AOE 3140 Vehicle Performance Problem Sheet 1 (ans)

2. At some altitude, the density is 4.100×10^{-4} slugs/ft^3 and the temperature is -60.0 deg F. What is the pressure altitude of this aircraft? What is the density altitude of this aircraft (ft)?
We need to get the pressure at this altitude:

$$\rho = 0.00041 \text{ slugs/ft}^3 \qquad T = -60°\text{F} + 459.688 = 399.688\ °\text{R}$$

We can determine P from the perfect gas law: $P = \rho R T$,

$$P = 0.00041\,(1716.5)\,399.688 = 281.3 \text{ lbs/ft}^2$$

Pressure Altitude: (Using Marchman's Standard Atmosphere)

h_L = 45,000 ft $\qquad$ P_L = 308.0 lbs/ft^2
h_u = 50,000 ft $\qquad$ P_u = 252.2 lbs/ft^2

$$h_{pres} = h_l + \frac{P - P_L}{P_u - P_L}(h_u - h_L)$$

$$= 45{,}000 + \frac{218.311 - 11308.0}{242.2 - 308.0}(50{,}000 - 45{,}000)$$

$$= 45{,}000 + 2028.9$$

$$= 47{,}028 \text{ ft}$$

Density Altitude:

h_L = 45,000 ft $\qquad$ ρ_L = 0.0004605 slugs/ft^3
h_u = 50,000 ft $\qquad$ ρ_u = 0.0003622 `` ``

$$h_\rho = h_L + \frac{\rho - \rho_L}{\rho_u - \rho_L}(h_u - h_L)$$

$$= 45{,}000 + \frac{0.00041 - 0.0004605}{0.0003622 - 0.0004605}(50{,}000 - 45{,}000)$$

$$= 45{,}000 + 2568.7$$

$$= 47{,}568.7 \text{ ft}$$

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3. Assume that the pressure conditions at sea-level are standard, and that the lapse rate is standard, but the sea-level temperature is elevated to 120 degrees (say the in the desert). If we are at a geopotential altitude of 5000 ft, what will be the pressure, density, and (loosely speaking), the temperature altitudes of this aircraft (ft)? (Note, when considering takeoff requirements, density altitude is important. What are the implications of the calculations that you just completed?)

We need to establish conditions at sea-level and then at 5000 ft.

$$P_{SL} = 2116.2 \text{ lbs/ft}^3 \qquad T_{SL} = 120° \text{ F} + 459.688 = 579.688 \text{ °K}$$

@ 5000 ft

$$T_5 = T_1 + K(h - h_1) = 579.688 + (-0.003566)(5000 - 0) = 561.858 \text{ °R}$$

$$P = P_1 \left(\frac{T}{T_1} \right)^{-\frac{g_0}{RK}} = 2116.2 \left(\frac{561.858}{579.688} \right)^{-\frac{32.174}{(1716.5(-0.003566)}} = 2116{,}2\,(0.8486) = 1795.7 \text{ lbs/ft}^2$$

calculate the density: $\rho = \dfrac{P}{R\,T} = \dfrac{1795.7}{(1716.5)(561.858)} = 0.001862 \text{ slugs/ft}^3$

From Standard Atmosphere (Marchman's)

h_L = 4000 ft	P_L = 1822.7 lbs/ft^2	h_L = 8000 ft	ρ_L = 0.001869 slugs/ft^3
h_u = 5000 ft	P_u = 1760.8 `` ``	h_u = 9000 ft	ρ_u = 0.001812 `` ``

$$h_p = h_1 + \frac{P - P_L}{P_u - P_L}(h_u - h_L) = 4000 + \frac{1795.7 - 1827.7}{1760.8 - 1727.7}(1000) = 4478.3 \text{ ft Pressure alt.}$$

$$h_\rho = h_1 + \frac{\rho - \rho_L}{\rho_u - \rho_L}(h_u - h_L) = 8000 + \frac{0.001862 - 0.001869}{0.001812 - 0.001869}(1000) = 8122.8 \text{ ft density alt.}$$

$$T = T_1 + K(h - h_1) = 561.858 = 518.688 + (-0.003566)\,h \quad \text{Standard temperature}$$

h = -12105 ft!!!

4. Determine at what geometric altitude that the error between the geometric altitude and the geopotential altitude is greater than 5% of the geometric altitude. The radius of the Earth is 6378.135 km

We have: $h = \frac{R_e}{R_e + h_G} h_G$. For a 5% error we can write:

$$0.05 = \frac{h_G - h}{h_G} = \frac{h_G - h_G\left(\frac{R_e}{R_e + h_G}\right)}{h_G} = 1 - \frac{R_e}{R_e + h_G} = \frac{R_e + h_G - R_e}{R_e + h_G}$$

$$0.05\,(R_e + h_G) = h_G \quad \Rightarrow \quad 0.05\,R_e = 0.95\,h_G \quad \Rightarrow \quad h_G = 0.0526\,R_e$$

$$h_G = 0.05263\,(6378.135) = 335.7 \text{ m} = 1{,}101{,}378 \text{ ft} = 208.6 \text{ miles}$$

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5. What would be the depth of the atmosphere if air were incompressible with a density equal to that of standard sea-level air? (In ft, miles, km)

We can apply the hydrostatic equation to the atmosphere assuming a constant density, and we will also assume a constant gravity so we will get the geopotential height.

$$\int_{P_{SL}}^{P_{h_{\max}}} dP = -\rho_0\, g_0 \int_0^{h_{\max}} dh$$

or

$$P_{h_{\max}} - P_{SL} = -\rho_0\, g_0\,(h_{\max} - 0) = -\rho_0\, g_0\, h_{\max}$$

But at the edge of the atmosphere, the pressure is = 0, so the above equation becomes:

$$-P_{SL} = -\rho_0\, g_0\, h_{\max} \quad \Rightarrow \quad h_{\max} = \frac{P_{SL}}{\rho_0\, g_0}$$

Putting in some numbers:

$$h_{\max} = \frac{101325}{1.2250\,(9.807)} = 8434.2 \text{ m} = 8.434 \text{ km}$$

$$h_{\max} = \frac{2116.2}{0.002377\,(32.174)} = 27670.8 \text{ ft} = 5.241 \text{ mi}$$

6. The small wind tunnel at Virginia Tech has a 3 ft diameter test section. It is open to the atmosphere so the temperature and pressure of the air in the test section is the same as that of the surrounding atmosphere. If we assume a standard atmosphere, (Blacksburg is at 2100 ft above sea-level), what is the mass rate of air flow (slugs/sec) through the tunnel if it is running at a speed of 60 miles/hr. (88 ft/sec)?

We need to determine the local density. From the standard atmosphere tables (Marchman)

h_L = 2000 ft ρ_L = 0.002242 slug/ft^3
h_u = 3000 ft ρ_u = 0.002177 slug/ft^3

$$\rho = \rho_L + \frac{h - h_L}{h_u - h_L}(\rho_u - \rho_L) = 0.002242 + \frac{2100 - 2000}{3000 - 2000}(0.002177 - 0.002242) = 0.002236 \text{ slugs/ft}^3$$

Calculate the mass flow rate:

$$\dot{m} = \rho A V = (0.002236)\, \pi \frac{3^2}{4}(88) = 1.39 \text{ slugs/ft}^3$$

We can also calculate the volumetric flow rate:

$$A V = \pi \frac{3^2}{4}(88) = 622.0 \text{ ft}^3\text{/sec}$$