned} C_L &= n\, C_w = C_{L_\alpha} \bar{\alpha} + C_{L_{\delta_e}} \delta_e + C_{L_q} \hat{q} \\ C_m &= 0 = C_{m_{0L}} + C_{m_\alpha} \bar{\alpha} + C_{m_{\delta_e}} \delta_e + C_{m_q} \hat{q} \end{aligned} \tag{20}$$

For level flight we have

$$\begin{aligned} C_L &= C_w = C_{L_\alpha} \bar{\alpha}_L + C_{L_{\delta_e}} \delta_{e_L} \\ C_m &= 0 = C_{m_{0L}} + C_{m_\alpha} \bar{\alpha}_L + C_{m_{\delta_e}} \delta_{e_L} \end{aligned} \tag{21}$$

where the subscript L stands for the level flight condition. If we subtract the equations in Eq. (21) from the corresponding equations in Eq. (20) we obtain:

$$(n-1)C_w = C_{L_\alpha}\Delta\bar{\alpha} + C_{L_{\delta_e}}\Delta\delta_e + C_{L_q}\hat{q}$$
$$0 = C_{m_\alpha}\Delta\bar{\alpha} + C_{m_{\delta_e}}\Delta\delta_e + C_{m_q}\hat{q} \tag{22}$$

where $\Delta\bar{\alpha} = \bar{\alpha} - \bar{\alpha}_L$, and $\Delta\delta_e = \delta_e - \delta_{e_L}$, the differences of the maneuvering values from the values in level flight.

We can solve these two equations for the two unknowns, $\Delta\bar{\alpha}$, and $\Delta\delta_e$. Of particular interest here is the elevator deflection:

$$\Delta\delta_e = -\frac{C_{m_\alpha}(n-1)C_w + \left(C_{L_\alpha}C_{m_q} - C_{m_\alpha}C_{L_q}\right)\hat{q}}{C_{L_\alpha}C_{m_{\delta_e}} - C_{m_\alpha}C_{L_{\delta_e}}} \tag{23}$$

For a pull-up we can recall from Eqs . (3) and (11a)

$$\hat{q} = \frac{q\bar{c}}{2V} = \frac{(n-1)g\bar{c}}{2V^2}\frac{m\rho S}{m\rho S} = (n-1)C_W\frac{\rho S\bar{c}}{4m} \tag{24}$$

For a horizontal turn this equation takes the form

$$\hat{q} = \frac{q\bar{c}}{2V} = \frac{\left(\frac{n^2-1}{n}\right)g\bar{c}}{2V^2}\frac{m\rho S}{m\rho S} = (n-1)C_W\left(\frac{n+1}{n}\right)\frac{\rho S\bar{c}}{4m} \tag{25}$$

We can combine the results of Eqs. (24) and (25), to give us an expression for “elevator angle per g”

$$\frac{\Delta\delta_e}{(n-1)} = -\frac{C_w\left[C_{m_\alpha} + \left(C_{L_\alpha}C_{m_q} - C_{m_\alpha}C_{L_q}\right)(\mathrm{Fac})\frac{\rho S\bar{c}}{4m}\right]}{C_{L_\alpha}C_{m_{\delta_e}} - C_{m_\alpha}C_{L_{\delta_e}}} \tag{26}$$

elevator angle per g

where $\mathrm{Fac} = 1$ for a pull-up, and $\mathrm{Fac} = \frac{n+1}{n}$ for a horizontal turn.

Note that 1-g flight is straight and level. A 2-g pull up requires an elevator deflection equal to level flight + (n-1= 1) times the elevator angle /g.

Stick-Fixed Maneuver Point

The next question we can ask is, is there a location of the center of mass that would make the elevator angle per g go to zero?. The answer of course is, “yes”. We can determine this point by noting that $C_{m_\alpha} = a(h - h_n)$, substitute this value into Eq. (26), set $\Delta\delta_e = 0$ and solve for h. We designate the solution to be h_{mp}, the stick-fixed maneuver point. We can also note that the denominator term, even though it contains C_{m_α} and C_{m_δ}, (both terms that depend on the center of mass location), is independent of the center of mass location. Hence we can just set the numerator equal to zero. The result is the stick fixed maneuver point:

$$h_{mp} = h_n - \frac{C_{m_q}}{\left(\frac{4m}{\rho S \bar{c}}\right) - C_{L_q}} \tag{27}$$

The quantity

$$h_{mp} - h \tag{28}$$

is called the stick fixed maneuver margin.

Equation (26) can be rewritten in terms of the stick fixed maneuver margin in the following form:

$$\frac{\Delta\delta_e}{n - 1} = -\frac{C_w C_{L_\alpha}\left(\left(\frac{4m}{\rho S \bar{c}}\right) - C_{L_q}\right)}{\left(\frac{4m}{\rho S \bar{c}}\right) - C_{L_q}}(h - h_{mp}) \tag{29}$$

Note that the stick fixed maneuver point is behind the stick fixed neutral point. So that for zero elevator angle per g, the aircraft must be statically unstable. Also if the center of mass were behind the stick fixed maneuver point, the elevator deflection to sustain a pull-up would be in the opposite direction! That is to sustain, say a 2-g pull-up would require a ***down*** elevator!

Stick Force Per g

Related to elevator angle per g is the stick force per g. The stick force per g is determined by the increment in the hinge moment. Again, we will look at the change in the stick force from straight and level flight. This change is caused by the change in the hinge moment and can be represented as:

$$\Delta F_s = G\, 1/2\, \rho\, V^2\, \eta_{ht}\, S_e\, \bar{c}_e\, \Delta C_{H_e} \tag{30}$$

Here the change in the hinge moment is given by:

$$\Delta C_{H_e} = b_1\, \Delta \alpha_{ht} + b_2\, \Delta \delta_e \tag{30}$$

The change in the tail angle of attack is given by

$$\Delta \alpha_{ht} = \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right) \Delta \bar{\alpha} + \frac{l_{ht}\, q}{V_{ht}} \tag{31}$$

Now $\Delta \delta_e$ is given by Eq. (23) and $\Delta \bar{\alpha}$ can be obtained from the same set of equations. Carring out these operations, we can obtain an expression for the stick force per g that yields the following result:

$$\frac{\Delta F_s}{n - 1} = -G\, \eta_{ht}\, S_e\, \bar{c}_e\, \frac{W}{S}\, \hat{a}\, b_2\, \frac{\left(\frac{4\, m}{\rho\, S\, \bar{c}} \right) - C_{L_q}}{\left(\frac{4\, m}{\rho\, S\, \bar{c}} \right) \Delta} \left(h - \hat{h}_{mp} \right) \tag{32}$$

where $\Delta = C_{L_\alpha} C_{m_{\delta_e}} - C_{m_\alpha} C_{L_{\delta_e}}$ and the ***stick free maneuver point*** is determined from

$$\hat{h}_{mp} = h_{mp} + \frac{\Delta}{\hat{a}\, b_2} \left[\frac{b_1 \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right)}{a} + \frac{2\, b_1\, (h_{ht} - h)}{\sqrt{\eta_{ht}} \left(\frac{4\, m}{\rho\, S\, \bar{c}} - C_{L_q} \right)} \right] \tag{33}$$

The stick force per g is also called the stick force gradient. It is clear from the equations that the stick force gradient (with respect to the g loading) has the following characteristics:

$$\text{the stick force per g is} \propto \begin{cases} \hat{h}_{mp} - h \\ l^3 \\ \dfrac{W}{S} \end{cases}$$

Note that if the vehicle densities were the same, the stick force per g would be proportional to the scale factor to the 4th power, l^4.