Problem Sheet 8 (ans)

41. A high speed aircraft has a parabolic drag polar with the coefficients varying according to Mach number. This variation is such that the values of the coefficients are constant prior to M = 0.8 and after M = 1.2. In between these two values assume that the coefficients vary as follows:

$$C_{D_{0L}} = C_{D_{0L_0}} + \left(C_{D_m} - C_{D_{0L_0}}\right) \sin^3\left[\left(\frac{M - 0.8}{0.6}\right)\pi\right] \tag{1}$$

$$e = e_0 - e_2 (M - 0.8)^2 + e_3 (M - 0.8)^3$$

Where $C_{D_{0L_0}}$ is the low speed (incompressible) value, $C_{D_{0L_m}}$ is the max value it obtains during transition, and $\boldsymbol{M}$ is the Mach number. Also the e_i, i = 0,2,3 are the constants associated with the Oswald efficiency factor, with e_0 the low speed (incompressible) value.

For this particular aircraft, the values of these constants are: $C_{D0L0} = 0.015$, $C_{Dm} = 3\ C_{D0L0}$, $e_0 = 0.7$, $e_2 = 7$, and $e_3 = 22$. The aircraft properties are: W = 10,000 lbs., S = 200 ft^2, and C_L stall is 1.24. The aspect ratio of the wing is 6.

a) Plot C_{D0L} vs M, and K vs M for M = 0 to 2.0
b) Plot the thrust required Vs. Mach number for M = M (stall) to M = 2.0 at sea level conditions.

a) The zero-lift drag coefficient is given by the expression above. The induced drag coefficient is given by

$$K = \frac{1}{\pi\, AR\, e}$$

Where the Oswald efficiency factor, e, is given above

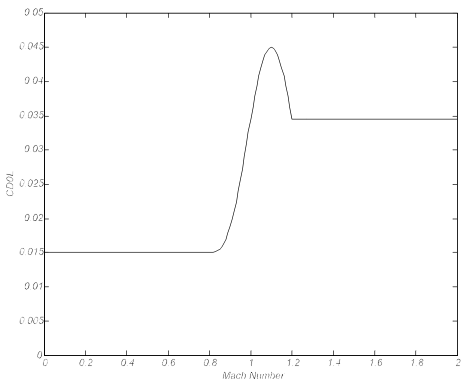
The following Matlab code will make the required plots:

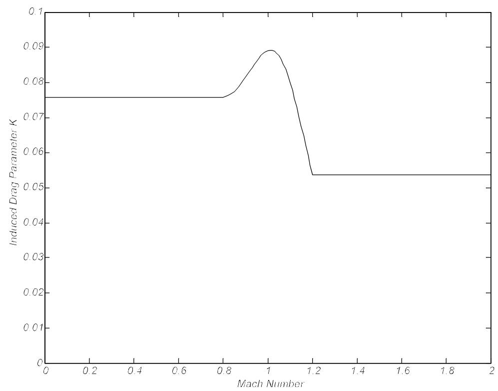
```
%Problem 41 (803) drag with compressibility effects
%Enter Constants
W = 10000; S = 200; cd0l0=0.015; cdm=3*cd0l0; e0=0.7; e2=7; e3=22; ar=6;
for i=1:201;
   M(i)=0.01*(i-1);
   if M(i)<=0.8 ;
      cd0l(i)=cd0l0;
      k(i)=1/(pi*ar*e0);
   elseif M(i)>=1.2;
      cd0l(i)=cd0l0+(cdm-cd0l0)*(sin(2*pi/3))^3;
      k(i)=1/(pi*ar*(e0-e2*(0.4)^2+e3*(0.4)^3));
   else
      cd0l(i)=cd0l0+(cdm-cd0l0)*(sin((M(i)-0.8)/0.6*pi)^3);
      k(i)=1/(pi*ar*(e0-e2*(M(i)-0.8)^2+e3*(M(i)-0.8)^3));
   end
```

```
end
figure(1)
    plot(M,cd0l);
    axis([0 2 0 .05]);
    xlabel('Mach Number');
    ylabel('CD0L');
figure(2)
    plot(M,k);
    axis([0 2 0 .1]);
    xlabel('Mach Number');
    ylabel('Induced Drag Parameter K');
```

The corresponding plots:

C_{D0L} K





b) For calculating the drag, we need to determine the lift coefficient. We can obtain it from

$$C_L = \frac{W}{1/2\,\rho\, V^2 S} = \frac{W}{1/2\,\gamma\, P\, M^2 S}$$

and the drag coefficient from:

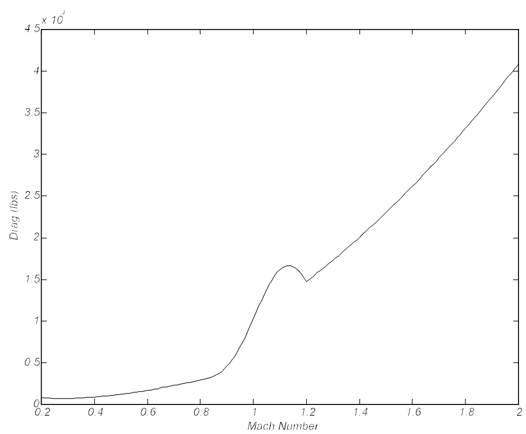
$$C_D = C_{D_{0L}}(M) + K(M)\, C_L^2$$

The original Matlab code was modified to do these calculations:

```
%Problem 41 (803) drag with compressibility effects
%Enter Constants
clear
W = 10000; S = 200; cd0l0=0.015; cdm=3*cd0l0; e0=0.7; e2=7; e3=22; ar=6; p = 2116.2;
for i=1:181;
    M(i)=0.01*(i+19);
    cl(i)=W/(0.5*p*1.4*S*M(i)^2);
    if M(i)<=0.8 ;
        cd0l(i)=cd0l0;
```

```
        k(i)=1/(pi*ar*e0);
    elseif M(i)>=1.2;
        cd0l(i)=cd0l0+(cdm-cd0l0)*(sin(2*pi/3))^3;
        k(i)=1/(pi*ar*(e0-e2*(0.4)^2+e3*(0.4)^3));
    else
        cd0l(i)=cd0l0+(cdm-cd0l0)*(sin((M(i)-0.8)/0.6*pi)^3);
        k(i)=1/(pi*ar*(e0-e2*(M(i)-0.8)^2+e3*(M(i)-0.8)^3));
    end
    cd(i)=cd0l(i)+k(i)*cl(i)^2;
    D(i)=cd(i)*0.5*1.4*p*S*M(i)^2
end
figure(1)
    plot(M,cd0l);
    axis([0 2 0 .05]);
    xlabel('Mach Number');
    ylabel('CD0L');
figure(2)
    plot(M,k);
    axis([0 2 0 .1]);
    xlabel('Mach Number');
    ylabel('Induced Drag Parameter K');
figure(3)
    plot(M,D);
    xlabel('Mach Number');
    ylabel('Drag (lbs)');
```

The following plot is obtained:



This is not a representative drag curve! It is unlikely that the drag will decrease with increased speed in this part of the drag curve. Once the speed passes the initial drag rise Mach number, the drag rate of increase may decrease, but not the drag!

42. If we define the ratio of V/V_{md} = n, and assume a parabolic drag polar with constant coefficients (C_{D0L}, K), find an expression for (L/D) / $(L/D)_{max}$ as a function of m only.

First, lets write an expression for L/D

$$\frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{D_{0L}} + K C_L^2} = \frac{1}{K}\frac{C_L}{\frac{C_{D_{0L}}}{K} + C_L^2} = \frac{1}{K}\frac{C_L}{C_{L_{md}^2} + C_L^2} = \frac{1}{K C_{L_{md}}}\frac{\frac{C_L}{C_{L_{md}}}}{1 + \frac{C_L^2}{C_{L_{md}^2}}}$$

Now $K C_{L_{md}} = K\sqrt{\frac{C_{D_{0L}}}{K}} = \sqrt{C_{D_{0L}} K}$ and $\frac{C_L}{C_{L_{md}}} = \frac{\frac{W}{1/2\,\rho\, S\, V^2}}{\frac{W}{1/2\,\rho\, S\, V_{md}^2}} = \frac{V_{md}^2}{V^2} = \frac{1}{n^2}$

Then,

$$\frac{L}{D} = \frac{1}{\sqrt{C_{D_{0L}} K}}\left[\frac{\frac{1}{n^2}}{1 + \frac{1}{n^4}}\right],$$

But, $\left.\frac{L}{D}\right)_{max} = \frac{1}{2\sqrt{C_{D_{0L}} K}}$

We can multiply numerator and denominator by n^4 to get:

$$\frac{\frac{L}{D}}{\left.\frac{L}{D}\right)_{max}} = \frac{2n^2}{n^4 + 1}$$

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43. An aircraft is powered by a turbojet engine where the thrust is independent of speed. The

aircraft weighs 35,000 lbs and its wing area is 500 ft^2. The drag polar is given by $C_D = 0.016 + 0.045\ C_L^2$. At sea level the maximum rate of climb is 5250 ft/min and occurs at a flight speed of 500 ft/sec. Calculate the rate of climb at the same *angle of attack* with a rocket motor giving 10,000 lbs additional thrust.

Here we note that if the angle-of-attack is constant, we can use that information to imply that the lift coefficient and consequently, the drag coefficient is constant. This in turn implies that the L/D is a constant. Since $D = \frac{W}{(L/D)}$, then we see that the drag is a constant!. Furthermore, since $V = \sqrt{\frac{W}{1/2\,\rho\, S\, C_L}}$, and C_L = const, it follows that V must remain constant. We can now consider the problem at hand. The original rate-of-climb is given by:

$$R/C_0 = \left(\frac{T_0 - D_0}{W} \right) V_0$$

If we add an additional amount of thrust we have:

$$R/C_0 = \left(\frac{T_0 + \Delta T - D_0}{W} \right) V_0$$, since we have established that D and V are constant.

Then

$$R/C_{new} = \left(\frac{T_0 - D_0}{W} \right) V_0 + \frac{\Delta T\, V_0}{W}$$

$$= 5250 + \frac{10{,}000\ (500)}{35{,}000} (60\ \text{sec/min}$$

$$= 5250 + 8571$$

$$= 13{,}821\ \text{ft/min} = 230\ \text{ft/sec}$$

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44. The same aircraft as in Problem 43 has the same rocket motor attached, but this time the same *flight-path angle* (not angle of attack) is held constant. Determine the rate of climb under these conditions.

This is a significantly tougher problem then the previous one. Since the flight path angle is constant we can make the following observations;

$$\sin\gamma = \frac{T_0 - D_0}{W} = \frac{T + \Delta T - D_0 + \Delta D}{W} \qquad \Rightarrow \qquad \Delta T = \Delta D$$

We also have,

$$\dot{h} - V\sin\gamma = \frac{5250}{60} = 500\sin\gamma \qquad \Rightarrow \qquad \sin\gamma = 0.175 \qquad \Rightarrow \qquad \gamma = 10.08 \text{ deg}$$

From the original flight conditions, we can determine the drag, and hence the thrust. The lift coefficient is given by:

$$C_{L_0} = \frac{W}{1/2\,\rho\, S\, V^2} = \frac{35000}{1/2\,(0.002377)\,500\,(500^2)} = 0.02353$$

and

$$C_{D_0} = 0.016 + 0.045\,C_L^2 = 0.016 + 0.045\,(0.2353)^2 = 0.01849$$

Then

$$D = C_D\, 1/2\,\rho\, V^2 S = 0.01849\,(1/2)\,(0.002377)\,(500^2)\,(500) = 2750.6 \text{ lbs}$$

From the climb angle equation, we have

$$T = W\sin\gamma + D = 35000\,(0.1750) + 2750.6 = 8875.6 \text{ lbs}$$

Now we can compute the new lift coefficient from:

$$\sin\gamma = \frac{T}{W} - \frac{D}{W} = \frac{T}{W} - \frac{C_D}{C_L} = \frac{T}{W} - \frac{C_{D_{0L}} + K C_L^2}{C_L} \qquad \text{Where } T = T_0 + \Delta T$$

Rearranging we have:

$$K C_L^2 + \left(\sin\gamma - \frac{T}{W}\right) C_L + C_{D_{0L}} = 0$$

and putting in the numbers:

$$0.045\,C_L^2 + (0.1750 - \frac{18875.6}{35000})\,C_L + 0.016 = 0 \qquad \Rightarrow \qquad C_L = 0.0442, \quad 8.0514$$

$$V = \sqrt{\frac{W}{1/2\,\rho\, S\, C_L}} = \sqrt{\frac{35000}{1/2\,(0.002377)\,500\,(0.0442)}} = 1153.62 \text{ ft/sec}$$

$R/C = V \sin\gamma - 1153.62\,(0.1750) = 201.9 \text{ ft/sec} = 12113.0 \text{ ft/min}$

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45. Our class executive jet has a thrust of 2000 lbs at sea-level and is independent of speed. The wing area is 200 ft^2, its weight is 10,000 lbs, and it has a parabolic drag polar of $C_D = 0.02 + 0.05\,C_L^2$. We want to estimate the time to climb to altitude by two methods.

a) The first method will use a single straight-line rate-of-climb curve fit with the data points calculated at h = 10,000 ft and h = 30,000 ft. Assume a maximum rate-of-climb climb schedule and using this fit, calculate the time-to-climb from sea-level to 10, 20, and 30,000 ft.

b) The second method will use two straight-line approximations to the rate-of-climb curve. The data points to calculate the first straight line segment are h = 0 and h = 15000 ft. The data points for the second straight line segment are at h = 15000 ft and at 30,000 ft.. Again, use the maximum rate-of-climb climb schedule. Using these fits, calculate the time to climb from sea-level to 10, 20 and 30, 000 ft.

a) The strategy here is to find the airspeed and the flight path angle and to calculate the rate of climb from $\dot{h} = V \sin\gamma$

Since the thrust is independent of airspeed, the maximum rate-of-climb lift coefficient is given by:

$$C_L = \frac{-T/W + \sqrt{(T/W)^2 + 12\,C_{D_{0L}}\,K}}{2\,K}$$

At 10,000, $T = T_{SL}\dfrac{\rho}{\rho_{SL}} = 2000\,\dfrac{0.001756}{0.002377} = 1477.5 \text{ lbs}, \quad T/W = 1477.5/10000 = 0.1477$

$$C_L = \frac{-0.1477 + \sqrt{0.1477^2 + 12\,(0.02)\,(0.05)}}{2\,(0.05)} = 0.3619$$

The corresponding drag coefficient is $C_D = 0.02 + 0.05\,(0.3619)^2 = 0.0265$

The corresponding airspeed is:

$$V = \sqrt{\frac{W}{1/2\,\rho\,S\,C_L}} = \sqrt{\frac{10000}{1/2\,(0.002377)\,(200)\,(0.3619)}} = 396.7 \text{ ft/sec}$$

And the flight path angle:

$$\sin\gamma = \frac{T - D}{W} = \frac{T}{W} - \frac{C_D}{C_L} = 0.1477 - \frac{0.0265}{0.3619} = 0.0745$$

The rate of climb at 10,000 ft is

$$\dot{h} = V\sin\gamma = 396.7\,(0.0745) = 29.55 \text{ ft/sec} = 1773 \text{ ft/min}$$

At h = 30,000 ft,

$$T = T_{SL}\frac{\rho}{\rho_{SL}} = 2000\,\frac{0.000890}{0.002377} = 748.84 \text{ lbs}$$ T/W = 748.84/10000 = 0.07488:

The lift coefficient for max R/C at 30,000 ft is given by

$$C_L = \frac{-0.07488 + \sqrt{0.07488^2 + 12\,(0.02)\,(0.05)}}{2\,(0.05)} = 0.5781$$

The corresponding drag coefficient is $C_D = 0.02 + 0.05\,(0.5781)^2 = 0.0367$

The corresponding airspeed is:

$$V = \sqrt{\frac{W}{1/2\,\rho\,S\,C_L}} = \sqrt{\frac{10000}{1/2\,(0.002377)\,(200)\,(0.5781)}} = 440.86 \text{ ft/sec}$$

And the flight path angle:

$$\sin\gamma = \frac{T - D}{W} = \frac{T}{W} - \frac{C_D}{C_L} = 0.0.07488 - \frac{0.0367}{0.5781} = 0.0114$$

The rate of climb at 30,000 ft is

$$R/C = V\sin\gamma = 440.86\,(0.0114) = 5.0258 \text{ ft/sec} = 301.55 \text{ ft/min}$$

Now we can compute the sea-level intercept, and the ceiling intercept:

Define: $h_a = 10000$ ft, $\dot{h}_a = 29.55$ ft/sec $h_b = 30000$ ft $\dot{h}_b - 5.0258$ ft/min

Then:

$$\dot{h}_0 = \frac{h_b\,\dot{h}_a - h_a\,\dot{h}_b}{h_b - h_a} = \frac{30000\,(29.55) - 10000\,(5.0258)}{30000 - 10000} = 41.81 \text{ ft/sec}$$

$$H = \frac{h_a \dot{h}_b - h_b \dot{h}_a}{\dot{h}_b - \dot{h}_a} = \frac{10000\,(5.0258) - (30000)\,(29.55}{5.0258 - 29.55} = 34098.6 \text{ ft}$$

The time to climb can be determined from:

$$t_2 - t_1 - \frac{H}{\dot{h}_0} \ln\left(\frac{H - h_1}{H - h_2}\right) \qquad \text{From sea-level} \quad TOF = \frac{34098.6}{41.81} \ln\left[\frac{34098.6}{34098.6 - h}\right]$$

We get:

Alt	TOF sec	TOF min
0	0	0
10K	283.1	4.72
20K	720.3	12.00
30K	1727.8	28.80

b) Here we need to find the equations of the two straight lines representing the rates of climb. We obtain that using the same procedure as we did previously, here at 0, 15k and 30k ft.

At sea-level: T/W = 0.2

$$C_{L_{\max R/C}} = \frac{-T/W + \sqrt{(T/W)^2 + 12\, C_{D_{0L}} K}}{2\,K} = \frac{-0.2 + \sqrt{(0.2)^2 + 12\,(0.02)\,(0.05)}}{2\,(0.05)} = 0.2804$$

$$C_D = 0.02 + 0.05\,(0.2804)^2 = 0.0239;$$

$$V = \sqrt{\frac{W}{1/2\, \rho\, S\, C_L}} = \sqrt{\frac{10000}{1/2\,(0.0023777)\,(200)\,(0.2804)}} = 387.3 \text{ ft/sec}$$

$$\sin\gamma = \frac{T - D}{W} = T/W - \frac{C_D}{C_L} = 0.2 - \frac{0.0239}{0.2804} = 0.1148$$

$$\dot{h} = V \sin\gamma = 387.3\,(0.1148) = 44.45 \text{ ft/sec} = 2667 \text{ ft/min}$$

At 15000 ft:

$$T = T_{SL} \frac{\rho}{\rho_{SL}} = 200 \frac{0.001497}{0.002377} = 1250.57 \qquad T/W = \frac{1250.57}{2000} = 0.1260$$

The lift coefficient for max rate of climb is then:

$$C_L = \frac{-0.1260 + \sqrt{(0.1260)^2 + 12\,(0.02)\,(0.05)}}{2\;0.05} = 0.4096$$

$$C_D = 0.02 + 0.05\,(0.4096)^2 = 0.0284$$

$$V = \sqrt{\frac{W}{1/2\,\rho\,S\,C_L}} = \sqrt{\frac{10000}{1/2\,(0.001497)\,(200)\,(0.4096}} = 403.84 \text{ ft/sec}$$

$$\sin\gamma = \frac{T}{W}\frac{C_D}{C_L} = 0.1260 - \frac{0.0284}{0.4096} = 0.0567$$

$$\dot{h}_{15K} = V\sin\gamma = 403.84\,(0.0567) = 22.88 \text{ ft/sec} = 1373 \text{ ft/min}$$

We now need to generate the equations of the two lines:

(0 to 15K ft) $h_a = 0$ ft, $\dot{h}_a = 44.45$ ft/sec, $h_b = 15000$ ft, $\dot{h}_b = 22.88$ ft/sec

Then:

$$\dot{h}_{0_1} = \frac{15000\,(44.45) - 15000\,(22.88)}{15000 - 0} = 44.45 \text{ ft/sec}$$

$$H_1 = \frac{0\,(22.88) - 15000\,(44.45)}{22.88 - 44.45} = 30910.99 \text{ ft}$$

15K to 30000+) $h_a = 15000$ ft, $\dot{h}_a = 22.88$ ft/sec, $h_b = 30000$ ft, $\dot{h}_b = 5.0258$ ft/sec

$$\dot{h}_{0_2} = \frac{30000\,(22.88) - 15000\,(5.0258)}{30000 - 15000} = 40.73 \text{ ft/sec}$$

$$H_2 = \frac{15000\,(5.0258) - 30000\,(22.88)}{5.0258 - 22.88} = 34222.37 \text{ ft}$$

We have to use line 1 to calculate times from sea-level to 15000 ft. and use both lines from sea-level to altitudes above 15000 ft. The calculations are as follows:

0 to 15K ft

$$TOF = \frac{\dot{h}_0}{H_1} \ln\left[\frac{H_1 - h_1}{H_1 - h_2}\right] = \frac{30910.99}{44.45} \ln\left[\frac{30910.99}{30910.99 - h}\right]$$

h	TOFsec	TOFmin
0	0	0
10K	271.8	4.52
15K	461.8	7.70

0 to 15000+ ft

$$TOF = TOF_{15K} + \frac{H_2}{\dot{h}_{0_2}} \ln\left[\frac{H_2 - 15000}{H_2 - h}\right] = 461.8 + \frac{34222.37}{40.7342} \ln\left[\frac{34222.37 - 15000}{34222.37 - h}\right]$$

h	TOFsec	TOFmin
15k	461.8	7.70
20K	714.9	11.91
30K	1735.18	28.92