Longitudinal Aircraft Dynamics

Previously we looked at a "pinned" aircraft motion in pitch and found the following differential equation of motion:

$$\Delta\ddot{\alpha} - \frac{M_q + M_{\dot{\alpha}}}{I_y}\Delta\dot{\alpha} - \frac{M_\alpha}{I_y}\Delta\alpha = 0\,, \tag{1}$$

and its associated characteristic equation:

$$\lambda^2 - \frac{M_q + M_{\dot{\alpha}}}{I_y}\lambda - \frac{M_\alpha}{I_y} = 0\,. \tag{2}$$

We now seek a method to extend our capabilities beyond a second order equation. To do this, we will see how we can represent this same motion (pinned aircraft in pitch) in another form that is easily extended to a higher dimension system. The method of approach is to replace the single second order equation with two first order ordinary differential equations. We do this as follows:

$$\begin{aligned} \Delta\dot{\alpha} &= \Delta q \\ \Delta\dot{q} &= \frac{M_q + M_{\dot{\alpha}}}{I_y}\Delta q + \frac{M_\alpha}{I_y}\Delta\alpha \end{aligned} \tag{3}$$

Equation (3) is equivalent to Eq. (1). However in this form it is convenient to write these equations in matrix form:

$$\begin{Bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{M_\alpha}{I_y} & \dfrac{M_q + M_{\dot{\alpha}}}{I_y} \end{bmatrix} \begin{Bmatrix} \Delta\alpha \\ \Delta q \end{Bmatrix} \tag{4}$$

We consider the variables whose derivatives appear to be "state" variables since they tell us the "state" of the system. Here the state variables are $\Delta\alpha$ and Δq. In order to determine a solution to these ordinary differential equations, we can proceed just as we did for the second order equation, we will assume solutions of the form:

$$\Delta\alpha = C_1 e^{\lambda t} \qquad\qquad \Delta q = C_2 e^{\lambda t} \tag{5}$$

If we substitute these guesses into Eq.(4) and note that the scalars $e^{\lambda t}$ divide or cancel out (since they are non-zero), we arrive at the following :

$$\begin{Bmatrix} C_1 \lambda \\ C_2 \lambda \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{M_\alpha}{I_y} & \dfrac{M_q + M_{\dot\alpha}}{I_y} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix}. \tag{6}$$

We can rearrange this matrix by putting everything on the left hand side of the equation to get:

$$\begin{bmatrix} \lambda & -1 \\ -\dfrac{M_\alpha}{I_y} & \lambda - \dfrac{M_q + M_{\dot\alpha}}{I_y} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{7}$$

Here we note that this matrix is just the negative of the original matrix with λs added to the diagonals. If we seek a solution to Eq. (6) for the C_i, i = 1,2 , for the case where the right hand side is zero, then the only way we can get non-zero solutions for C_i is if the determinant of the coefficient matrix is zero. That is we require

$$\begin{vmatrix} \lambda & -1 \\ -\dfrac{M_\alpha}{I_y} & \lambda - \dfrac{M_q + M_{\dot\alpha}}{I_y} \end{vmatrix} = 0 . \tag{8}$$

Performing the determinant operation we get the characteristic equation:

$$\lambda^2 - \frac{M_q + M_{\dot\alpha}}{I_y} \lambda - \frac{M_\alpha}{I_y} = 0, \tag{9}$$

which is exactly the same as Eq. (2), the characteristic equation of the second order system! Consequently, although the procedures were different, we arrived at the same result. However in this form, the method can be extended to higher order systems since the matrix can be of any dimension.

We can summarize the above results in a concise manner. If we consider the state vector x to be a vector whose components are the state variables. In this case $\Delta\alpha \ and \ \Delta q$. Then we can write Eq. (4) as:

$$\dot{x} = A x \tag{10}$$

where *A* is called the system matrix (here it is a 2 x 2 matrix), All the properties of the solution are contained in the matrix *A*. These are extracted by forming the characteristic equation as described in Eqs. (7) and (8), that takes the form:

$$| \lambda I_n - A | = 0, \tag{11}$$

where I_n is an n x n identity matrix that is a matrix with 1s on its diagonal, and zeros elsewhere. So the matrix in Eq. (11) has λs on the diagonals with the negative of the elements of the system matrix elsewhere and on the diagonals.

We can now extend this approach to the general longitudinal dynamics of an aircraft. The procedure is similar to that used to develop all the equations of motion we used previously. First we write the general equations of motion, then we examine a reference flight condition, and then look at small motions away from that reference flight condition.

Longitudinal Flight Equations

The longitudinal flight equations of motion can be written in the following fashion using the force equations along and perpendicular to the velocity. Then the general equations of motion become:

$$\begin{aligned} T - D - mg\sin\gamma &= m\dot{V}, \\ L - mg\cos\gamma &= mV\dot{\gamma}, \\ M &= I_y\dot{q}, \\ q &= \dot{\theta}, \end{aligned} \tag{12}$$

where V = airspeed
γ = flight path angle (angle between velocity and local horizontal)
T = thrust
D = drag
m = mass
M = pitch moment
q = pitch rate
θ = pitch angle

Using the definition of state variables we introduced earlier, the state here is x = $[V, \gamma. q, \theta]^T$. The other variables in Eq. (12) are not state variables, but are functions of the state variables (and another type called control variables). It is necessary for us to assume what variables these functions contain. Based on past experience, we will assume that the functions that appear in Eq.(12) take the following form:

$$\begin{aligned} T &= T(V, \alpha, \delta_T) \\ L &= L(V, \alpha, q, \delta_e) \\ D &= D(V, \alpha, q, \delta_e) \\ M &= M(V, \alpha, q, \delta_e) \\ \alpha &= \alpha(\gamma, \theta) = \theta - \gamma \end{aligned} \tag{13}$$

Note that in the functions we introduced three new variables, δ_T, δ_e. *and* α. The first two are control variables since their derivatives do not appear and they do not depend on any other variables. α on the other hand is a ***function*** of other state variables. Since the angle of attack appears in all the other functions, it might me convenient (but not necessary) to use it as a state variable and replace either the flight path angle or pitch angle as a state. We will make this change by using the relation $\alpha = \theta - \gamma \quad and \quad \dot{\alpha} = \dot{\theta} - \dot{\gamma}$.

If we rearrange Eq. (12) with the derivatives on the left and the remainder of the terms on the right hand side of the equations, and make the substitution for $\dot{\alpha}$, we will get the equations of motion in the following form:

$$\begin{aligned} \dot{V} &= \frac{1}{m}(T - D) - g\sin\gamma = f_1, \\ \dot{\alpha} &= q - \frac{L}{mV} - \frac{g}{V}\cos\gamma = f_2, \\ \dot{q} &= \frac{M}{I_y} = f_3, \\ \dot{\theta} &= q = f_4. \end{aligned} \tag{14}$$

The functions now take the form:

$$\begin{aligned} T &= T(V, \alpha, \delta_T), \\ L &= L(V, \alpha, q, \delta_e), \\ D &= D(V, \alpha, q, \delta_e), \\ M &= M(V, \alpha, q, \delta_e), \\ \gamma &= \gamma(\alpha, \theta) = \theta - \alpha. \end{aligned} \tag{15}$$

Note that the only change is in the last function which is now the flight path angle since the angle of attack is now a state. This may same like a small detail, but it is an important one. We need to establish the (four) states, and the functions. So now we have the following:

$$\begin{aligned} &\text{State variables :} && V, \alpha, q, \theta \\ &\text{Control variables:} && \delta_T, \delta e \\ &\text{Functions:} && T, L, D, M, \gamma \end{aligned} \tag{16}$$

We are now ready to proceed. We need to establish a reference flight condition. We will select a steady state reference flight condition which means that the state derivatives are zero. Hence we get:

$$\frac{1}{m}(T_{ref} - D_{ref}) - g\sin\gamma_{ref} = 0,$$

$$q_{ref} - \frac{L_{ref}}{m V_{ref}} - \frac{g}{V_{ref}}\cos\gamma_{ref} = 0, \tag{17}$$

$$\frac{M_{ref}}{I_y} = 0,$$

$$q_{ref} = 0.$$

These equations represent the conditions for a steady state reference flight condition. In principle the contain six variables, the four state and the two control variables. In theory we can pick any two and solve the four equations for the remaining four variables. Fortunately we do not have to do that for what we are interested in doing. It is sufficient to say that such conditions can exist.

Equations (14) and (17) are in the form

$$\begin{aligned} \dot{x} &= f(x, u), \\ 0 &= f(x_{ref}, u_{ref}), \end{aligned} \tag{18}$$

where x is the four component state vector (V, α, q, θ) and u is the two component control vector (δ_T, δ_e). We now let the variables take on the value at the reference flight condition plus a small disturbance:

$$\begin{aligned} \dot{x}_{ref} + \Delta\dot{x} &= f(x_{ref} + \Delta x, u_{ref} + \Delta u) \\ &= f(x_{ref}, u_{ref}) + \left.\frac{\partial f}{\partial x}\right|_{ref} \Delta_x + \left.\frac{\partial f}{\partial u}\right|_{ref} \Delta u + \cdots \end{aligned} \tag{19}$$

We can ignore the higher order terms, and note that the reference conditions are zero. What is left is a linear ordinary differential equation with constant coefficients that has the form:

$$\Delta\dot{x} = A\Delta x + B\Delta u \tag{20}$$

where A is called the system matrix, and B is called the control matrix. Here we will hold the control at the reference flight value so that $\Delta u = 0$. Hence we will be dealing with the equation in the form:

$$\Delta\dot{x} = A\,\Delta x \tag{21}$$

where the matrix A is given by:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial V} & \frac{\partial f_1}{\partial \alpha} & \frac{\partial f_1}{\partial q} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial V} & \frac{\partial f_2}{\partial \alpha} & \frac{\partial f_2}{\partial q} & \frac{\partial f_2}{\partial \theta} \\ \frac{\partial f_3}{\partial V} & \frac{\partial f_3}{\partial \alpha} & \frac{\partial f_3}{\partial q} & \frac{\partial f_3}{\partial \theta} \\ \frac{\partial f_4}{\partial V} & \frac{\partial f_4}{\partial \alpha} & \frac{\partial f_4}{\partial q} & \frac{\partial f_4}{\partial \theta} \end{bmatrix}_{ref} \tag{22}$$

where the f_i are the right hand sides in Eq. (14).

We can evaluate this system matrix by carrying out the operations specified in Eq.(22). The calculations follow. We will do the calculations by column, taking the derivatives of all the functions with respect to the same state variable:

V derivatives:

$$\frac{\partial f_1}{\partial V} = \frac{1}{m}\left(\frac{\partial T}{\partial V} - \frac{\partial D}{\partial V} \right)_{ref} \tag{24}$$

$$\frac{\partial f_2}{\partial V} = \frac{1}{m V^2}\left[L - mg\cos\gamma \right] - \frac{1}{mV}\frac{\partial L}{\partial V}\bigg|_{ref} = -\frac{1}{m V}\frac{\partial L}{\partial V}\bigg|_{ref} \tag{25}$$

Note that the term in [ ] is zero when evaluated at the reference flight condition (see Eq. (17)).

$$\frac{\partial f_3}{\partial V} = \frac{1}{I_y}\frac{\partial M}{\partial V}\bigg|_{ref} \qquad \frac{\partial f_4}{\partial V} = 0 \tag{26}$$

α derivatives:

$$\begin{aligned} \frac{\partial f_1}{\partial \alpha} &= \frac{1}{m}\left(\frac{\partial T}{\partial \alpha} - \frac{\partial D}{\partial \alpha} \right) - g\cos\gamma\frac{\partial \gamma}{\partial \alpha} \\ &= \frac{1}{m}\left(\frac{\partial T}{\partial \alpha} - \frac{\partial D}{\partial \alpha} \right) + g\cos\gamma\bigg|_{ref} \end{aligned} \tag{27}$$

Here we see that the known function $\gamma = \theta - \alpha$ appears in the derivative so we need to use the chain rule: $\frac{\partial f}{\partial \alpha} = \frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial \alpha} = \frac{\partial f}{\partial \gamma}(-1)$.

$$\frac{\partial f_2}{\partial \alpha} = -\frac{1}{m\,V}\frac{\partial L}{\partial \alpha} + \frac{g}{V}\sin\gamma\bigg|_{ref} \tag{28}$$

and

$$\frac{\partial f_3}{\partial \alpha} = \frac{1}{I_y}\frac{\partial M}{\partial \alpha}\bigg|_{ref} \qquad \frac{\partial f_4}{\partial \alpha} = 0 \tag{29}$$

q derivatives:

$$\frac{\partial f_1}{\partial q} = -\frac{1}{m}\frac{\partial D}{\partial q}\bigg|_{ref} \; ; \qquad \frac{\partial f_2}{\partial q} = 1 - \frac{1}{m\,V}\frac{\partial L}{\partial q}\bigg|_{ref} \tag{40}$$

$$\frac{\partial f_3}{\partial q} = \frac{1}{I_y}\frac{\partial M}{\partial q}\bigg|_{ref} \; ; \qquad \frac{\partial f_4}{\partial q} = 1 \tag{41}$$

θ derivatives:

$$\frac{\partial f_1}{\partial \theta} = -g\cos\gamma\big|_{ref} \; ; \quad \frac{\partial f_2}{\partial \theta} = -\frac{g}{V}\sin\gamma\bigg|_{ref} \; ; \quad \frac{\partial f_3}{\partial \theta} = \frac{\partial f_4}{\partial \theta} = 0 \tag{42}$$

These results can be assembled back into the system matrix. We will use the short-hand notation for the dimensional derivatives, $\frac{\partial L}{\partial V} = L_V$, etc.

$$A = \begin{bmatrix} \frac{1}{m}(T_V - D_V) & \frac{1}{m}(T_\alpha - D_\alpha) + g\cos\gamma & \frac{1}{m}D_q & -g\cos\gamma \\ -\frac{1}{m\,V}L_V & -\frac{1}{m\,V}L_\alpha + \frac{g}{V}\sin\gamma & 1 - \frac{L_q}{m\,V} & -\frac{g}{V}\sin\gamma \\ \frac{1}{I_y}M_V & \frac{1}{I_y}M_\alpha & \frac{1}{I_y}M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{ref} \tag{43}$$

Equation (43) is a general longitudinal system matrix for an aircraft (or any other atmospheric vehicle). The only assumptions made were that the thrust is along the velocity vector, and that there are no functions that depend on $\dot{\alpha}$.

The individual terms are calculated by performing the indicated operation and evaluating the result and the reference flight condition. The result of all this activity will be a matrix full of numbers! Some examples follow that will also define new non-dimensional stability derivatives. For example:

$$\begin{aligned} D_V &= \frac{\partial D}{\partial V} = \frac{\partial C_D}{\partial V}\bar{q}S + C_D \rho V S \\ &= \frac{\partial C_D}{V_{ref}\,\partial\left(\frac{V}{V_{ref}}\right)}\bar{q}S + \frac{2 C_D \rho V^2 S}{2V} \\ &= \left.\frac{C_{D_V}\bar{q}S}{V} + \frac{2D}{V}\right|_{ref} \end{aligned} \tag{44}$$

The exact same forms hold for the lift coefficient :

$$\begin{aligned} L_V &= \frac{\partial L}{\partial V} = \frac{\partial C_L}{\partial V}\bar{q}S + C_L \rho V S \\ &= \frac{\partial C_L}{V_{ref}\,\partial\left(\frac{V}{V_{ref}}\right)}\bar{q}S + \frac{2 C_L \rho V^2 S}{2V} \\ &= \left.\frac{C_{L_V}\bar{q}S}{V} + \frac{2L}{V}\right|_{ref} \end{aligned} \tag{45}$$

and the moment coefficient:

$$M_V = C_{m_V}\left.\frac{\bar{q}S\bar{c}}{V}\right|_{ref} \tag{46}$$

Note that in the reference flight condition $M_{ref} = 0$, and hence the second term is missing.

The alpha derivatives are straight forward:

$$\begin{aligned} D_\alpha &= C_{D_\alpha}\bar{q}S\big|_{ref} \\ L_\alpha &= C_{L_\alpha}\bar{q}S\big|_{ref} \\ M_\alpha &= C_{m_\alpha}\bar{q}S\bar{c}\big|_{ref} \end{aligned} \tag{47}$$

The q derivatives are not so straight forward, but we have encountered them before:

$$
\begin{aligned}
D_q &= C_{D_q} \frac{\bar{q} S \bar{c}}{2V}\bigg|_{ref} \\
L_q &= C_{L_q} \frac{\bar{q} S \bar{c}}{2V}\bigg|_{ref} \\
M_q &= C_{m_q} \frac{\bar{q} S \bar{c}^2}{2V}\bigg|_{ref}
\end{aligned}
\qquad (48)
$$

Example: We can illustrate the procedure by doing an example. The non-dimensional derivatives will be supplied as needed. First we will assume that the aircraft is a jet, and that the thrust is independent of speed and angle of attack, thus T_V *and* $T_\alpha = 0$. Further, the mass of the vehicle is given by $m = \frac{W}{g} = \frac{38200}{32.174} = 1187.29 \text{ slugs}$. The aircraft is at sea level and flying at 223.28 ft/sec. Consequently we have $\bar{q}S = 1/2\,(0.00238)\,223.28^2\,(542.5) = 32184.47 \text{ lbs}$. The drag is calculated from: $D = C_D \bar{q} S = 0.095\,(32184.47) = 3057.52 \text{ lbs}$

$$a_{11} = -\frac{1}{m}(T_V - D_V) = \frac{1}{1187.29}\left(0 - \frac{2(3057.52)}{223.28}\right) = -0.0231$$

Here we assumed no compressibility effects so that $C_{D_V} = 0$.

$$a_{12} = -\frac{1}{m}D_\alpha + g\cos\gamma = \frac{1}{1187.29}(0.75)\,32184.47 + 32.174(1) = 11.8434$$

Here we assume level flight.

$$a_{13} = \frac{1}{m}D_q = 0 \;; \qquad a_{14} = -g\cos\gamma = -32.174$$

Here we assume no drag dependence on pitch rate ($C_{D_q} = 0$), and level flight.

$$a_{21} = -\frac{1}{mV}L_V = -\frac{1}{mV}\frac{2L}{V} = -\frac{2g}{V^2} = -\frac{2(32.174)}{223.28^2} = -0.0013$$

Here we assume no compressibility effects and level flight, lift = weight.

$$a_{22} = -\frac{1}{mV}L_\alpha + \frac{g}{V}\sin\gamma = -C_{L_\alpha}\frac{\bar{q}S}{mV} = -5.0\frac{32184.47}{1187.29(223.28)} = -0.6070$$

Here we assume level flight ($\gamma = 0$).

$$a_{23} = 1 - \frac{L_q}{mV} = 1 \; ; \qquad a_{24} = -\frac{g}{V}\sin\gamma = 0$$

Here we assume that $C_{L_q} = 0$, and level flight ($\gamma = 0$)

$$a_{31} = \frac{M_v}{I_y} = \frac{1}{I_u} C_{m_V} \frac{\bar{q}S\bar{c}}{V} = 0$$

Here we assume no compressible effects and hence $C_{m_V} = 0$.

$$a_{32} = \frac{M_\alpha}{I_y} = C_{m_\alpha} \frac{\bar{q}S\bar{c}}{I_y} = -0.8 \frac{32184.47(10.93)}{35773} = =2.0733$$

$$a_{33} = \frac{M_q}{I_y} = C_{m_q} \frac{\bar{q}s\bar{c}^2}{2V} = -8.0 \frac{32184.47(10.93^2)}{2(223.28} = -0.5073$$

and $a_{34} = 0$.

If we substitute these numbers into the matrix we have:

$$A = \begin{bmatrix} -0.0231 & 11.8434 & 0 & -32.174 \\ -0.0013 & -0.6070 & 1.0 & 0 \\ 0 & -2.0733 & -0.5073 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation is determined from the determinant of $[\lambda I_4 - A]\ set{=}0$.

$$[\lambda I_4 - A] = \begin{bmatrix} \lambda+0.0231 & -11.8434 & 0 & 32.174 \\ 0.0013 & \lambda+0.6070 & -1.0 & 0 \\ 0 & 2.0733 & \lambda+0.5073 & 0 \\ 0 & 0 & -1 & \lambda \end{bmatrix}$$

The determinant of this matrix will give a fourth order polynomial in λ, which, when set equal to zero is the characteristic equation for this system:

$$\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e = 0$$

Consequently there are four routes to this equation. There can be 4 real roots, 2 real roots and a complex conjugate pair, or two complex conjugate pairs. Real roots indicate the motion is like a first order system, while complex conjugate pairs indicate an oscillatory motion.

The solution to this 4th order polynomial, in this case turns out to be two complex conjugate pairs:

$$\lambda_{1,2} = -0.0046 \pm i\,0.1910$$
$$\lambda_{3,4} = -0.5641 \pm i\,1.4343$$

Hence the longitudinal motions (in the variables, V, α, q, and θ) are oscillatory. There two "modes" of motion, each one having the properties associated with one of the eigenvalue pairs. For example the motion associated with these values of λ have the following properties:

$\lambda_{1,2}$:

$$t_{1/2} = \frac{\ln 2}{0.0046} = 150.7 \text{ sec} \qquad T_p = \frac{2\,\pi}{0.1910} = 32.90 \text{ sec} \qquad \tau = \frac{1}{0.0046} = 217.2 \text{ sec}$$

$\lambda_{3,4}$

$$t_{1/2} = \frac{\ln 2}{0.5650} = 1.23 \text{ sec} \qquad T_p = \frac{2\,\pi}{1.4343} = 4.38 \text{ sec} \qquad \tau = \frac{1}{0.564} = 1.77 \text{ sec}$$

The first motion is referred to as the long period motion or the "***Phugoid***" mode of motion, while the second motion is referred to as the ***short period*** mode of motion. Hence in the usual case, the longitudinal motion consists of two oscillatory modes, the short period and the phugoid. We can now as the question as to how much each variable participates in the motion.

Eigenvectors

The answer to how much each variable participates in each motion is obtained by solving for the eigenvectors associated with each characteristic (or eigen) value. Actually, all we are doing is solving for the values of the C_i that appear in Eq. (7). Lets repeat what we are trying to do. We started with the set of equations that can be concisely written as in Eq. (10):

$$\Delta \dot{x} = A\,\Delta x \qquad (49)$$

where *x* is written as Δx to remind us that we are dealing with small changes away from the reference flight condition.

Then we assumed a solution where $\Delta x = C\,e^{\lambda t}$, where $C = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$, a constant associated with each of the states, $\Delta x = \begin{Bmatrix} V \\ \alpha \\ q \\ \theta \end{Bmatrix}$. If substitute this solution into the original equation, Eq.(49), divide through by the scalars $e^{\lambda t}$, and rearrange, the resulting algebraic equation can be put in the form:

$$\left[\lambda I_4 - A \right] \{ C \} = \{ 0 \} \tag{50}$$

Equation (50) is the generalization of Eq. (7) to (in this case) four variables (or dimensions). The only way we can solve for *C*, is for the determinant of the coefficient matrix to be zero:

$$\left| \left[\lambda I_4 - A \right] \right| = | \lambda I_4 - A | = 0 \tag{51}$$

Equation (51) is the characteristic equation for this system and its roots are the characteristic values or eigenvalues of the system. If, indeed, we set λ in Eq. (50) equal to one of these characteristic values, we should then be able to solve Eq. (50) for unique values of C_i, i=1,2,3, 4. These values form the four components of the eigenvector associated with that particular eigenvalue. Since there are, in this case, four eigenvalues, there are four different eigenvectors. Details on how to obtain these eigenvectors will not be developed here. Suffice it to say that for all practical purposes, every square matrix (*A*), has a set of eigenvalues and corresponding eigenvectors. It's the eigenvectors that tell us the participation of each state variable in a particlar motion.

If we are given a square matrix, then the MATLAB function [U, D} = eig(A) will provide the eigenvalues (the diagonal elements of D) and the corresponding eigenvectors (the columns of U). We can note that if the eigenvalue is real, the corresponding eigenvector will be real. If the eigenvalue is complex, the corresponding eigenvalue will be complex. Interpreting the complex component of velocity, angle of attack, pitch rate, and pitch angle requires some thought. The easiest way to discuss this idea is with an example.

For our example we had the phugoid mode with the eigenvalue and eigenvector given by (from MATLAB}:

$$\lambda_1 = -0.0046 \pm i\,0.1910$$

$$\vec{U}_1 = \begin{matrix} -1.0 & + & & = 1.0 & @ & 180\,° \\ 0.0004 & + & i\,0.0059 & = 0.0059 & @ & 86.12\,° \\ -0.0011 & + & i\,0.0001 & = 0.0011 & @ & 174.80\,° \\ 0.0007 & + & i\,0.0060 & = 0.00604 & @ & 83.34\,° \end{matrix} \qquad \vec{U} = \begin{Bmatrix} \Delta V \\ \Delta \alpha \\ \Delta q \\ \Delta \theta \end{Bmatrix}$$

Note that λ_2 and $\vec{U}_2$ are just the complex conjugates of the above and represent the same motion. SO we only need to consider one of the roots.

and the short period mode with the eigenvalue and eigenvector given by:

$$\lambda_3 = -0.5641 \pm i\,1.4343$$

$$\vec{U}_3 = \begin{matrix} 0.9908 & & = 0.9908 \;@\; 0 \text{ deg } (0 \text{ rad}) \\ -0.0244 - i\,0.0188 & & = 0.0308 \;@\; 217.61 \text{ deg } (3.798 \text{ rad}) \\ 0.0617 + i\,0.0915 & & = 0.1104 \;@\; 17.54 \text{ deg } (0.306 \text{ rad}) \\ 0.0406 - i\,0.0590 & & = 0.0716 \;@\; 304.53 \text{ deg } (5.315 \text{ rad}) \end{matrix} \qquad \vec{U} = \begin{Bmatrix} \Delta V \\ \Delta \alpha \\ \Delta q \\ \Delta \theta \end{Bmatrix}$$

Again, λ_4 and $\vec{U}_4$ are just the complex conjugates of the above and represent the same motion, so we only have to consider one of the roots.

To interpret these eigenvectors, it is convenient to look at the phase angle form of the complex numbers. Here we have a magnitude and phase angle and can think of each component of the eigenvector in itself a vector with a magnitude and direction. If we recall that our original equations had dimensions, we can think of the magnitudes of each component as the participation of that state in the motion.

The short period mode:

The third (and fourth) root is the short period motion. Here (from the eigenvalue) we see that the variables will oscillate with a period of 4.381 sec. Further if the amplitude of the velocity oscillation is 0.9908 ft/sec. Then the amplitude of the angle of attack will be 0.0308 rad (1.76 deg), the amplitude of the pitch rate is 0.1104 rad/sec (6.32 deg/sec) and the ptich angle amplitude is 0.0716 rad (4.10 deg). Hence we have a (relatively) large participation in the pitch angle and pitch rate, significant participation in angle of attack, and little (1 ft/sec compared to the reference velocity off 223 ft/sec). Hence this motion is primarily a pitching motion about the cg. Although all the variables are oscillating at the same frequency, they all do not reach their maximums or minimums at the same time. The difference in time when these maximums occur is (loosely) called the phase. Recalling material from vibrations and control, we can write the oscillation of a variable as

$$\Delta x = C\cos(\omega t + \phi) \tag{52}$$

where C is the amplitude of the motion, and ϕ is the phase angle of the motion. Here we can identify C with the amplitude of a component of the eigenvector, and ϕ with the argument of the component of the eigenvector. For our short period example we can write:

$$\Delta V = 0.9908 \cos (1.4343 \cdot t)$$
$$\Delta \alpha = 0.0303 \cos (1.4343 \cdot t + 3.798)$$
$$\Delta q = 0.1104 \cos (1.4343 \cdot t + 0.306)$$
$$\Delta \theta = 0.0716 \cos (1.4343 t + 5.315)$$

These values must be multiplied by the exponential $e^{nt} = e^{-0.5641 \cdot t}$ to make the amplitude decrease in time.

All these results can be summarized in a simple diagram in the complex plane. Just plot the components of the eigenvector in the complex plane and draw a vector from the origin to each point plotted (four vectors, one for each component) Then consider all of these vectors to be rotating at the same (positive) angular rate ω (here = 1.4343 rad/sec) and shrinking according to the exponential $e^{n \cdot t}$. The projection of these rotating vectors on the real axis represents the oscillatory motion of each state. The relative amplitudes of the oscillations are given by the amplitudes of the eigenvector components. The angles between them are called the phase angles. In our example, all phase angles are measured with respect to the velocity vector (MATLAB'S choice). We can say for example, that th angle of attack leads the velocity by 217.61 degrees, The pitch rate leads the velocity by 17.54 degrees and lags the angle of attack by 200.07 degrees, etc.

The Phugoid Mode

The phugoid mode can be examined in the same fashion. In particular, we can say that all the states will oscillate with a period of 32.90 seconds. The amplitudes of the oscillations are given by the magnitudes of the components of the eigenvector. Again, recalling that these are dimensional equations, we see that the deviations of the states from their reference conditions are: velocity 1.0 ft/sec, angle of attack, 0.33 degrees, pitch rate, 0.063 deg/sec, and for the pitch angle, 0.346 deg. The numbers here are significantly smaller than those for the short period motion. We can conclude that there is little pitch motion and that velocity is a dominant contributor to this motion. We can also discuss the phase lead or lag of the various components with each other. For example the angle of attack leads the velocity by 86.12 degrees and the pitch angle by 2.78 degrees, etc.

Further study (not done here) indicates that the phugoid mode is close to a constant energy motion, with an interchange between kinetic and potential energy. Consider a vehicle eith the controls set to balance a 223 ft/sec as in our example aircraft. Now suppose nothing was changed except that the speed was lowered slightly (same angle of attack). Then the lift would be less than the weight and the aircraft would drop. As it does so the speed will pick up (potential to kinetic energy) and eventually, the speed will increase enough to make the lift greater than the weight and the aircraft will start to climb. As it climbs, it loses speed and hence lift so it falls again and the cycle is repeated. Since the motion is lightly damped, this oscillatory motion can continue for several minutes before it damps out. Here for example, the time to half amplitude is 150 secondss or well over two minutes. Four time constants would be about 14 minutes!

Lateral-Directional Motions

The lateral-directional equations are developed in a similar manner, although the calculations are slightly more difficult because the derivatives of more than one state variable appear in the same equation. After considerable effort, the lateral-directional matrix for the state variables: $x = [\Delta\beta,\ \Delta p,\ \Delta r,\ \Delta\phi\,]^T$ is given by:

$$A = \begin{bmatrix} \dfrac{Y_\beta}{m} & \dfrac{Y_p}{mV} + \sin\alpha & \dfrac{Y_r}{mV} - \cos\alpha & \dfrac{g}{V}\cos\theta \\ \dfrac{1}{I}[I_z L_\beta + I_{xz} N_\beta] & \dfrac{1}{I}[I_z L_p + I_{xz} N_p] & \dfrac{1}{I}[I_z L_r + I_{xz} N_r] & 0 \\ \dfrac{1}{I}[I_{xz} L_\beta + I_x N_\beta] & \dfrac{1}{I}[I_{xz} L_p + I_x N_\beta] & \dfrac{1}{I}[I_{xz} L_r + I_x N_r] & 0 \\ 0 & 1 & \tan\theta & 0 \end{bmatrix} \tag{53}$$

For the light aircraft called the Navion, we can fill in the numbers to get:

$$A = \begin{bmatrix} -0.2545 & 0 & -1 & -0.1823 \\ -16.0472 & -8.4172 & 2.1967 & 0 \\ 4.5710 & -0.3505 & -0.7618 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The eigenvalues and eigenvectors are given by:

<u>Rolling Convergence</u>

$\lambda_1 = -8.4498$

$$\vec{U}_1 = \begin{Bmatrix} -0.0076 \\ -1.0 \\ -0.0410 \\ 0.1183 \end{Bmatrix} \qquad \vec{U} = \begin{Bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \end{Bmatrix}$$

Hence the motion is a pure subsidence with a time to half amplitude of $t_{1/2} = 0.082$ seconds. Further more the participation of each state variable in the mode is for each rad/sec of roll rate (p = 1 rad/sec , 57.3 deg/sec), the sideslip angle is 0.44 degrees, yaw rate, r = 2.34 deg/second, and the roll angle is 6.78 degrees. It is clear that the main motion associated with this motion is roll rate. This mode is called the rolling convergence mode of motion. It justifies that the pure roll approximation we used earlier is a good approximation.

The Spiral Mode

$$\lambda_2 = -0.0082$$

$$\vec{U}_2 = \begin{Bmatrix} 0.0283 \\ -0.0082 \\ 0.1754 \\ 1.0 \end{Bmatrix} \qquad \vec{U} = \begin{Bmatrix} \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \end{Bmatrix}$$

Here it is clear that the roll angle exceeds the sideslip angle and the yaw rate exceeds the roll rate. Furthermore the roll and yaw are in the same direction. Remember, all the magnitudes are shrinking since the mode is stable. The result is a banked turn that has little sideslip. This mode is called the spiral mode and in many aircraft it is unstable. However since it takes so long to half (or double) amplitude, an unstable vehicle is not a problem. Here the time to half amplitude is 84.5 sec.

The Dutch Roll Mode

$$\lambda_3 = -0.4879 + i\,2.3516 \qquad (\lambda_4 = -0.4879 - i\,2.3516)$$

$$\vec{V}_3 = \begin{Bmatrix} -0.1084 + i\,0.4370 = 0.4503 \ @ \ 103.93 \text{ deg} \\ 0.2255 - i\,0.8534 = 0.8827 \ @ \ -75.2 \text{ deg} \\ 0.9355 + i\,0.3533 = 1.0 \ @ \ 20.69 \text{ deg} \\ -0.3670 - i\,0.0197 = 0.3675 \ @ \ -176.92 \text{ deg} \end{Bmatrix}$$

Here we have an oscillatory motion with the a large yaw rate and an almost as large a roll rate. Further the magnitudes of the sideslip angle and roll angle are nearly the same, with the sideslip angle slightly larger. This mode then is yaw mode in beta and yaw rate (similar to our pinned aircraft) but there is also a significant amount of roll oscillation in the roll rate and roll angle. Hence our pinned approximation is not too good for this mode. We see that the sideslip angle leads the roll angle by 79.15 degrees which says that when the sideslip angle is at max deflection, the roll angle is near zero deflection. Similarly, the yaw angle leads the roll angle by 95.89 degrees and a similar statement can be made regarding the max yaw rate and roll rate. This mode is a roll-yaw oscillation and is called the Dutch Roll.