**Performance**

## 14. Range and Endurance

Both the range and endurance of an aircraft depends on rate of fuel consumption of the engine, and therefore, on the type of engine that is involved - output measured in terms of ***thrust*** or output measured in terms of ***power***. We generally consider the range to be the distance the aircraft can fly from a given speed and altitude until it runs out of fuel and the endurance as the time it takes to run out of fuel. As one might expect, there is a flight condition that will give us the best range for a given aircraft, and a different flight condition that will give us maximum endurance. In this section we are interested in determining:

1. Flight conditions for maximum endurance
2. Flight conditions for maximum range
3. Computing range and endurance for any given flight condition

Of interest to us are two measures of fuel consumption. For ***maximum endurance***, we are interested in determining the fuel consumed per unit time:

$$\frac{dW_f}{dt} \qquad \Rightarrow \qquad \left( \frac{\text{lb fuel}}{\text{sec}}, \qquad \frac{\text{N fuel}}{\text{sec}} \right) \qquad \text{Minimize for maximum endurance}$$

and for ***maximum range*** we are interested in determining the fuel consumed per unit distance:

$$\frac{dW_f}{dS} \qquad \Rightarrow \qquad \left( \frac{\text{lb fuel}}{\text{ft}}, \qquad \frac{\text{N fuel}}{\text{ft}} \right) \qquad \text{Minimize for maximum range}$$

As indicated previously, the fuel consumption is related to the type of power plant with which an aircraft is equipped. The results being different depending on if the aircraft is equipped with an engine whose output is measured in terms of thrust or in terms of power.

***<u>Range and Endurance for Aircraft whose Engine Performance is given in Terms of Thrust (Jets)</u>***

Here we will define a measure of fuel consumption. For our problems we need this measure in terms of proper (basic) units. Unfortunately, it is usually not given in these units. We can define:

**Definition: Thrust specific fuel consumption:** The thrust specific fuel consumption can be defined in proper units as:

$$c_t = \frac{\text{lbs fuel / sec}}{\text{lbs thrust}} \quad \text{or} \quad \frac{1}{\text{sec}}, \qquad \frac{\text{N fuel / sec}}{\text{N thrust}} \quad \text{or} \quad \frac{1}{\text{sec}} \tag{1}$$

Unfortunately, the numbers above are rarely given. The information is usually given in the following terms:

$$\bar{c}_t = \frac{\text{lbs fuel / hr}}{\text{lbs thrust}} \quad \text{or} \quad \frac{1}{\text{hr}}, \qquad \frac{\text{N fuel / hr}}{\text{N thrust}} \quad \text{or} \quad \frac{1}{\text{hr}} \tag{2}$$

A typical value of this parameter is 1 lb fuel/hr per lb thrust (1 Newton fuel/hr pr Newton thrust). All equations that follow will be written for the use of ***basic units***! Consequently the normally given number must be converted. (There are some cases where such a conversion is not necessary. However one is best off using basic units).

Recall, for straight and level flight, under our usual assumptions, T = D, and L = W. Then we can determine the rate of fuel consumption as:

$$\frac{dW_f}{dt} = c_t T = c_t D \tag{3}$$

**Maximum Endurance Flight Condition**

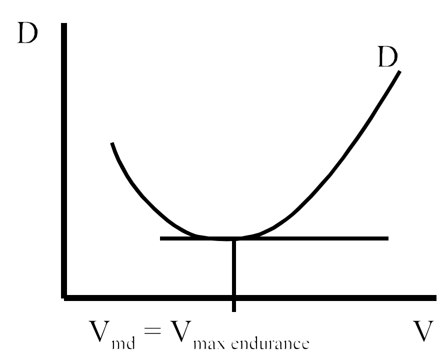
Clearly, from Eq. (3) the rate of fuel consumption is a minimum when drag, D, is a minimum. Hence the ***maximum endurance flight condition of a jet is at the minimum drag condition.*** Then the maximum endurance for a jet occurs at the maximum L/D.

**Maximum Endurance Flight Condition**

$$\left.\frac{L}{D}\right)_{\max} \Rightarrow \underline{\text{minimum drag condition}} \tag{4}$$

For the ***general case***, the minimum drag flight condition must be determined by selecting the minimum point on the Drag vs. Airspeed plot. However, for the special case of a low performance parabolic drag polar,

$C_D = C_{D_{0L}} + K C_L^2$, then we can determine the minimum drag flight conditions from:




$$C_{L_{md}} = \sqrt{\frac{C_{D_{0L}}}{K}}, \qquad C_{D_{md}} = 2\,C_{D_{0L}}$$

and

$$\left(\frac{L}{D}\right)_{max} = \frac{1}{2\sqrt{C_{D_{0L}} K}}$$

**Maximum Range Flight Condition**

The maximum range flight condition can be determined by noting that in level flight, the airspeed is given by: $V = \frac{dS}{dt}$, where *S* is the range variable. Then we can compute the fuel burned per unit range as follows:

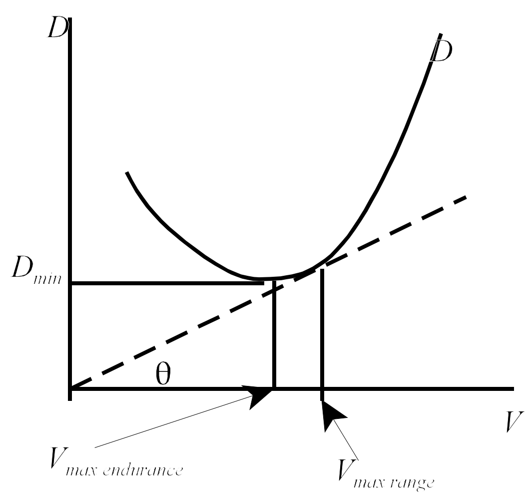
$$\frac{dW_f}{dS} = \frac{dW_f/dt}{dS/dt} = \frac{c_t T}{V} = c_t \frac{D}{V} \qquad (5)$$

Then, it is clear from Eq. (5) that the maximum range occurs when D/V is a minimum. For the general case of an arbitrary drag polar, we can determine this flight condition from the basic Drag vs Airspeed plot in the following way.

Note that from the diagram we can draw a line from the origin that in general can intersect the drag curve in two places. If we take the minimum value of the angle so that it is just tangent to the drag curve we can see that:

$$\tan\theta = \frac{D}{V}$$

and hence when θ is a minimum (or tangent to the drag curve) then the tangent point will be the maximum range flight condition. Further, we can note that the airspeed for maximum range is greater than the airspeed for maximum endurance.




For the ***general case*** we have:

$$\frac{D}{V} = \frac{C_D\, 1/2\, \rho V^2 S}{V} = 1/2\, \rho S V C_D = 1/2\, \rho S \sqrt{\frac{W}{1/2\, \rho S}}\, \frac{C_D}{C_L^{1/2}} \qquad (6)$$

Then for a given altitude and weight, ***the range is maximized when*** $\frac{C_D}{C_L^{1/2}}$ ***is a minimum***. This is ***a general result***.

*Special Case - Maximum Range, Low Performance Parabolic Drag Polar*

For the special case of a low performance aircraft (drag parameters constant) with a parabolic drag polar, we can find the flight condition for the maximum range of a jet type aircraft. We can simply use some introductory calculus:

$$\frac{d\left(\dfrac{C_D}{C_L^{1/2}}\right)}{d\,C_L} = \frac{d}{d\,C_L}\left(\frac{C_{D_{0L}} + K\,C_L^2}{C_L^{1/2}}\right) = \frac{C_L^{1/2}\,(2\,K\,C_L) - (C_{D_{0L}} + K\,C_L^2)\,1/2\,C_L^{-1/2}}{C_L} = 0$$

or

**Maximum Range Flight Conditions for Engine Performance Measured in Terms of Thrust**

$$\boxed{C_{L_{\max\ range}} = \sqrt{\frac{C_{D_{0L}}}{3\,K}}} \quad \text{and} \quad \boxed{C_{D_{\max\ range}} = \frac{4}{3}\,C_{D_{0L}}} \tag{7}$$

**General Results for Endurance and Range of an Aircraft whose Performance is Measured in Terms of Thrust**

The following results are true for all aircraft whose engine performance is measured in terms of thrust. We are interested in calculating the range and endurance of the aircraft. In order to do that, we must specify how the aircraft is to be flown so we can evaluate the required integrals. First, we will develop general integral expressions for endurance and range.

***Endurance***

The endurance equation, Eq. (3) can be rearranged in the following form:

$$d\,t = \frac{d\,W_f}{c_t\,T} \tag{8}$$

We can also note that the rate of fuel burn is the same as the rate of aircraft loss of weight. Hence we can write, $d\,W_f = -\,d\,W$, so the time equation can be rewritten as:

$$d\,t = -\frac{d\,W}{c_t\,T} = -\frac{d\,W}{c_t\,D} \tag{9}$$

We can integrate Eq. (9) over the time-of-flight on the left and over the change in weight on the right to get:

**Endurance Equation**

$$E = t_2 - t_1 = -\int_{W_1}^{W_2} \frac{dW}{c_t D} \qquad (10)$$

**Range**

In a similar manner, we can rearrange Eq. (5) to get the range equation:

**Range Equation**

$$R = S_2 - S_1 = -\int_{W_1}^{W_2} \frac{V\,dW}{c_t D} \qquad (11)$$

**Integration of the Endurance and Range Equations**

In order to integrate the above equations, we need to know how the variables in the integrand vary with the integral independent variable weight. That is, we need to know $c_tD$ as a function of W or $V/(c_tD)$ as a function of W. These functions are not unique and depend on the ***flight schedule***. Typical flight schedules are:

1) flight using the maximum endurance flight conditions
2) flight using the maximum range flight conditions
3) flight at constant airspeed
4) flight at constant angle-of-attack
5) flight at constant altitude and constant airspeed

Etc.

Other flight schedules or any combination of schedules could be used. Here, however we will assume one flight schedule is used throughout the flight.

***Endurance (Engines whose output is measured in terms of thrust)***

In order to be able to integrate Eq. (10), we will have to make some assumptions. These assumptions involve how the thrust specific fuel consumption behaves and what flight schedule is used. Generally, the engine manufacturer provides tables that indicate how the thrust specific fuel consumption varies with altitude and Mach number. Here, we will assume that these variations are small and that we can assume $c_t$ is constant. In addition, we will assume that the flight path is nearly level so that we can assume lift = weight. The last assumption we will make has to do with the flight schedule. Here we will assume constant angle-of-attack.

***Assumption:*** The thrust specific fuel consumption is constant, $c_t$ ***- const***.

***Assumption: L = W***

***Assumption: Constant angle-of-attack,*** $\alpha$ ***= const***

This last assumption has many ramifications. Constant angle-of-attack implies that the lift coefficient is constant, $C_L = \text{const}$, that in turn implies the drag coefficient ( $C_D = f(C_L)$), that in turn implies $\frac{C_L}{C_D} = \text{const}$, or $\frac{L}{D} = \text{const}$. We these assumptions we can integrate Eq. (10) as follows:

$$E = -\int_{W_1}^{W_2} \frac{dW}{c_t D} = -\int_{W_1}^{W_2} \frac{dW}{c_t D}\frac{W}{W} \\ = -\int_{W_1}^{W_2} \frac{1}{c_t}\left(\frac{C_L}{C_D}\right)\frac{dW}{W} \tag{12}$$

With the assumptions we have made, everything in the integrand other than the weight, is constant. Consequently, we can integrate Eq. (12) to obtain:

**Endurance - Thrust-rated Vehicle at Constant Angle-of-Attack, and Constant $c_t$**

$$E = \frac{1}{c_t}\frac{C_L}{C_D}\ln\frac{W_1}{W_2} = \frac{1}{c_t}\left(\frac{L}{D}\right)\ln\frac{W_1}{W_2} \tag{13}$$

From Eq. (13) we can see that for long endurance we want to have:

1) $c_t$ as small as possible
2) L/D as large as possible $L/D|_{max}$
3) $W_1/W_2$ as large as possible ($W_2 = W_1 - W_{fuel}$ as small as possible - lots of fuel!)

Note that this result (dependent on the constant angle-of-attack flight schedule)is independent of altitude and is a general result independent of the form of the drag polar!

As indicate previously, the result of carrying out the integration of Eq. (10) depends on the flight schedule used. To demonstrate this idea, we will consider a flight schedule where we use a constant altitude, constant airspeed flight schedule. We will still assume that the thrust specific fuel consumption, $c_t$ is a constant. Under this flight schedule we can write the drag as

$$D = C_{D_{0L}} 1/2 \rho S V^2 + \frac{K W^2}{1/2 \rho S V^2} = E + F W^2$$

where

$$E = C_{D_{0L}} 1/2 \rho V^2 S = C_{D_{0L}} \bar{q} S \qquad F = \frac{K}{1/2 \rho V^2 S} = \frac{K}{\bar{q} S}$$

Here, we can see that if the altitude and airspeed are constant, the E and F are constant. The endurance integral becomes:

$$E = -\int_{W_1}^{W_2} \frac{dW}{c_t \left( E + F W^2 \right)}$$

The integral is well known and can be written as:

$$E = \frac{1}{c_t \sqrt{E F}} \left[ \tan^{-1} \left( \sqrt{\frac{F}{E}} W_1 \right) - \tan^{-1} \left( \sqrt{\frac{F}{E}} W_2 \right) \right]$$

If we substitute in for E and F, we have:

**Endurance for Thrust Rated Aircraft Using Constant Altitude, Constant Velocity Flight Schedule with a Parabolic Drag Polar**

$$E = \frac{1}{c_t \sqrt{C_{D_{0L}} K}} \left[ \tan^{-1} \left( \frac{C_{L_1}}{C_{L_{md}}} \right) - \tan^{-1} \left( \frac{C_{L_2}}{C_{L_{md}}} \right) \right] \qquad (14a)$$

or

$$E = \frac{2}{c_t} \left. \frac{L}{D} \right)_{\max} \left[ \tan^{-1} \left( \frac{C_{L_1}}{C_{L_{md}}} \right) - \tan^{-1} \left( \frac{C_{L_2}}{C_{L_{md}}} \right) \right] \qquad (14b)$$

where

$$C_{L_1} = \frac{W_1}{\bar{q}\,S} \qquad \text{and} \qquad C_{L_2} = \frac{W_2}{\bar{q}\,S}$$

Even though L/D$|_{max}$ and $C_{L_{md}}$ appear in Eq. (14b) it is ***not specifically for maximum endurance***!

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Example:

A 600,000 lb aircraft has a drag polar, $C_D = 0.017 + 0.042\,C_L^2$, and a wing area of 5128 ft$^2$. The Thrust Specific Fuel Consumption (TSFC) = 0.85 (lbs/hr)/lb. The total fuel on board is $W_f$ = 180,000 lbs. Find the endurance for a constant angle-of-attack flight schedule and for a constant speed, constant altitude flight schedule. Assume the initial conditions to be at 30,000 ft.

The first thing we need to do is to convert TSFC to basic units:

$$c_t = \frac{TSFC}{3600} = \frac{0.85}{3600} = 0.0002361\ \frac{\text{lbs fuel/sec}}{\text{lbs thrust}}$$

(Note that we don't have to do this)

We will start the flight with the maximum endurance conditions (L/D$|_{max}$ or min drag)

$$C_{L_{md}} = \sqrt{\frac{C_{D_{0L}}}{K}} = \sqrt{\frac{0.017}{0.42}} = 0.6362\,, \qquad C_{D_{md}} = 2\,C_{D_{0L}} = 0.034\,, \qquad \left.\frac{L}{D}\right)_{\text{max}} = 18.7120$$

The weight at the end of the flight is the initial weight minus the fuel weight:

$$W_2 = W_1 - W_f = 600{,}000 - 180{,}000 = 420{,}000\ \text{lbs}$$

For Endurance with a flight schedule of a constant angle-of-attack, we have

$$E = \frac{1}{c_t}\left(\frac{L}{D}\right)\ln\frac{W_1}{W_2} = \frac{1}{0.0002361}(18.712)\ln\frac{600000}{420000} = 28268\ \text{sec} = \underline{7.85\ \text{hrs}}$$

If we use a constant airspeed, constant altitude strategy we need to calculate the conditions at the initial and final times. We will assume maximum endurance conditions at the initial point. (Note that we can not maintain the best endurance conditions throughout the flight). At the initial point, at 30,000 ft we have:

$$C_{L_1} = C_{L_{md}} \qquad \text{and} \qquad \bar{q}\,S = \frac{W}{C_{L_{md}}} = \frac{600000}{0.6362} = 943099.6\ \text{lbs}$$

( V = 642.9 ft/sec )

Then at the final time we have:

$$C_{L_2} = \frac{W_2}{\bar{q}\,S} = \frac{420000}{943099.6} = 0.4453 \qquad (\text{ Note that this is not } C_{L_{md}} = 0.6362)$$

We can now substitute into Eq. (14b)

$$E = \frac{2}{0.0002361}(18.7120)\left[\tan^{-1}(1) - \tan^{-1}\left(\frac{0.4453}{0.6362}\right)\right] = 27693.8 \text{ sec } = \underline{7.69 \text{ hrs}}$$

If we had arranged our flight so that we ended up at the optimal min drag condition we would have the conditions:

$$\bar{q}\,S = \frac{W_2}{C_{L_{md}}} = \frac{420000}{0.6362} = 660169.8 \text{ lbs} \qquad (\text{V} = 537.8 \text{ ft/sec})$$

and $$C_{L_1} = \frac{W_1}{\bar{q}\,S} = \frac{600000}{660169.8} = 0.9089.$$

The endurance is given by:

$$E = \frac{2}{0.0002361}(18.7120)\left[\tan^{-1}\left(\frac{0.9089}{0.6362}\right) - \tan^{-1}(1)\right] = 27690.6 \text{ sec } = \underline{7.69 \text{ hrs}}$$

Although the results of looking at imposing the max endurance conditions at the beginning and at the end give close to the same result, one can ask if there is a “best point” at which to impose the maximum endurance flight conditions so that under this schedule of flight, the endurance will be the greatest? One might guess at the midpoint, when the fuel is half gone. Is this correct and can you prove it or some other result?

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***Range (Engines whose Output is in Terms of Thrust)***

We now will investigate the range integral, Eq. (11). Again, the integral can be evaluated if we make appropriate assumptions, and pick a flight schedule. The integral of interest is:

$$R = -\int_{W_1}^{W_2} \frac{V\,dW}{c_t\,D} \tag{15}$$

In a similar manner with the endurance equation, in order to integrate this integral we need to know the airspeed, drag, and TSFC in terms of the weight. How these items depend on weight depends on some assumptions and on the ***flight schedule*** we select. We will look at several flgiht schedules. In all of them, however, we will assume constant TSFC.

**Assumption:** The thrust specific fuel consumption, $c_t$ = const

Here we will look at the various flight schedules and their associated assumptions.

**Flight Schedule A**: Constant altitude and constant angle-of-attack

**Assumption:** Angle-of-attack is a constant, $\alpha$ = const
**Assumption:** The altitude is a constant, $\rho$ = const

Under these assumptions we can rearrange the range integral in the following way in order to allow us to implement the assumptions:

$$R = -\int_{W_1}^{W_2} \frac{V\,dW}{c_t D} = -\int_{W_1}^{W_2} \sqrt{\frac{W}{1/2\,\rho\,S\,C_L}}\,\frac{1}{c_t}\,\frac{C_L}{C_D}\,\frac{dW}{W}$$
$$= -\int_{W_1}^{W_2} \frac{1}{c_t}\left(\frac{2}{\rho\,S}\right)^{1/2}\frac{C_L^{1/2}}{C_D}\,\frac{dW}{W^{1/2}} \qquad (16)$$

Then, from our assumptions, everything in the integrand except the weight W, is a constant. Recall that the lift coefficient is a function of angle-of-attack and hence is constant if angle-of-attack is constant. The drag coefficient depends only on the lift coefficient, hence it is a constnnt. Further, at constant altitude, the density is constant. Finally we assumed $c_t$ to be constant.

Under these assumptions and flight schedule, we can carry out the integration to give:

**Range for Constant Altitude, Constant Angle-of-Attack**, **Thrust-Rated Vehicle**

$$R = \frac{2\sqrt{2}}{c_t\sqrt{\rho\,S}}\left(\frac{C_L^{1/2}}{C_D}\right)\left(\sqrt{W_1} - \sqrt{W_2}\right) \qquad (17)$$

We should note that to fly this flight schedule, airspeed is not constant. In addition, by inspecting Eq. (17) we can note the following requirements for a long range:

1) A small $c_t$, the lower the TSFC, the farther you can fly!

2) We would like to fly at a high altitude in order to have a low density

3) We would like to maximize $\dfrac{C_L^{1/2}}{C_D}$

4) We would like to carry lots of fuel ($W_2 = W_1 - W_f$), then $\sqrt{W_1} - \sqrt{W_2}$ will be large.

**Flight Schedule B:** Constant angle-of-attack, and constant airspeed

**Assumption:** Constant angle-of-attack, $\alpha$ = const
**Assumption:** Constant airspeed, $V$ = const

Under these assumptions we can rearrange the integral, Eq. (15) slightly to get (recall L = W)

$$R = -\int_{W_1}^{W_2} \frac{V\,dW}{c_t D} = -\int_{W_1}^{W_2} \frac{V}{c_t}\frac{C_L}{C_D}\frac{dW}{W} \tag{18}$$

Everything in the integrand is constant except W so we can easily evaluate it to get the range:

**Range for Constant Airspeed, Constant Angle-of-Attack, Thrust-Rated Vehicle**

$$R = \frac{V}{c_t}\frac{C_L}{C_D}\ln\frac{W_1}{W_2} \tag{19}$$

We can note that this expression is just the endurance equation (for the same conditions, i.e. constant angle-of-attack, Eq. (13)) multiplied by the constant airspeed. This equation is valid for all flight conditions. However, to maximize range we need to maximize $V\dfrac{C_L}{C_D}$ or equivalently maximize $\dfrac{C_L^{1/2}}{C_D}$ as we have suggested previously..

**Flight Schedule C:** Constant airspeed, and constant altitude

**Assumption:** Constant airspeed, $V$ = const
**Assumption:** Constant altitude, $\rho$ = const
**Assumption:** Parabolic drag polar, $C_D = C_{D_{0L}} + KC_L^2$

Under these assumptions we can rearrange the range integral, Eq. (15) to get:

$$R = -\int_{W_1}^{W_2} \frac{V}{c_t} \frac{dW}{D} = -\frac{V}{c_t}\int_{W_1}^{W_2} \frac{dW}{C_{D_{0L}}\, 1/2\,\rho\, V^2 S + \dfrac{K W^2}{1/2\,\rho\, V^2 S}} = -\frac{V}{c_t}\int_{W_1}^{W_2} \frac{dW}{E + F W^2} \qquad (20)$$

where E and F are defined the same as in Eq. (14). The integration is the same and leads to the result:

$$R = \frac{V}{c_t \sqrt{E F}} \left[ \tan^{-1}\left( \sqrt{\frac{F}{E}}\, W_1 \right) - \tan^{-1}\left( \sqrt{\frac{F}{E}}\, W_2 \right) \right]$$

or:

**Range for Constant Airspeed, Constant Altitude, Parabolic Drag Polar, Thrust-Rated Vehicle**

$$R = \frac{2V}{c_t} \left.\frac{L}{D}\right)_{\max} \left[ \tan^{-1} \frac{C_{L_1}}{C_{L_{md}}} - \tan^{-1} \frac{C_{L_2}}{C_{L_{md}}} \right] \qquad (21)$$

where $C_{L_i} = \dfrac{W_i}{1/2\,\rho\, V^2 S}$, $\qquad i = 1, 2$

***Consequences of Assumptions***

In the previous developments, we found that certain characteristics of the flight were assumed constant. For the three different flight schedules, we had three different combinations of these constant variables. One would think that in each case there are different combinations of variables that are changing. Here we want to look into what is happening to the non-constant variables. Further, we did not examine if the assumptions were consistent with each other, of even if they were possible, and if possible what they imply on the aircraft flight path of flight controls.

Flight Schedule: Constant angle-of-attack and constant airspeed.

This flight schedule leads to $C_L$ = const and V = const. From the equation for V we have

$$V = \sqrt{\frac{W}{1/2\,\rho\, S\, C_L}} = \text{const} \qquad \Rightarrow \qquad \sqrt{\frac{W}{\rho}} = \text{const} \tag{22}$$

Under this flight schedule (V, α = const), we see as fuel is used up and the vehicle gets lighter, the density is required to decrease. Therefore the altitude must increase as fuel is burned up and the aircraft gets lighter. In addition, it may be required to adjust the throttle so that the airspeed remains constant In the stratosphere, where the temperature is constant, if the thrust available is proportional to the density, the engine thrust will drop off with altitude at the same rate that the drag is reduced with altitude (with constant $C_D$ and $C_L$) so that the throttle can remain unchanged. This flight technique is called the “drift up” flight schedule.

Flight Schedule: Constant angle-of-attack and altitude

This flight schedule gives us α = const, and ρ = constant. The airspeed equation becomes:

$$V = \sqrt{\frac{W}{1/2\,\rho\, S\, C_L}} \qquad \Rightarrow \qquad V \propto \sqrt{W} \tag{23}$$

Here as the flight continues and the fuel is burned up, the airspeed decreases. Generally, to fly this schedule, the throttle must be reduced as the fuel is consumed and the weight decreases.

Flight Schedule: Constant altitude and airspeed

This flight schedule gives us V = const and ρ = const. The airspeed equation becomes:

$$V = \sqrt{\frac{W}{1/2\,\rho\, S\, C_L}} \qquad \Rightarrow \qquad C_L \propto W \tag{24}$$

Here the lift coefficient is proportional to the weight in order to satisfy the constant airspeed and constant altitude constraint. As the weight decreases, so must the lift coefficient. Consequently the angle-of-attack must decrease as the flight continues. As a result, the drag will decrease slightly (smaller induced drag, zero-lift drag is unchanged) and the throttle may have to be continually reduced during the flight.

***<u>Range and Endurance for Aircraft whose Engine Performance is given in Terms of Power (Piston Engines and Turboprops)</u>***

The fuel consumption for engines who's output is measured in terms of power (piston engines) or equivalent power ( turboprop engines), is measured in terms of power specific fuel consumption (PSFC) which is defined as:

$$c_p = \frac{\text{lbs fuel / sec}}{\text{ft} - \text{lbs/sec}} \qquad \frac{\text{N fuel/sec}}{\text{watts}} \tag{25}$$

As with the TSFC, the usual information is given in non-proper units:

$$\bar{c}_p = \frac{\text{lbs fuel}}{\text{HP hr}} \quad \Rightarrow \quad \frac{\text{N fuel}}{\text{Watt hr}}$$

Here, all equations will be developed using the proper units. Here we will assume that all flight conditions are level or near level (quasi-level) so that L = W, and T = D, or $P_{av} = P_{req}$. We can relate power required to the engine shaft power through the propulsive efficiency. In level (or quasi level flight) we have

$$P_s \eta_p = P_{av} = D V \quad \Rightarrow \quad P_s = \frac{D V}{\eta_p} \tag{26}$$

We can now develop the ***endurance equation***

$$\frac{d W_f}{dt} = c_p P_{req} = \frac{c_p P_s}{\eta_p} = \frac{c_p D V}{\eta_p} \tag{27}$$

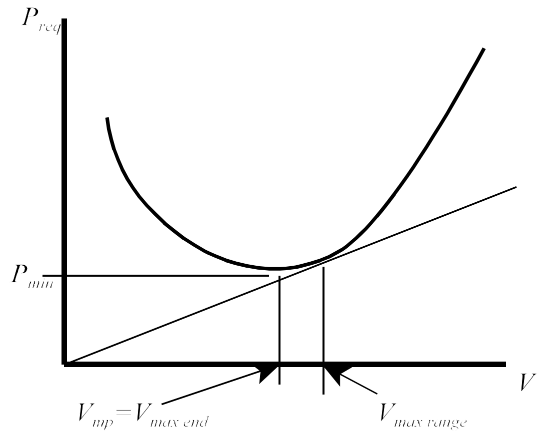
and the ***range equation***

$$\frac{d W_f}{d S} = \frac{d W_f / d t}{d S / d t} = \frac{c_p D V}{\eta_p V} = \frac{c_p D}{\eta_p} \tag{28}$$

***General results (point performance)***

For maximum endurance, we would like to minimize Eq. (27) and for maximum range, minimize Eq. (28). We can determine the minimum of each equation directly from the basic plot of power required vs airspeed. It is clear from Eq. (27) that the minimum of that equation occurs where power required, $P_{req} = DV$, is a minimum, and that the minimum of Eq. (28) occurs when drag, D is a minimum (assuming $c_p$ and $\eta_p$ are weak functions of airspeed or are constant).

From the power required curves, we can simply draw a horizontal tangent to the power required curve and the tangent point will provide the minimum power required and the corresponding airspeed for maximum endurance. If we draw a line from the origin through the power required curve, the points of intersection give




$$\tan\theta = \frac{D\,V}{V} = D$$

Hence the smallest angle obtained by drawing the tangent line as shown in the figure will give the point where the drag is a minimum, and the corresponding airspeed for the maximum range flight condition.

To summarize, for a ***power rated aircraft***, the maximum endurance and range conditions are as follows:

**Maximum Endurance: Minimum power required flight condition**

**Maximum Range: Minimum drag (max L/D) flight condition**

***Endurance for Power-Rated Aircraft (Integral Performance)***

We can compute the endurance for a power-rated aircraft by rearranging Eq. (27). As before we will do the calculations in terms of the aircraft weight instead of the fuel weight by noting that $d\,W = -\,d\,W_f$. Then the endurance equation becomes:

**Endurance Equation - Power-Rated Vehicle**

$$E = -\int_{W_1}^{W_2} \frac{\eta_p\, d\,W}{c_p\, D\, V} \qquad (29)$$

We can now investigate the endurance if we make a few assumptions, and prescribe some flight schedule. ***In all that follows we will assume that $c_p$ = const and $\eta_p$ = const.***

**Flight Schedule A**: Constant angle of attack, $\alpha$ = const and constant altitude, $\rho$ = const.

Since angle-of-attack is constant, this implies that the lift coefficient is constant that in turn implies that the drag coefficient is constant. We can use this information to simplify the integral in Eq. (29) so that the integrand only depends on the weight, W. For quasi level flight we can write:

$$D = W\frac{D}{L} = W\frac{C_D}{C_L}$$

Then the endurance equation, Eq. (29) can be written as:

$$E = -\int_{W_1}^{W_2} \frac{\eta_p}{c_p}\frac{1}{V}\frac{C_L}{C_D}\frac{dW}{W} \qquad \text{and} \qquad V = \sqrt{\frac{W}{1/2\,\rho\,S\,C_L}}$$

Combining the two equations above, we can write the endurance as:

$$E = -\int_{W_1}^{W_2} \frac{\eta_p}{c_p}\sqrt{\frac{\rho S}{2}}\frac{C_L^{3/2}}{C_D}\frac{dW}{W^{3/2}} \tag{30}$$

With our assumption regarding $c_p$ = const, and our specified flight schedule, all the items in the integrand are constant except for the weight. Consequently we can integrate Eq.(30) to get:

**Endurance for Power rated Aircraft, Constant Angle-of-Attack, Constant Altitude**

$$E = \frac{\eta_p}{c_p}\sqrt{2\rho S}\frac{C_L^{3/2}}{C_D}\left(\frac{1}{\sqrt{W_2}} - \frac{1}{\sqrt{W_1}}\right) \tag{31}$$

Here we can note from Eq. (31) that to maximize the endurance, we need to:

1) have a small power specific fuel consumption, $c_{p.}$
2) operate at a low altitude
3) use a flight condition that maximizes $\frac{C_L^{3/2}}{C_D}$
4) have a lot of fuel, $W_1$ - $W_2$ should be large.

**Flight Schedule B:** Constant angle-of-attack, $\alpha$ = const, and constant speed, V = const

Constant angle-of-attack implies that the lift and drag coefficients are constant throughout the flight. Under these circumstances we can rearrange the endurance equation to appear as:

$$E = -\int_{W_1}^{W_2} \frac{\eta_p}{c_p} \frac{1}{V} \frac{C_L}{C_D} \frac{dW}{W} \tag{32}$$

In Eq. (32), under our assumptions and flight condition, everything is constant except W so that we can easily integrate it to get:

**Endurance for Power-Rated Aircraft, Constant Angle-of-Attack, Constant Airspeed**

$$E = \frac{\eta_p}{c_p} \frac{1}{V} \frac{C_L}{C_D} \ln \frac{W_1}{W_2} \tag{33}$$

Here we note that for long endurance we need

1) small $c_p$

2) large $\frac{1}{V} \frac{C_L}{C_D} \quad \Rightarrow \quad \max \frac{C_L^{3/2}}{C_D}$

3)have a large amount of fuel

From our previous discussion regarding constant angle-of-attack and constant airspeed, this flight schedule requires a "drift up" flight trajectory so maintain these constants.

***Range for a Power-Rated Aircraft (Integral Performance)***

The range integral can be established from Eq. (28) to be (using L = W)

$$R = -\int_{W_1}^{W_2} \frac{\eta_p \, dW}{c_p D} = -\int_{W_1}^{W_2} \frac{\eta_p}{c_p} \frac{C_L}{C_D} \frac{dW}{W} \tag{34}$$

**Flight Schedule A and B:** Angle-of-Attack constant, $\alpha$ = const, and either airspeed constant V=const, ***or*** altitude constant, $\rho$ = const.

If we use any flight schedule that includes a constant angle-of-attack, we can integrate the above equation. Although the details of the flight path will be different, (constant altitude or constant airspeed) the range will be the same:

**Range- Power-Rated Aircraft, Angle-of-Attack Constant, Either Airspeed or Altitude Constant**

$$R = \frac{\eta_p}{c_p} \frac{C_L}{C_D} \ln \frac{W_1}{W_2} \qquad (35)$$

This equation is known as the Breguet Range Equation. However virtually all the equations that have this general from are called Breguet equations, even the one for thrust-rated aircraft that were not around during his time.

**Flight schedule C:** Airspeed and Altitude Constant, V = const and ρ = const. (Parabolic drag polar)

Under these conditions the range integral looks like:

$$R = - \int_{W_1}^{W_2} \frac{\eta_p}{c_p} \frac{dW}{E + F W^2}$$

and the range becomes:

$$R = \frac{2 \eta_p}{c_t} \left.\frac{L}{D}\right)_{max} \left[ \tan^{-1} \frac{C_{L_1}}{C_{L_{md}}} - \tan^{-1} \frac{C_{L_2}}{C_{L_{md}}} \right] \qquad (36)$$

***Summary*** : For a power rated aircraft:

Maximum Endurance - requires minimum power required flight condition. For a parabolic drag polar that is:

$$C_{L_{mp}} = \sqrt{\frac{3 C_{D_{0L}}}{K}} \qquad C_{D_{pm}} = 4 C_{D_{0L}} \qquad V_{mp} = \sqrt{\frac{W}{1/2 \, \rho \, S \, C_{L_{mp}}}}$$

Maximum Range - requires minimum drag (or maximum L/D) conditions. For a parabolic drag polar that is:

$$C_{D_{md}} = \sqrt{\frac{C_{D_{0L}}}{K}} \qquad C_{D_{md}} = 2 C_{D_{0L}} \qquad V_{md} = \sqrt{\frac{W}{1/2 \, \rho S \, C_{L_{md}}}}$$

**Effect of Wind on Range and Endurance**

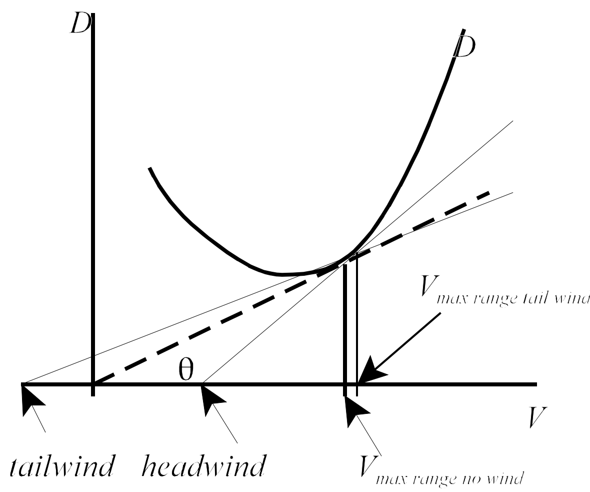
Since time aloft doesn't depend on location with respect to the ground, ***wind has no effect on endurance.*** However, range can be considerably affected by the wind. If we assume that the wind is blowing along the flight trajectory, the range can be given by:

$$R = R_{no\ wind} \pm V_w E \tag{37}$$

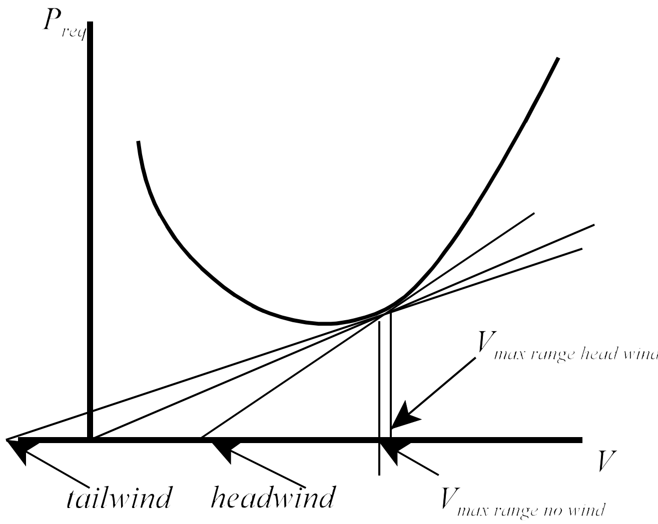
where: $R_{no\ wind}$ = the range for given flight schedule with no wind
$E$ = the endurance for the same flight schedule
$V_w$ = the component of wind along the trajectory
+ = tail wind
- = head wind

Consequently, one might expect to fly at a different airspeed to maximize the range for a given wind condition. If the wind was a tail wind, one would want to fly slower to take advantage of the tail wind to add to the range. The longer the time aloft, the more the wind aids in the range. On the other hand, a head wind reduces the range, so one might want to reduce the time aloft to reduce the effect of the headwind on the range. So one would fly faster then normal in a head wind. The actual speeds at which to fly can be determined graphacally.

Consider the case of an thrust rated aircraft. We can make the basic plot of drag vs airspeed just as we did previously. Then we can mark off a location on the airspeed axis that corresponds to a headwind or tailwind This point is the location where the ground speed would be zero. For example, if we had 20 knot head wind, then we would mark off a positive 20 knots on the airspeed axis, and if a 20 know tail wind we would mark off a negative 20 knots. This point then acts as the "origin" for the ground speed axis. We can then draw our tangent to the drag curve, and the tangent point will give us the airspeed for maximum range. And, as expected, it is clear with a headwind you fly faster, and with a tailwind you prolong your flight by flying slower.



The same arguments can be made for power rated aircraft. Here, however, we use the power required curves to pick off the tangent points. Again, the headwind leads to a higher airspeed and the tail wind to a lower one. Note that not the aircraft is not flying the flight schedule for maximum range. The procedure for obtaining the maximum range flight condition is as follows: Determine the best airspeed graphically from the appropriate figure, drag or power required curves for a thrust rated or power rated vehicle respectively. Draw the tangents and read the airspeed of the airspeed axis. Then determine the lift coefficient for that airspeed from:




$$C_L = \frac{W}{1/2\,\rho\,V^2\,S}$$

Determine the drag coefficient from $C_D = C_{D_{0L}} + K C_L^2$ or from some other drag polar $C_D = f(C_L)$. Determine *L/D* from $C_L / C_D$. Then use these values in the range and endurance equations to determine the no-wind range, and endurance. Substitute the no-wind range and endurance into Eq. (37) to obtain the wind-effected range.