

16. Turning Flight

One of the more interesting aspects of performance is turning flight. In aerial combat, the victor is often the aircraft that has the smallest turning radius or the highest turn rate. Here we will consider level turning flight, with the key word being level. Level flight requires the forces in the vertical direction to balance. As we will see shortly, this requirement puts a limit on our turning performance. If we draw the free-body diagram we can sum the forces in the vertical and horizontal (radial) directions.

The vertical force balance gives:

$$L \cos \phi = W \quad (1)$$

And the horizontal (radial) force balance gives us:

$$L \sin \phi = \frac{W}{g} a_r \quad (2)$$

where a_r = the radial acceleration. We can determine a few different expressions for the acceleration by looking at a top view of the horizontal turn. From kinematics, we have

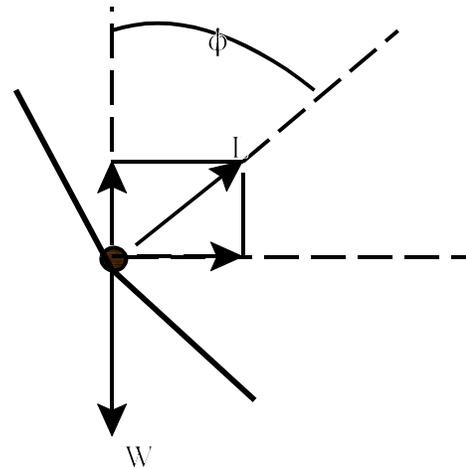
$$V = R \dot{\psi} \quad (3)$$

We can now write the radial acceleration in several forms:

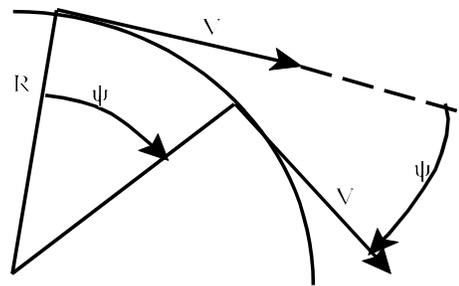
$$a_r = \frac{V^2}{R} = V \dot{\psi} \quad (4)$$

Hence Eq. (2) becomes:

$$L \sin \phi = \frac{W}{g} \frac{V^2}{R} = \frac{W}{g} V \dot{\psi} \quad (5)$$



Rear View of Turn



Top View of Turn

We can now rearrange Eq.(5) to determine either the radius of the turn or the turn rate. However, before doing that we need to introduce a definition.

Definition: Load Factor

The load factor (sometimes called the normal load factor) is defined as the lift divided by the weight.

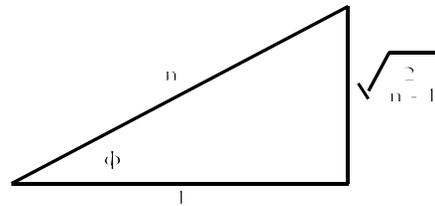
$$n \equiv \frac{L}{W} \tag{6}$$

Then from the vertical force balance, Eq. (1) we can write

$$\frac{L}{W} = n = \frac{1}{\cos \phi} \quad \text{or} \quad \cos \phi = \frac{1}{n} \tag{7}$$

Therefore a 60 degree banked turn requires a load factor of 2, often called a “2 g” turn. Level flight is a “1 g” flight condition. If the load limit on a vehicle is 6 g, then the steepest banked turn allowable would be found from $\cos \phi = 1/6$, or $\phi = 80.4$ deg.

We can now use a common trick to determine other trig functions of the bank angle. Form a triangle with the known trig function assigned to the appropriate sides. Here 1 and n to give us $\cos \phi$. Then calculate the third side using the pathagorian theorem. We can now form the remaining trig functions:



$$\sin \phi = \frac{\sqrt{n^2 - 1}}{n} \tag{8}$$

$$\tan \phi = \sqrt{n^2 - 1}$$

Turn Radius

We can rearrange Eq. (5) to solve for the range.

$$\begin{aligned} R &= \frac{W V^2}{g L \sin \phi} = \frac{V^2}{g} \frac{1}{n} \frac{n}{\sqrt{n^2 - 1}} \\ &= \frac{V^2}{g} \frac{1}{\sqrt{n^2 - 1}} \end{aligned} \tag{9}$$

We can remove airspeed from the equation using the lift coefficient to get

$$V = \sqrt{\frac{nW}{1/2 \rho S C_L}} = \sqrt{\frac{W}{1/2 \rho S C_L}} \sqrt{n} \quad (10)$$

Note that lift does not equal the weight since we are not in straight and level flight. Here, $L = nW$. In any case, we may now write an expression for the turn radius:

$$R = \frac{1}{1/2 \rho g C_L} \left(\frac{W}{S} \right) \left(\frac{n}{\sqrt{n^2 - 1}} \right) \quad (11)$$

From Eq. (11) it is clear that if we want have a low turn radius, that we would expect to need

1. Large C_L (C_{Lmax})
2. Low altitude (high density)
3. High load factor n (n_{max})
4. Low wing loading (W/S)

Turn Rate

We can establish an expression for the turn rate from the same Eq. (5).

$$\begin{aligned} \dot{\psi} &= \frac{g}{V} \frac{L}{W} \sin \phi = \frac{g}{V} n \frac{\sqrt{n^2 - 1}}{n} \\ &= \frac{g}{V} \sqrt{n^2 - 1} \end{aligned} \quad (12)$$

If we eliminate the airspeed as we did previously, using Eq. (10), we obtain the expression for turn rate:

$$\dot{\psi} = g \sqrt{\frac{1/2 \rho S C_L}{W}} \sqrt{\frac{n^2 - 1}{n}} \quad (13)$$

For a high turn rate, it is clear we require:

1. Large C_L
2. Low altitude (high density)
3. High load factor
4. Low wing loading

Instantaneous and Sustained Turn Rates

From the above equation for turn rate it seems clear that the maximum turn rate can occur when the lift coefficient has its maximum value and the load factor has its maximum value. In general, one can think of the load factor as a structural limit, although as we will see later, it also has an aerodynamic limit. For now, let us consider the limiting load factor as that below which the structure will fail, typically $n = 6$ for civil aircraft and $n = 10$ or 12 for military aircraft. A turn rate can be sustained if we can maintain the airspeed at which it occurs. For this we require thrust to equal the drag. Hence the thrust available must be capable of equaling the thrust required (drag). A simple example will illustrate all the ideas presented above.

Example

Consider an aircraft with the following properties: $W/S = 59.88$ lbs/ft². $W = 10,000$ lbs, $S = 167$ ft², $C_{L_{max}} = 1.5$, $n_{max} = 6$, $C_D = 0.018 + 0.064 C_L^2$, and $T_{max} = 5000$ lbs. Find the extreme turn rate and turn radius, and the speed at which they occur. Are these sustainable turn rates?

First, find the maximum turn rate by assuming maximum lift coefficient and maximum load factor are occurring at the same time. For turning rate we have:

$$\begin{aligned}\dot{\psi} &= g \sqrt{\frac{1/2 \rho S C_{L_{max}}}{W}} \sqrt{\frac{n_{max}^2 - 1}{n_{max}}} \\ &= 32.174 \sqrt{\frac{1/2 (0.002377) 167 (1.5)}{10000}} \sqrt{\frac{6^2 - 1}{6}} \\ &= 0.424 \text{ rad/sec} = 24.3 \text{ }^\circ/\text{sec}\end{aligned}$$

The airspeed for the maximum turn rate is given by:

$$V = \sqrt{\frac{n W}{1/2 \rho S C_{L_{max}}}} = \sqrt{\frac{6 (10000)}{1/2 (0.002377) 167 (1.5)}} = 448.6 \text{ ft/sec}$$

The corresponding minimum turning radius is given by:

$$R_{min} = \frac{W}{1/2 \rho g S C_{L_{max}}} \frac{n_{max}}{\sqrt{n_{max}^2 - 1}} = \frac{10000}{1/2 (0.002377) 32.174 (167) 1.5} \frac{6}{\sqrt{6^2 - 1}} = 1058 \text{ ft}$$

The question now arises if we can sustain this turn rate. Is the thrust large enough to overcome the drag at this turn rate? We can calculate the drag coefficient from:

$$C_D = 0.018 + 0.64 C_{L_{\max}}^2 = 0.018 + 0.064 (1.5^2) = 0.162$$

Then the thrust required (drag) is calculated from

$$T = D = C_D \frac{1}{2} \rho V^2 S = 0.162 (1/2) 0.002377 (446.6^2) 167 = 6479 \text{ lbs}$$

The thrust available is given as 5000 lbs. Hence it is not possible to sustain this turn rate since the thrust available is less than the thrust required.

The V-n Diagram

There are two primary uses for a V-n diagram. One for turning flight, and one for straight and level flight. The diagrams are the same, but the information obtained is different, or at least has a different interpretation.

The level turn V-n diagram consists of showing both the aerodynamically limited load factor and the structurally limited load factor on the same graph, as a function of airspeed. To determine the aerodynamically limited load factor we can consider the following:

In level flight we have the definition of stall speed:

$$L = W = C_{L_{\max}} \frac{1}{2} \rho V_{stall}^2 S \quad (14)$$

In turning flight we can define the maximum (aerodynamically limited) load factor by the following equation for any V:

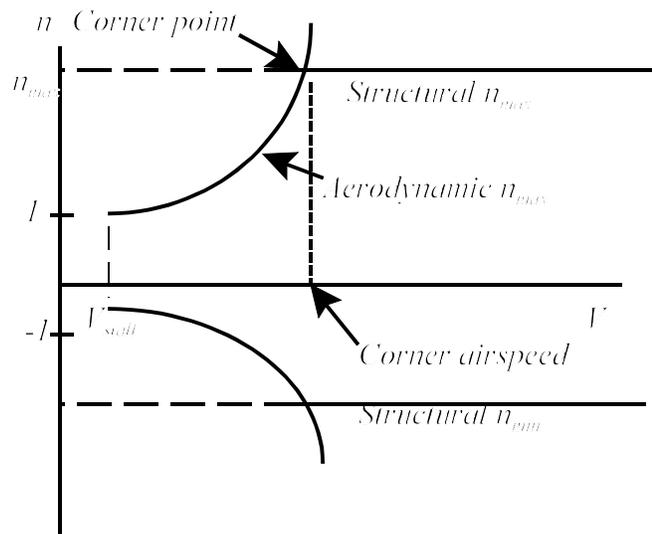
$$L = n_{\max_{aero}} W = C_{L_{\max}} \frac{1}{2} \rho V^2 S \quad (15)$$

If we divide Eq. (15) by Eq.(14), we get:

$$n_{\max_{aero}} = \frac{V^2}{V_{stall}^2} \quad (16)$$

We can now construct our level turn V-n diagram.

On the graph to the right we see both the maximum (positive) structurally limited load factor, and the minimum (negative) structurally limited load factor. These are typically different in magnitude with six being a reasonable positive maximum load factor and two or three being a reasonable minimum (negative) load factor. The other curved lines show the aerodynamically limited load factor due to stall. At any give speed on these curved lines, the load factor cannot exceed the aerodynamic limit that can be considerably less than the structural limit. For example, in level flight, the maximum load factor that can be achieved is one at stall speed.



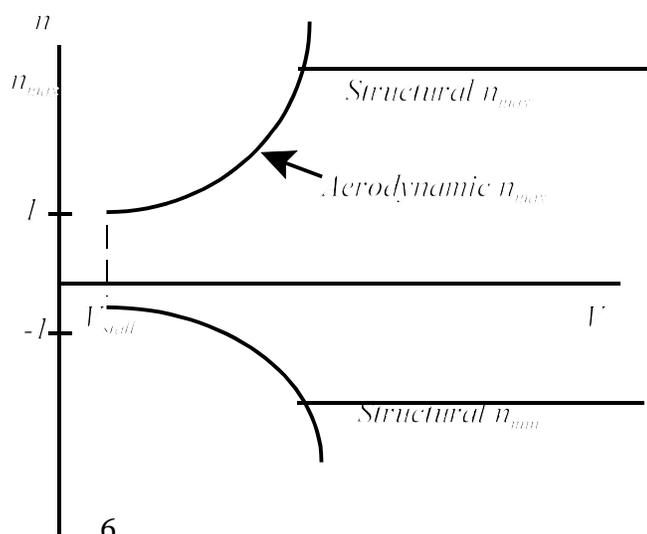
If we increase our speed slightly, we can do a turn that will generate a load factor greater than one. However if we try and turn too steep, the required load factor will exceed the stall lift coefficient and we will be unable to achieve our turn. Hence we will be restricted again by the maximum lift coefficient. If we continue to increase our speed, we will eventually be able to generate a load factor (in a turn) that will equal the maximum allowable structural load factor. The flight condition on the V-n diagram at which this occurs is called the **corner point**. The associated airspeed is called the **corner velocity**. At the corner point we can obtain the max instantaneous turn rate and the minimum turn radius. It will be sustainable only if we have the thrust available to balance the drag at that load factor and speed.

We can also generate the same V-n diagram and explore it from the point of view of level flight. The curves are the same, but we now are looking at the effect on the loads due to wind gusts. The effect of a vertical gust is to increase the angle-of-attack. The change can be estimated if we assume small angles to be:

$$\Delta \alpha = \frac{W_g}{V} \quad (17)$$

where W_g is the vertical wind gust. The change in the lift coefficient is then given by:

$$\Delta C_L = a \alpha = a \frac{W_g}{V} \quad (18)$$



where a is the lift-curve slope.

The change in the lift is given by:

$$\Delta L = \Delta C_L \frac{1}{2} \rho V^2 S = a \frac{1}{2} \rho V^2 S \frac{W_g}{V}$$

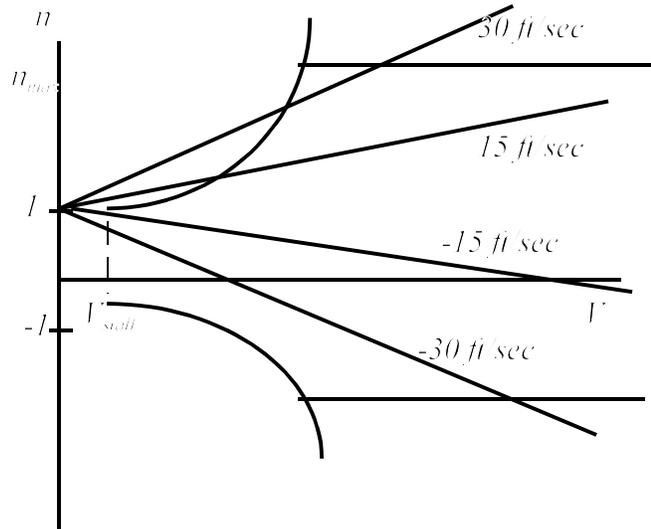
The change in the load factor is then given by:

$$\Delta n = \frac{\Delta L}{W} = a \frac{1}{2} \frac{\rho S W_g}{W} V \quad (19)$$

The total load factor caused by a gust is then determined by adding the increment given by Eq. (19) to the level-flight load factor of 1.

$$n = n_0 + \Delta n = 1 + a \frac{1}{2} \frac{\rho S W_g}{W} V \quad (20)$$

Equation (20) is that of a straight line with respect to airspeed and can be plotted on the V-n diagram. These straight lines indicate the load factor encountered by the aircraft due to the gusts. When the line intersects the maximum (or minimum) load factor limit, the corresponding airspeed is the maximum one that should be flown in gusts of that magnitude. Otherwise, at airspeeds greater than that speed where the line intersects the max load factor line, a gust of the specified magnitude would exceed the structural load factor and that can't be good. On the other hand, if one is flying slower than the "corner speed" then any gust will cause an angle-of-attack to exceed the stall angle-of-attack and cause the aircraft to stall, but not break! In this context, the "corner speed" is known as the **maneuvering speed**. If the aircraft is flown at or below maneuvering speed, then neither a gust or sudden control movement will cause the aircraft to exceed its maximum load factor. Consequently, when encountering turbulent air, the aircraft should be slowed to or below maneuvering speed to prevent "bending" the aircraft.



Sustained Turning Rate

Previously, we indicated that the sustained turn rate can be different from instantaneous maximum turn rate if the thrust available is not greater than or equal to the thrust required. In this section we will impose the requirement that thrust equals drag on the turn rate and determine the airspeed required for the maximum sustained turn rate. Hence we seek to maximize

$$\dot{\psi} = \frac{g}{V} \sqrt{n^2 - 1} \quad (21)$$

subject to the constraint:

$$T = D = C_{D_{0L}} \bar{q} S + \frac{K n^2 W^2}{\bar{q} S} \quad (22)$$

where $\bar{q} = 1/2 \rho V^2$. The assumption we will make here is that the thrust, T, is independent of airspeed. Our plan of attack is to write all equations in terms of the dynamic pressure and to find the dynamic pressure that minimizes the turn rate, subject to the constraint, T = D. To this end, we can rewrite Eq (21) in the form:

$$\dot{\psi}^2 = \frac{g^2 (n^2 - 1)}{V^2} = \frac{1/2 \rho g^2 (n^2 - 1)}{\bar{q}} \quad (23)$$

We can differentiate this equation implicitly with respect to \bar{q} , to get:

$$2 \dot{\psi} \frac{\partial \dot{\psi}}{\partial \bar{q}} = \frac{\bar{q} \left[1/2 \rho g^2 2 n \frac{\partial n}{\partial \bar{q}} \right] - 1/2 \rho g^2 (n^2 - 1)}{\bar{q}^2} = 0$$

or

$$\bar{q} 2 n \frac{\partial n}{\partial \bar{q}} - (n^2 - 1) = 0 \quad (24)$$

We can no use the constraint equation, Eq. (22) to help evaluate the derivative in Eq. (24). In particular, we can solve the T = D equation for n^2 ,

$$n^2 = \frac{T \bar{q} S}{K W^2} - \frac{C_{D_{0L}} \bar{q}^2 S^2}{K W^2} \quad (25)$$

We can take the derivative with respect to \bar{q} to get:

$$2n \frac{\partial n}{\partial \bar{q}} = \frac{TS}{KW^2} - \frac{2C_{D_{0L}} \bar{q} S^2}{KW^2} \quad (26)$$

this equation can be substituted into the left hand side of Eq. (24) to give:

$$\bar{q} \left(\frac{TS}{KW^2} - \frac{2C_{D_{0L}} \bar{q} S^2}{KW^2} \right) - \left[\left(\frac{T\bar{q}S}{KW^2} - \frac{2C_{D_{0L}} \bar{q}^2 S^2}{KW^2} \right) - 1 \right] \quad (27)$$

The terms containing thrust cancel out! This leaves us with the remaining terms:

$$- \frac{C_{D_{0L}} \bar{q}^2 S^2}{KW^2} + 1 = 0$$

or

Dynamic Pressure for Maximum Sustained Turn Rate (no constraint on $C_{L_{max}}$ or n_{max})

$$\bar{q} = \frac{W}{S} \sqrt{\frac{K}{C_{D_{0L}}}} \quad (28)$$

Note that this dynamic pressure corresponds to the level-flight minimum drag speed. However Eq.(28) was derived with only a constraint on the $T = D$ and it does not guarantee that the structural limit, n_{max} , or that the aerodynamic limit, $C_{L_{max}}$ are violated.

If $C_{L_{max}}$ is violated, we need to calculate the sustained turning rate with that constraint as well as the $T = D$ constraint. Under these conditions we can combine Eqs. (16 and 25) to get

$$n^2 = \frac{T\bar{q}S}{KW^2} - \frac{C_{D_{0L}} \bar{q}^2 S^2}{KW^2} = \frac{V^4}{V_{stall}^4} \quad (29)$$

Equation (29) can be solved for V, the airspeed for the turn at $C_{L_{max}}$ with T = D:

Airspeed for max turn rate with $C_{L_{max}}$ and T = D constraints active

$$V^2 = \frac{T \frac{1}{2} \rho S}{C_{D_{0L}} (1/2 \rho S)^2 + \frac{K W^2}{V_{stall}^4}} = \frac{T}{C_{D_{0L}} + K C_{L_{max}}^2} \quad (30)$$

or

$$T = (C_{D_{0L}} + K C_{L_{max}}^2) \frac{1}{2} \rho S V^2$$

The final case to consider is that where the turn rate is limited by the structural load limit, n_{max} . Here we must satisfy the constraints $n = n_{max}$ and T = D. We can simply take Eq. (22) and set $n = n_{max}$ and solve for the dynamic pressure or airspeed at which the minimum radius turn would occur with these constraints. The result is a quadratic equation in the dynamic pressure, \bar{q} .

Dynamic Pressure (or V) for max turn rate with n_{max} and T = D constraints active

$$C_{D_{0L}} S^2 \bar{q}^2 - T S \bar{q} + K n_{max}^2 W^2 = 0 \quad (31)$$

Note that it is possible that there is no solution to Eq. (31). In that case, the thrust available is not sufficient to balance the drag at the maximum load factor. In general, the conditions at maximum C_L generally yield the highest turn rate if the unconstrained result, Eq. (28) is not valid.

Example

We can continue our example from the previous problem. The dynamic pressure and airspeed can be determined from Eq. (28)

$$\bar{q} = \frac{W}{S} \sqrt{\frac{K}{C_{D_{0L}}}} = \frac{10000}{167} \sqrt{\frac{0.064}{0.018}} = 112.91 \text{ lbs.ft}^2$$

$$V = \sqrt{\frac{2 \bar{q}}{\rho}} = \sqrt{\frac{2 (112.91)}{0.002377}} = 308.22 \text{ ft/sec}$$

We can calculate the load factor from:

$$n^2 = \frac{T \bar{q} S}{K W^2} - \frac{C_{D_{0L}} \bar{q}^2 S^2}{K W^2} = \frac{5000 (112.91^2) 167 - 0.018 (112.91^2) 167^2}{0.064 (10000^2)} = 13.73$$

$$n = 3.705$$

However, lets check the lift coefficient:

$$C_L = \frac{n W}{\bar{q} S} = \frac{3.705 (10000)}{112.91 (167)} = 1.965 > 1.5 \quad \text{Hence not a viable solution.}$$

Should it have been ok, then

$$\dot{\psi} = \frac{g}{V} \sqrt{n^2 - 1} = \frac{32.174}{308.22} \sqrt{3.705^2 - 1} = 0.372 \text{ rad/sec} = 21.31 \text{ deg/sec}$$

Lets now assume that the max turn rate occurs at $C_{L_{\max}}$. Then we can determine the drag from

$$T = D = \left(C_{D_{0L}} + K C_{L_{\max}}^2 \right) \bar{q} S \quad \text{and}$$

$$\bar{q} = \frac{1}{S} \frac{T_{\max}}{C_{D_{0L}} + K C_{L_{\max}}^2} = \frac{1}{167} \frac{5000}{0.018 + 0.064 (1.5^2)} = 184.82 \text{ lbs/ft}^2$$

$$V = \sqrt{\frac{2 \bar{q}}{\rho}} = \sqrt{\frac{2 (184.82)}{0.002377}} = 394.34 \text{ ft/sec}$$

$$n = \frac{C_L \bar{q} S}{W} = \frac{1.5 (184.82) 167}{10000} = 4.63 = \left(\frac{V}{V_{\text{stall}}} \right)^2$$

$$\dot{\psi} = \frac{g}{V} \sqrt{n^2 - 1} = \frac{32.174}{394.34} \sqrt{4.63^2 - 1} = 0.369 \text{ rad/sec} = 21.13 \text{ deg/sec}$$

We can also see what would happen if we used the maximum structural load factor. Here we need to determine the dynamic pressure that would allow the T = D constraint to be satisfied. Hence we have

$$T = C_{D_{0L}} S \bar{q} + \frac{K n_{\max}^2 W^2}{S \bar{q}} \quad \Rightarrow \quad C_{D_{0L}} S^2 \bar{q}^2 - T_{\max} S \bar{q} + K n^2 W^2 = 0$$

or

$$0.018 (167^2) \bar{q}^2 - 5000 (167) \bar{q} + 0.064 (6^2) 10000^2 = 0$$

$$502.0 \bar{q}^2 - 83500 \bar{q} + 2.304 \times 10^8 = 0 \quad \bar{q} = 349.2, 1314 \text{ lbs/ft}^2$$

$$V = \sqrt{\frac{2 \bar{q}}{\rho}} = \sqrt{\frac{2 (349.2)}{0.002377}} = 542.0 \text{ ft/sec}$$

$$\dot{\psi} = \frac{g}{V} \sqrt{n^2 - 1} = \frac{32.174}{542.0} \sqrt{6^2 - 1} = 0.351 \text{ rad/sec} = 20.11 \text{ deg/sec}$$

Hence for this aircraft, the maximum sustainable turn rate occurs at the maximum C_L , where thrust available equals drag. This result is generally true if the corner turn rate is not sustainable. In that case the highest turn rate occurs by using $C_{L_{\max}}$.
