

## Performance

### 10. Thrust Models

In order to determine the maximum speed at which an aircraft can fly at any given altitude, we must solve the simple-looking equations for  $V_{\max}$ :

$$\begin{aligned} T &= D \\ L &= W \end{aligned} \quad (1)$$

We have previously developed a model for the drag assuming a parabolic drag polar. We now need to develop a mathematical model for the thrust. Typically there are two significantly different types of power plants that are used in aircraft. One utilizes a propeller that is driven by either a piston engine or a gas turbine engine (turboprop), and the other uses the exhaust gases of a gas turbine directly and is called a turbojet. Each of these engines develops thrust in a different manner that is affected by altitude (density and temperature), and speed. A mathematical model must include these effects on the thrust.

#### *Propeller Power Plant*

Both a piston driven propeller and a turbine driven propeller propulsive engine can be modeled using the same expression.

$$T = SP_{SL} \left( A_p \frac{\rho}{\rho_{SL}} - B_p \right) \frac{\eta_p}{V_\infty} \quad (2)$$

where:	T	=	thrust (lbs, N)
	SP	=	shaft power (equivalent shaft power for turboprop)(ft lb/sec. Watts)
	$\rho$	=	density (slugs/ft <sup>3</sup> , kg/m <sup>3</sup> )
	$\eta_p$	=	propeller efficiency factor
	$V_\infty$	=	speed of aircraft (free stream airspeed) (ft/sec. m/s)
	$(\cdot)_{SL}$	=	sea-level value
	$A_p$	=	a constant, typically = 1.132 (But often taken as 1.0)
	$B_p$	=	a constant = ( $A_p - 1.0$ ), typically 0.132 (But often taken as 0.0)

In general the shaft power is weakly dependent on the airspeed while the propulsive efficiency is strongly dependent on the airspeed so that the thrust can become a complicated function of airspeed.

From Eq. (2) we can see that at altitude, the total shaft power is linear with density.

However, if the piston engine is supercharged, then the shaft power remains constant up until the “critical” altitude after which it declines linearly with density. The “critical altitude is that altitude above which the supercharger can no longer supply sea-level density air. So generally, the shaft power at the “critical altitude” is the same as the shaft power at sea-level. Consequently the shaft power available from sea-level to the critical altitude is constant at the sea-level value. Above the critical altitude, the shaft power varies linearly with density and its variation can be represented by:

$$SP = \left( \frac{\sigma - D_p}{\sigma_c - D_p} \right) SP_c \quad (3)$$

where:

$SP_c$	=	shaft power at “critical altitude”
$SP$	=	shaft power at altitudes above “critical altitude”
$\sigma$	=	$\rho/\rho_{SL}$
$\sigma_c$	=	$\rho_c/\rho_{SL}$
$(\cdot)_c$	=	evaluated at “critical altitude”
$D_p$	=	a constant, typically 0.117 (But often taken as = 0.0)

### ***Shaft Power***

In an actual piston engine, the shaft power depends on the mean effective pressure in the cylinders, the length of the stroke, the area of the piston, and the rotational speed. When all these factors are combined, the shaft power ends up depending on two parameters, the manifold air pressure (MAP) and the propeller rotational rate (rpm). These are controlled by the pilot in way that depend on the propeller type:

### **Fixed Pitch Propeller**

For a fixed-pitch propeller, the MAP and rpm are tied together so there is only one control, the throttle. The only engine gage is the tachometer (rpm indicator). Consequently the power of the engine is set by moving the throttle until the rpm reads the value you want. Typically at take-off the throttle is pushed all the way in to get maximum rpm. As the aircraft speeds up, so does the propeller rpm so the pilot must be careful not to let the engine “over rev” or exceed the redline on the engine tachometer. As altitude is gained, in order to maintain a given rpm, the throttle must be continually advanced. Once it is fully advanced, any additional increase in altitude will create a decrease in power. If one was to take off at full throttle, and continue to climb at full throttle, the engine rpm would continually decrease during the climb, and the maximum shaft power would be continually decreasing. Typically one climbs out to some specified altitude and sets the engine rpm until about 60-75 % power is set (usually the proper rpm at altitude is specified in the pilot’s operating hand book (POH)).

## **Variable -Pitch Propeller (constant speed)**

In an aircraft equipped with a variable pitch propeller, the pilot has direct control over both the MAP, using the throttle, and the engine rpm, using the propeller pitch setting. Consequently the selection of desired power is slightly more complicated. High MAP and rpm lead to high engine power. However at cruise, 60-70% power there may be several different combinations of MAP (throttle setting) and rpm that would give the same output power. Usually the MAP indicator is in inches of mercury, and the rpm in revolutions per second (neither are “basic” units!). For takeoff, the pilot would set the rpm to some high value, usually just below redline, and advance the throttle to increase the MAP. As the aircraft climbs, the rpm stays constant (the pilot has direct control of it and s/he set it just below redline), but the MAP will drop because of the increase in altitude. Consequently, if the throttle is in the maximum open position, the maximum available power is reduced as the altitude is increased. If the engine is turbocharged, the pilot may not be able to use full throttle at takeoff because it may “over boost” (make the MAP too high) the engine. In this case the pilot will set the throttle to maximum allowable MAP. As the pilot climbs out, s/he can maintain that MAP (and hence power, recall rpm is constant) by continually advancing the throttle. This can be continued until the throttle hits the stop. The altitude at which this occurs is called the “critical altitude” as mentioned previously. Above this altitude, the maximum power must decrease.

One more comment about the throttle and the MAP. If an aircraft is a sea-level, the atmospheric pressure is at 29.92 inches of hg. If the aircraft engine is idling, the throttle valve is closed, and the action of the pistons and valves creates a low pressure in the intake manifold. When the throttle is open, the manifold pressure approaches the atmospheric pressure (it can't get any higher unless turbocharged). Hence an open throttle means close to the atmospheric pressure, and a closed throttle means low pressure. Clearly, as the altitude is increased, the maximum manifold pressure decreases. However, the aircraft is moving through the air, so we must modify what we said slightly. The pressure seen at the intake of the engine may be considered to be related to the stagnation pressure of the aircraft. Hence the maximum MAP is affected slightly by the airspeed. (Hence the weak dependence on airspeed mentioned previously).

## **Shaft Power Math Model**

In general, the shaft power depends on the airspeed, and the throttle setting. The altitude dependence is accounted for in Eqs. (2&3). We will assume (unless otherwise specified) That the shaft power is independent of airspeed and is constant with a given throttle setting. Hence, at any given altitude and throttle setting, the *shaft power will be assumed constant*.

## **Propeller Efficiency Model**

The propeller efficiency depends on many things, the primary one being the *advance ratio* of the propeller given by  $J = \frac{V}{nD}$ , where  $V$  is the free stream airspeed,  $D$  the diameter of the propeller, and  $n$  the revolutions per second of the propeller (not a proper unit!). If we assume

a constant rotational speed, and constant diameter, the efficiency depends on the airspeed. If we assume a constant speed propeller, then the propeller adjusts its pitch angle with speed so that the propeller efficiency is virtually a constant. However, the propeller efficiency at zero airspeed is always zero. Consequently we need a model that will take us from zero efficiency at zero airspeed (advance ratio) to some constant efficiency at some specified airspeed (advance ratio), and constant at higher speeds. One such curve that we could use is a sine curve. For preliminary calculations, we can write the following mathematical model for the propeller efficiency:

$$\eta_p = \eta_{p_{\max}} \sin\left(\frac{\pi}{2} \cdot J\right) \quad J < 1 \quad (4a)$$

and

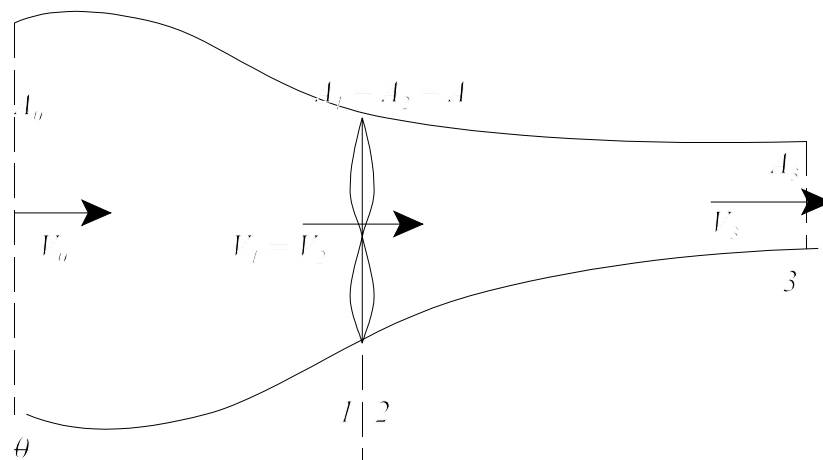
$$\eta_p = \eta_{p_{\max}} \quad J \geq 1 \quad (4b)$$

One should also note that if the speed gets too high (as it might in a dive), then the efficiency can go to zero again. Here we are assuming that we don't reach such a speed in our problems of interest. In fact for the complete range of airspeeds of interest for most of our problems, we will assume a constant propeller efficiency.

In the limit, as the airspeed  $V$  goes to zero, then the thrust in Eq. (2) won't go to infinity. It takes on a finite value. Exactly what value the static thrust is cannot be evaluated from Eq. (2) since it is not a very good model at the low speeds.

### ***Estimating the Static Thrust of a Propeller Power Plant***

We will need to know the static thrust (thrust at zero airspeed) for a propeller driven aircraft in order to estimate the take-off distance. Here we will develop an method that will be optimistic in its estimate. The method will be to apply the momentum theory and Bernoulli's equation for incompressible flow to the flow through the propeller.



The figure above represents a stream tube of all the air that flows through the propeller. If we apply the momentum equation, using the stream tube from point 0 to point 3 as the control volume, we can note that the force on the fluid in the “x” direction is equal to the momentum flux out in the “x” direction. Furthermore, the thrust is the negative of the force on the fluid, and acts in the opposite direction. Putting this all together, we have

$$T = \rho A_3 V_3 (V_3 - V_0) \quad (5)$$

The thrust is also equal to the pressure difference across the propeller times the area swept out by the propeller:

$$T = A (P_2 - P_1) \quad (6)$$

We can now apply Bernoulli’s equation for incompressible (low speed where propellers are used) flow between stations 0 and 1 and between stations 2 and 3, but not across the propeller! (Why?). If we designate the static pressure at station 0 and 3 to be atmospheric pressure,  $P_a$ , we can write:

$$P_a + \frac{1}{2} \rho V_0^2 = P_1 + \frac{1}{2} \rho V_1^2$$

$$P_a + \frac{1}{2} \rho V_3^2 = P_2 + \frac{1}{2} \rho V_2^2$$

If we subtract the first from the second, and note that  $V_1 = V_2$ , we obtain the result:

$$P_2 - P_1 = \frac{1}{2} \rho (V_3^2 - V_0^2) \quad (7)$$

Since the mass flow rate is constant, we can use the relation  $A V_1 = A_3 V_3$  in Eq. (5). Equating Eq. (5) to Eq. (6) with Eq. (7) substituted in for the pressure difference, we can obtain the result:

$$V_1 = \frac{V_0 + V_3}{2} \quad (8)$$

That is, ***the velocity through (at) the propeller is the average of the velocity ahead of the propeller and the velocity behind the propeller.*** Hence half the increment in velocity occurs in front of the propeller, and half of it occurs behind the propeller.

If we define the propeller-induced velocity as  $w = V_1 - V_0$  or equivalently  $V_1 = V_0 + w$ , and  $V_3 = V_0 + 2w$ , we can rewrite Eq. (7) in the form:

$$T = 2 \rho A (V_0 + w) w \quad (9)$$

Furthermore we can define the power that the propeller adds to the flow as the thrust times the velocity through the propeller:

Power applied to flow:

$$P = T(V_0 + w) \quad (10)$$

This equation can be split into two parts: The useful power,  $P_{\text{useful}} = T V_0$ , and the induced power,  $P_i = T w$ .

We can solve Eq. (9) for  $w$  using the quadratic formula to get:

$$w = \frac{1}{2} \left[ -V_0 + \sqrt{V_0^2 + \left( \frac{2T}{\rho A} \right)} \right] \quad (11)$$

Equation (11) is an expression for the induced velocity at the propeller disc. We can compute it for the case where the aircraft is sitting still,  $V_0 = 0$ . Then we get the static case,

$$w_0 = \sqrt{\frac{T}{2 \rho A}} \quad (12)$$

We can substitute this back into Eq. (10) for the case where  $V_0 = 0$ . The result is the power input into the air in the static case. There is no useful power so we are left with the induced power:

$$P_{i0} = \frac{T^{3/2}}{\sqrt{2 \rho A}} \quad (13)$$

We can invert this equation and solve for the static thrust:

$$T = P_{i0}^{2/3} (2 \rho A)^{1/3} \quad (14)$$

Equation (14) is the maximum static thrust that one would expect to get from an engine driving a propeller with zero forward airspeed. The actual thrust would be less than that given by Eq. (14) since the momentum theory approach does not include any propellor blade drag or any losses at the tips of the propellor blades (like a 3-D wing).

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### Example

A 200 Horsepower engine drives a 6.5 ft diameter propeller. What would be an estimate of the thrust in the static condition?

200 hp x 550 ft lbs/sec hp = 110,000 ft lbs/sec

The area of the propeller disc is  $A = \pi D^2 / 4 = \pi 6.5^2 / 4 = 33.18 \text{ ft}^2$

The estimated *maximum* static thrust is:

$$T = P^{2/3} (2 \rho A)^{1/3} = 110,000^{2/3} [2 (0.002377) 33.18]^{1/3} = 1240 \text{ lbs}$$

We can estimate the actual static thrust is 90% of that or 1116 lbs.

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### *Ideal Efficiency*

We can conclude this section by defining the ideal efficiency of a propeller. The efficiency of the propeller is the useful power out divided by the input power. The useful power out is the thrust times the free stream airspeed, while the power in is the thrust times the airspeed through the propeller disk. Hence the ideal efficiency is:

$$\eta_i = \frac{T V_0}{T V_1} = \frac{T V_0}{T (V_0 + w)} = \frac{1}{1 + \frac{w}{V_0}} \quad (15)$$

Introducing Eq. (11) we can write the ideal efficiency (an un-obtainable upper limit) as:

$$\eta_i = \frac{2}{1 + \sqrt{1 + \frac{T}{\bar{q} A}}} \quad (16)$$

The quantity  $T/A$  is called the *disc loading*. At zero disc loading, the efficiency is 1.0.

### *Turbine Power Plant*

The thrust in a turbine power plant can be estimated from momentum theory. From previous work, it is given by:

$$T = \rho A_{en} V_{en} (V_{ex} - V_{en}) \quad (17)$$

if we neglect the fuel added to the airflow. The leading three terms are the mass flow rate, and the terms in parentheses give the difference of exhaust and inlet velocities. Both of these terms

are affected by the airspeed. The higher the airspeed, the more mass flow through the engine, but the smaller the difference between the exit and entrance velocities. Hence increasing the speed of an aircraft can make the thrust increase or decrease. Usually there is an initial increase in thrust, followed by a decrease with increasing speed. The other factor in the equation is the inlet air density which may increase with airspeed at a given altitude.

As a result of these observations, we can propose a thrust model for a turbojet, or turbofan engine. The model suggests that the thrust depends on the airspeed and throttle setting at a given altitude, and also changes with altitude through the density. The functional form of the model can then be written as:

$$T = T_{h_{ref}}(V, \delta_T) \left( \frac{\rho}{\rho_{h_{ref}}} \right)^x \quad (18)$$

where:

- $T(V, \delta_T)$  = some function of  $V$  (or  $M$ ) and  $\delta_T$  to be determined
- $\delta_T$  = throttle setting ( $0 \leq \delta_T \leq 1.0$ )
- $h_{ref}$  = some reference altitude usually sea-level or the tropopause
- $\rho$  = air density
- $x$  = some exponent (typically 1). See below.

***Operation in the stratosphere:***

For operation in the constant temperature stratosphere, the exponent in Eq. (18) is generally equal to  $x = 1$  for all types of jet engines.

**Operation in the troposphere:**

For operation in the troposphere, the exponent can be different from  $x = 1$ , depending somewhat on what type of engine you have:

Engine type	approximate lower limit	approximate upper limit
Turbojet	0.3	1.0
Turbojet with afterburner	0.6	1.1
Turbofan	0.8	1.2

***Thrust Model Summary***



Generally the mathematical model for any engine must be given or generated by a curve fit of data. If such is not the case, the simplest model that can be used is as follows:

We will use the exponent  $x = 1$  for all altitudes so that the basic model is:

$$T = T_{SL} \frac{\rho}{\rho_{SL}} = \sigma T \quad (19)$$

The thrust will be assumed to be independent of speed so that we can represent the thrust as

$$T_{SL} = T_{\max_{SL}} \delta_T \quad (20)$$

Consequently the thrust at any speed and altitude would be given by:

Simplest Model

$$T = T_{SL_{\max}} \delta_T \left( \frac{\rho}{\rho_{SL}} \right) \quad (21)$$

Generally we can add Mach number dependence for high subsonic flight and modify Eq. (21) slightly in the following fashion:

$$T = T_{SL_{\max}} \delta_T \left( \frac{\rho}{\rho_{SL}} \right) (1 + K_T M) \quad (22)$$

where  $K_T$  = some constant, typically between 0.1 and 0.9  
 $M$  = Mach number