

Performance

4. Fluid Statics, Dynamics, and Airspeed Indicators

From our previous brief encounter with fluid mechanics we developed two equations: the one-dimensional continuity equation, and the differential form of Bernoulli's equation. These are repeated here:

Continuity (1-D):

The *general form* of the 1-D continuity equation is:

$$\begin{aligned}\dot{m} &= \text{const} = \rho A V \\ \rho_1 A_1 V_1 &= \rho_2 A_2 V_2\end{aligned}\tag{1}$$

For incompressible flow, $\rho = \text{const} = \rho_1 = \rho_2$, and Eq. (1) reduces to a simpler form.

Incompressible flow 1-D continuity equation:

$$A V = \text{const} = A_1 V_1 = A_2 V_2\tag{2}$$

Differential form of Bernoulli's equation:

$$\frac{dP}{\rho} + V dV + g dh_G = 0\tag{3}$$

We can examine an example of using the continuity equation for incompressible flow.

Example: The Virginia Tech Stability is a low speed facility and hence the air may be treated as an incompressible fluid. The air flow into a 6ft x 6 ft test section from a region of the tunnel (plenum) that is 24 ft x 24 ft. If the air in the plenum is moving at 10 ft/sec, what is the speed of the air in the test section?

Since the flow is considered incompressible, then we can use the incompressible continuity equation to estimate the speed. Let's designate the plenum section as location 1, and the test section as location 2. Since no air leaves the wind tunnel we have:

$$A_1 V_1 = A_2 V_2 = 24^2 (10) = 6^2 (V_2) \quad \Rightarrow \quad V_2 = \frac{5760}{36} = 160 \text{ ft/sec}$$

At the propeller (fan) the tunnel is circular and has a diameter of 15 ft. What is the velocity through the propeller?

Again we can use the continuity equation to get:

$$A_1 V_1 = A_3 V_3 = 24^2 (10) = \pi \left(\frac{15}{2} \right)^2 V_3 \quad \Rightarrow \quad V_3 = 32.59 \text{ ft/sec}$$

Furthermore the amount of air moving through the tunnel is:

$$A V = 24^2 (10) = 6^2 (160) = \pi 7.5^2 (32.59) = 5760 \text{ ft}^3/\text{sec}$$

The Measurement of Pressure:

In the previous example problem, as the air flows around the wind tunnel, there are variations of pressure. We determine how the pressure varies in the next section. Here, we want to discuss some ways that are use to measure the pressure in a wind tunnel or any other air flow situation, such as flow over a wing surface. We measure the pressure of any flow over a surface (such as a wing surface, or wind tunnel wall) by making a hole perpendicular to that surface, called a **pressure tap**, and measuring the pressure in that hole. The resulting pressure is called the **static pressure** of the fluid at that point.

Manometers

One way to measure pressure is to use a manometer. A manometer is a U tube filled with a liquid (such as mercury, water, or alcohol). One end of the tube is exposed to the atmosphere, and the other end is attached, using flexible tubing to the pressure tap. The difference in pressure will cause the fluid to rise in one leg of the U (the lower pressure side) and drop in the other (the higher pressure side). The difference in the heights of the two sides is supported by the pressure difference. This pressure difference is related to the height of the fluid by **the hydrostatic equation** (Bernoulli's equation with velocity equal to zero):

$$dP = -\rho g dh \tag{4}$$

From Eq. (4) we can see that for any given (non-moving) fluid, the pressure at any level, h, is the same. Hence the differences in the levels of the fluid in the U tube can be related directly to the difference in pressure by Eq. (4). Since the fluids in manometers are incompressible, and gravity may be considered constant over the length of the manometer, we can easily integrate Eq. (4) to get:

$$P_2 - P_1 = -\rho_f g (h_2 - h_1) \tag{5}$$

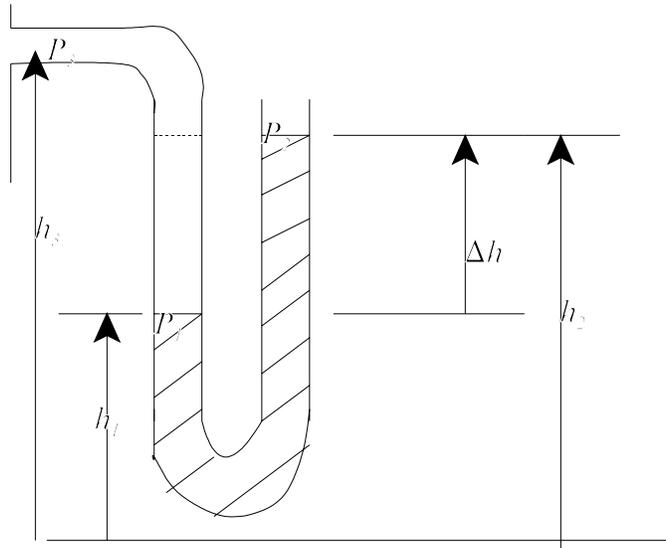
where ρ_f = density of the fluid in manometer

P_1 = pressure at level h_1

P_2 = pressure at level h_2

The figure at the right represents a pressure tap at point 3 attached to a U tube manometer that is open to the atmosphere, P_2 . Hence P_3 is the static pressure in the wind tunnel, P_2 is atmospheric pressure, and P_1 is some intermediate pressure (that we will show shortly is approximately equal to P_3).

Here we can apply the hydrostatic equation twice, once to the column of air between P_3 and P_1 , once again to the column of manometer fluid between P_1 and P_2 .



Just applying the equation strictly as it is written, we have:

$$P_3 - P_1 = -\rho_a g (h_3 - h_1)$$

$$P_1 - P_2 = -\rho_w g (h_1 - h_2)$$

where $\rho_a =$ density of tunnel fluid (in this case air).

$\rho_w =$ density of manometer fluid (in this case water).

If we combine the equations we can determine the tunnel pressure from:

$$P_3 - P_2 = -\rho_a g (h_3 - h_1) + \rho_w g (h_2 - h_1)$$

This equation tells us the difference between atmospheric pressure (P_2) and the tunnel pressure, (P_3). However, if we look at the equation carefully, we can see that if the tunnel fluid is air, the density of air ρ_a is much, much less than the density of the manometer fluid, ρ_w . Assuming the manometer tubes are not too long, $(h_3 - h_1)$ is of order $(h_2 - h_1)$, then we can neglect the first term and arrive at the required result. It is equivalent to assuming that $P_1 = P_3$, so we have:

$$P_3 - P_2 = -\rho_w g (h_1 - h_2) \tag{6}$$

Specific Gravity (σ_{sg})

The density of a fluid divided by the density of an equivalent amount of water is called the specific gravity, or

$$\rho_{fluid} = \sigma_{sg} \times \rho_w \quad (7)$$

Example:

A mercury barometer works by putting mercury into a closed tube and inverting it and putting the open end in a reservoir of mercury. Hence the pressure on the reservoir surface is atmospheric, and the pressure on the upper surface of the column of mercury is zero since it is in a vacuum. If we designate the surface of the reservoir as point 1, and the upper surface of the mercury column as point 2, we can write:

$$P_2 - P_1 = -\rho_{hg} g (h_2 - h_1)$$

We would like to find, the barometric reading for the standard atmosphere at sea-level. For our problem, $P_2 = 0$ since it is a vacuum, (we will assume it is a vacuum) and the above equation becomes:

$$-P_1 = -P_{atm} = -\rho_{hg} g (h_2 - h_1)$$

The specific gravity of mercury is 13.598. If we use US customary units we have:

$$2116.217 \text{ lbs.ft}^2 = 13.598 (1.940) (32.174 \text{ lbs/ft}^3) (h_2 - h_1)$$

$$(h_2 - h_1) = 2.4933 \text{ ft} = 29.92 \text{ inches of hg} = 0.0.760 \text{ m}$$

Hence the “pressure” at sea-level in a standard atmosphere is designated as 29.92 inches of mercury. To get the real pressure you need to convert that number to feet, and then multiply by the “weight density” of mercury (or the specific gravity times the density of water times the gravitational constant).

Some specific gravities of typical manometer fluids are:

Water	1.00	And in case your wondering:	
Mercury	13.595	Ice	0.92
Ethyl Alcohol	0.81	Lead	11.3
Benzene	0.8846	Platinum	21.4
Gasoline	0.68	Crude Oil	0.87

The density of water is: US - 1.940 slugs/ft³ SI - 1000 kg/m³

Integrating Bernoulli's Equation

The differential form of Bernoulli's equation is given by Eq. (3). We would now like to apply that equation to aerodynamic flow problems. In this case V will not be zero, and in fact can be quite large. There are three terms in the equation, dP/ρ , $V dV$, $-g dh_G$. Since we know g as a function of h_G as established in the discussion of the atmosphere, the last two terms can be integrated directly. However, the first cannot be integrated until we know how the density varies with pressure. Consequently there must be some relation $\rho(P)$ that is known before we can integrate the equation.

We realize that in many cases of interest, we may be dealing with relatively high speeds with little change in altitude. Recall that the last time we used this equation we had zero velocity and huge changes in altitude! Hence the last term was large because of the large changes in altitude. With small changes in altitude, up to several tens of meters, the last term in the equation can be ignored. Hence we have the reduced form of the differential form of Bernoulli's equation:

$$\frac{dP}{\rho} + V dV = 0 \quad (8)$$

We can integrate this equation under assumptions associated with two special cases. The first somewhat restrictive, and the second less so.

Case 1 - Special Case - Incompressible Flow

The most common form of the integrated Bernoulli's equation is for the special case of incompressible flow. Under this assumption, $\rho = \text{constant}$, and Eq. (8) is easily integrated (thus the popularity of this case). We should also remember that Eq. (3) and hence Eq. (8) was derived for flow along a streamline so that Bernoulli's equation is restricted to that situation. All that being said, if the density is assumed constant, then the above equation is integrated to give:

Incompressible Bernoulli's Equation

$$\begin{aligned} \frac{P}{\rho} + \frac{V^2}{2} &= \text{constant} \\ P + \frac{1}{2} \rho V^2 &= P_0 = \text{constant} \\ P_1 + \frac{1}{2} \rho V_1^2 &= P_2 + \frac{1}{2} \rho V_2^2 = P_0 \end{aligned} \quad (9)$$

Assumptions:

1. Inviscid (no viscosity) - since only forces due to pressures were considered
2. Incompressible flow - $\rho = \text{const}$
3. Flow along a streamline
4. Steady flow

Definitions:

1. The pressure $P =$ Static pressure
2. The pressure $P_0 =$ Stagnation pressure
3. The quantity $\frac{1}{2} \rho V^2 =$ dynamic pressure $\equiv \bar{q}$

Definition (3) holds for all flow regimes, incompressible or compressible.

Finally we can note that if all the streamlines in the flow originate from some common flow condition, then Bernoulli's equation holds *throughout the flow*.

Case 2 - Special Case - Compressible Flow (Isentropic - Adiabatic)

In order to integrate Eq. (8) for the case where the density is not constant, we must determine how the density changes with pressure. If we assume certain conditions on the flow, such as 1) no viscosity (friction), called an isentropic process, and 2) no heat added or taken away, called an adiabatic process, then a rule under which the fluid variables behave is given by the equation:

$$\frac{P}{\rho^\gamma} = \text{constant} = C \tag{10}$$

where $P =$ pressure
 $\rho =$ density
 $\gamma =$ ratio of specific heats = 1.4 for air

It is only important to know that such a rule exists, and that it is a good approximation of how the pressure and density are related. If we solve for the density we get:

$$\rho = \left(\frac{P}{C} \right)^{\frac{1}{\gamma}} \tag{11}$$

We can now substitute Eq. (11) into Eq. (8) so that we can integrate it.

$$C^{\frac{1}{\gamma}} \frac{dP}{P^{\frac{1}{\gamma}}} + V dV = 0$$

This equation is easily integrated to obtain the following:

$$C^{\frac{1}{\gamma}} \frac{P^{1-\frac{1}{\gamma}}}{1-\frac{1}{\gamma}} + \frac{V^2}{2} = \text{constant}$$

$$C^{\frac{1}{\gamma}} \frac{P^{\frac{\gamma-1}{\gamma}}}{\frac{\gamma-1}{\gamma}} + \frac{V^2}{2} = \text{constant}$$

We can note that $C^{\frac{1}{\gamma}} = \frac{P^{\frac{1}{\gamma}}}{\rho}$, so that $C^{\frac{1}{\gamma}} P^{\frac{\gamma-1}{\gamma}} = \frac{P^{\frac{1}{\gamma}}}{\rho} P^{1-\frac{1}{\gamma}} = \frac{P}{\rho}$. When the smoke clears, we have the following integrated form of Bernoulli's equation:

Compressible Bernoulli's Equation

$$\begin{aligned} \frac{\gamma}{\gamma-1} \frac{P}{\rho} + \frac{V^2}{2} &= \text{constant} \\ \frac{\gamma R T}{\gamma-1} + \frac{V^2}{2} &= \text{constant} \\ \frac{\gamma R T}{\gamma-1} + \frac{V^2}{2} &= \frac{\gamma R T_0}{\gamma-1} \end{aligned} \tag{12}$$

Assumptions:

1. Non-viscous
2. Adiabatic, Isentropic
3. Steady flow

Definitions: The temperature T = local temperature (absolute)
The temperature T_0 = stagnation temperature (temperature when the fluid is slowed to rest)

Speed of Sound

Equation (12) takes on an interesting form if we introduce the concept of the speed of sound. Without proof, the speed of sound can be shown to be given by: $a^2 = \frac{dP}{d\rho}$. If we use the same adiabatic process that we used to get Eq. (12) it is easy to show that the speed of sound is determined from:

$$a^2 = \frac{\gamma P}{\rho} = \gamma R T \quad (13)$$

We can note that the *speed of sound depends only on the temperature!*

For example, in a standard sea-level atmosphere, the speed of sound is determined from:

$$a = \sqrt{\gamma R T} = 1.4 (287.05) (288.16) = 340.36 \text{ m/s} = 1116.7 \text{ ft/sec}$$

If we introduce the speed of sound expressions into the compressible Bernoulli's equation, we can perform the following operations:

$$\begin{aligned} \frac{\gamma R T}{\gamma - 1} + \frac{V^2}{2} &= \frac{\gamma R T_0}{\gamma - 1} \\ \frac{a^2}{\gamma - 1} + \frac{V^2}{2} &= \frac{a_0^2}{\gamma - 1} \\ \frac{1}{\gamma - 1} + \frac{1}{2} \frac{V^2}{a^2} &= \frac{a_0^2}{a^2} \frac{1}{\gamma - 1} \\ 1 + \frac{\gamma - 1}{2} M^2 &= \frac{a_0^2}{a^2} = \frac{T_0}{T} \end{aligned} \quad (14)$$

where a = local speed of sound
a₀ = speed of sound in stagnation region
T = local temperature
T₀ = stagnation temperature
V = local airspeed
γ = 1.4 (ratio of specific heats)

Definition: Mach number = the local airspeed over the local speed of sound: $M = \frac{V}{a}$.

Essentially, Eq.(14) gives us the temperature distribution in a compressible flow with Mach number. We need now to determine the pressure and density distributions with Mach number. To do this we need to use the process equation, Eq. (10), and the perfect gas law, $P = \rho RT$. The fundamental equation to arrive at all these results is given by

$$\frac{P}{\rho^\gamma} = \frac{P_0}{\rho_0^\gamma} \quad \Rightarrow \quad \frac{P_0}{P} = \frac{\rho_0^\gamma}{\rho^\gamma} = \frac{\rho_0 R T_0}{\rho R T}$$

Then

$$\frac{T_0}{T} = \left(\frac{\rho_0}{\rho} \right)^{\gamma-1} \quad \Rightarrow \quad \frac{\rho_0}{\rho} = \left(\frac{T_0}{T} \right)^{\frac{1}{\gamma-1}}$$

and

$$\frac{P_0}{P} = \left(\frac{T_0}{T} \right)^{\frac{\gamma}{\gamma-1}}$$

Putting it all together, we have the equations for compressible flow:

Equations for Compressible Flow

$$\begin{aligned} \frac{T_0}{T} &= 1 + \frac{\gamma-1}{2} M^2 \\ \frac{\rho_0}{\rho} &= \left(1 + \frac{\gamma-1}{2} M^2 \right)^{\frac{1}{\gamma-1}} \\ \frac{P_0}{P} &= \left(1 + \frac{\gamma-1}{2} M^2 \right)^{\frac{\gamma}{\gamma-1}} \end{aligned} \quad (15)$$

The first equation above is the integrated compressible Bernoulli's equation. The remaining equations relate pressure and density to the temperatures determined from Bernoulli's equation. The last equation (for the pressure) is the one that should mostly resemble the incompressible Bernoulli's equation.

We can examine that idea by expanding the pressure equation in a binomial series:

$$(1 + x)^n = 1 + nx + n(n-1)\frac{x^2}{2} + n(n-1)(n-2)\frac{x^3}{3!} + \dots$$

If we apply the binomial series to the pressure equation, we can determine the following:

$$P_0 = P + \frac{1}{2} \rho V^2 \left[1 + \frac{M^2}{4} + \frac{2-\gamma}{24} M^4 + \frac{(2-\gamma)(3-2\gamma)}{192} M^6 + \dots \right] \quad (16)$$

The terms in the square brackets together are considered the **compressibility factor**. Hence for the value of $M = 0$, the compressibility factor is 1, and the equation reduces to the incompressible Bernoulli's equation. For $M = 1$, the compressibility factor becomes 1.276. Hence the incompressible Bernoulli's equation would give a 27.6% error if it were used. For the value $\gamma = 1.4$, the above equation has the values:

$$P_0 = \frac{1}{2} \rho V^2 \left[1 + \frac{M^2}{4} + \frac{M^4}{40} + \frac{M^6}{1600} + \dots \right] \quad (16a)$$

Example: (High speed subsonic flow)

An aircraft is flying at sea-level at a speed of $M = 0.8$.

- a). Determine the speed of the aircraft.
- b) Determine the actual difference between the stagnation and static pressure sensed by the aircraft.
- c) Determine the speed of the aircraft based on incompressible flow with the same pressure difference.

a) $V = a M$, where a is the speed of sound.

$$a = \sqrt{\gamma R T} = \sqrt{1.4 (287.05) 288.16} = 340.36 \text{ m/s} = 1116.7 \text{ ft/sec}$$

$$V = (1116.7) 0.8 = 893.3 \text{ ft/sec}$$

$$= (340.36) 0.8 = 272.3 \text{ m/s}$$

b) Pressure difference: First, compute the pressure ratio:

$$\begin{aligned}
\frac{P_0}{P} &= \left(\frac{T_0}{T} \right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \frac{\gamma-1}{2} M^2 \right)^{\frac{\gamma}{\gamma-1}} \\
&= \left[1 + \frac{1.4-1}{2} (0.8^2) \right]^{\frac{1.4}{1.4-1}} \\
&= [1 + 0.2 (0.8^2)]^{3.5} \\
&= 1.5243
\end{aligned}$$

We can determine the pressure difference from:

$$\frac{P_0}{P} - 1 = \frac{P_0 - P}{P} = 1.5243 - 1 = 0.5243$$

$$\begin{aligned}
P_0 - P &= (0.5243) P = (0.5243) 2116.2 \quad \text{or} \quad (0.5243) 1.01325 \times 10^5 \\
&= 1109.5 \text{ lbs/ft}^2 \qquad \qquad \qquad 53124.7 \text{ Pa}
\end{aligned}$$

c) In this part it is assumed that we measure the pressure difference and that would be the true pressure difference we just calculated. However, we will calculate the airspeed assuming the flow is incompressible and use the incompressible form of Bernoulli's equation:

$$P_0 = P + \frac{1}{2} \rho V^2 \quad \Rightarrow \quad P_0 - P = \frac{1}{2} \rho V^2$$

then, substituting:

$$1109.5 = \frac{1}{2} (0.002377) V^2$$

$$V = 966.2 \text{ ft/sec} = 294.5 \text{ m/s}$$

compared to the actual airspeed of 893.3 ft/sec or 272.3 m/s or about an 8.2 % error!

(Note that if we calculated the pressure difference using the correct airspeed and density using the incompressible Bernoulli's equation, we would have encountered a 17 % error (too low) in the pressure difference, see Eq. (16a))

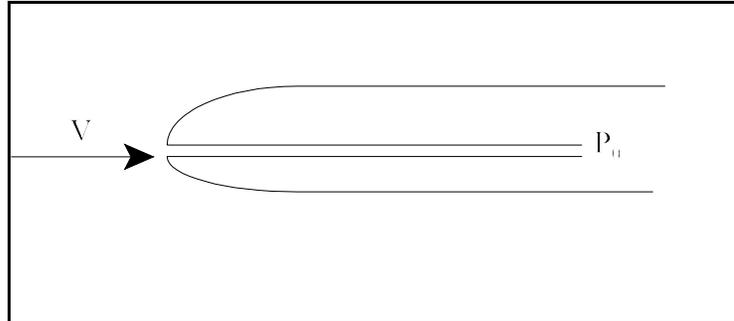
Measurement of Airspeed

The problem we wish to deal with now is how to measure airspeed in a wind tunnel or in an aircraft. Furthermore, in an aircraft the information available may not be the same as it is in the wind tunnel. In addition, we must consider what information is most useful to a pilot. . One must consider the sensors required to measure airspeed and what sensors are available in the different environments.

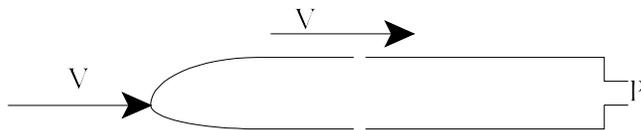
General Comments :

As can be seen from previous work, two key ingredients for measuring airspeed are the total (or stagnation) pressure, P_0 , and the static pressure, P . These pressures can be measured using a total pressure tube, or pitot tube. A pitot tube is a tube with a hole in its end that is aligned with the flow so the flow the strikes the end of the tube is brought to rest at the hole, and the pressure recorded at the hole will be total pressure.

A sketch of a pitot tube is shown to the right. The end of the center tube is attached to a pressure sensor and it will read the pressure P_0 since the flow will come to rest at the tip of the tube. These tube can be observed to be located at various points on different aircraft. In flight-test aircraft it is usually located at the nose on an “instrumentation boom.” On typical general aviation aircraft it is located on the outboard of the wing so as not to be in the propeller wash, and on jet propelled aircraft, it can generally be found on the side of the fuselage or on the top of the vertical tail, again, out of the region of jet wash or other jet effects.



Static pressure is measured by putting a pressure tap in a surface parallel to the flow. One way to do this is to use a static tube. A static tube is shaped like a pitot tube, but the pressure taps are along the side, rather than at the front. Here the flow is still moving at the free stream speed and the pressure will be the static pressure. In an aircraft, the static pressure taps can be located at points along the fuselage. In fact one of the pre-flight inspection checklist item is to be sure the static pressure ports are not clogged or obstructed. These pressure taps are the source of the static pressure for airspeed measurement, and for the altimeter discussed previously. Generally, as indicated in the drawing, the pressure taps are located on both sides (actually all around) the tube and on both sides of the fuselage. The reason for these locations is to account for any misalignment of the tube (or fuselage) with the wind.



Finally, in wind tunnel applications, it is convenient to combine the pitot tube with the

static tube to provide a pitot-static tube with a hole in the front to measure the total pressure, and holes around the side to measure the static pressure. We can make use of the total and static pressure to estimate the airspeed. How we do this depends on some assumptions that we make.

Incompressible Flow

For flow at Mach numbers in the range of $M < 0.3$, the errors in the total and static pressure difference is approximately 2% and can be neglected. Under these conditions, the flow may be considered incompressible, and we can use the incompressible form of Bernoulli's equation to determine the airspeed.

$$P_0 = P + \frac{1}{2} \rho V^2 \quad (17)$$

From which we can solve for V:

$V = \sqrt{\frac{2(P_0 - P)}{\rho}} \quad (18)$

V in this equation is called the ***true airspeed***.

In order to measure the true airspeed for low speed flow, we need the difference in the stagnation and static pressure and the density. We can generally get the density from measuring the temperature, and using the perfect gas law:

$$\rho = \frac{P}{RT} \quad (19)$$

In the wind tunnel we can generally use a pitot-static tube to measure P_0 and P and a temperature sensor to measure T. We can then compute the airspeed.

Incompressible Airspeed Indicator

In an aircraft, we generally attach the pitot tube and the static port to a pressure sensor so that the output of the sensor is the ***difference of the two pressures***. This difference is what the airspeed indicator receives, and that is all! So how is the airspeed determined, since we don't have T? The airspeed indicator is calibrated as if the density is sea-level density. So that what the airspeed indicator reads is airspeed that would give the same pressure difference at sea-level. Hence we have:

Incompressibly Calibrated Airspeed Indicator

$$V_{cal_{inc}} = \sqrt{\frac{2(P_0 - P)}{\rho_{SL}}} \quad (20)$$

Definition: Equivalent Air Speed - Equivalent air speed is defined by the following equation. The definition holds for *all flight regimes*, low speed, high speed and supersonic.

$$\bar{q} = \frac{1}{2} \rho_{SL} V_{eq}^2 = \frac{1}{2} \rho V^2 \quad \Rightarrow \quad V_{eq} = \sqrt{\frac{\rho}{\rho_{SL}}} V = \sqrt{\sigma} V \quad (21)$$

where:

\bar{q}	=	dynamic pressure
ρ_{SL}	=	density at standard sea-level conditions
ρ	=	local density
V	=	true airspeed
V_{eq}	=	equivalent airspeed
σ	=	ratio of density over standard sea-level density (often tabulated in standard atmosphere tables)

Example:

An aircraft is flying at 3000 m and has a true airspeed of 120 kts. What is the reading observed on the airspeed indicator? What is the dynamic pressure, the static pressure and the total pressure?

$$V = 120 \text{ kts} \times \frac{0.5144 \text{ m/s}}{1 \text{ kt}} = 61.728 \text{ m/s}$$

$$\begin{aligned} V_{cal_{inc}} &= V_{eq} = \sqrt{\sigma} V \\ &= \sqrt{\frac{0.9090}{1.2250}} (61.728) \text{ m/s} \\ &= (0.8614) (61.728) \\ &= 53.1736 \text{ m/s} = 103.4 \text{ kts} \end{aligned}$$

The *dynamic pressure* is given by:

$$\bar{q} = \frac{1}{2} \rho V^2 = \frac{1}{2} \rho_{SL} V_{eq}^2 = \frac{1}{2} 0.9090 (61.728)^2 = 1731.8 \text{ Pa} = 36.2 \text{ lbs/ft}^2$$

Note that at these low speeds, the dynamic pressure is \ll than the static pressure at 3000 m (70,100 Pa, or 1,464 lbs/ft²)

The *total or stagnation pressure* is determined from Bernoulli's equation:

$$P_0 = P + \frac{1}{2} \rho V^2 = P + \bar{q} = 70,100 + 1731.8 = 71831.8 \text{ Pa}$$

Compressible Airspeed Indicator

In order to have an airspeed indicator that accounts for compressibility effects at high subsonic Mach number we need to use the compressible form of Bernoulli's equation. Recall:

$$\begin{aligned} \frac{P_0}{P} &= \left[1 + \frac{\gamma - 1}{2} M^2 \right]^{\frac{\gamma}{\gamma - 1}} = [1 + 0.2 M^2]^{3.5} \\ M^2 &= \frac{2}{\gamma - 1} \left[\left(\frac{P_0}{P} \right)^{\frac{\gamma - 1}{\gamma}} - 1 \right] = 5 \left[\left(\frac{P_0}{P} \right)^{0.2871} - 1 \right] \\ V^2 &= \frac{2 a^2}{\gamma - 1} \left[\left(\frac{P_0}{P} \right)^{\frac{\gamma - 1}{\gamma}} - 1 \right] \\ &= \frac{2 a^2}{\gamma - 1} \left[\left(\frac{P_0 - P}{P} + 1 \right)^{\frac{\gamma - 1}{\gamma}} - 1 \right] \end{aligned} \quad (22)$$

Hence *true airspeed is given by*: (numbers for $\gamma = 1.4$)

$$V^2 = \frac{2 a^2}{\gamma - 1} \left[\left(\frac{P_0 - P}{P} + 1 \right)^{\frac{\gamma - 1}{\gamma}} - 1 \right] = 5 a^2 \left[\left(\frac{P_0 - P}{P} + 1 \right)^{0.2857} - 1 \right] \quad (23)$$

Unfortunately, only $P_0 - P$ is available to the airspeed indicator. We now define the calibrated airspeed by using the above equation with the value of the speed of sound, and the lone pressure in the denominator defined at sea-level:

$$V_{cal_{comp}}^2 = \frac{2 a_{SL}^2}{\gamma - 1} \left[\left(\frac{P_0 - P}{P_{SL}} + 1 \right)^{\frac{\gamma - 1}{\gamma}} - 1 \right] = 5 a_{SL}^2 \left[\left(\frac{P_0 - P}{P_{SL}} + 1 \right)^{0.2857} - 1 \right] \quad (23)$$

Calibrated airspeed for compressible subsonic flow ($M > 0.3$).

Example

An aircraft is at a pressure altitude of 10 km altitude where the temperature is measured to be 230 deg K. The stagnation pressure is measured to be 4.24×10^4 N/m². Find the true airspeed, calibrated airspeed, and equivalent airspeed.

At a pressure altitude of 10k, the static pressure is $P = 26420$ N/m². We can calculate the Mach number from Eq. (22):

$$\begin{aligned} M^2 &= \frac{2}{\gamma - 1} \left[\left(\frac{P_0}{P} \right)^{\frac{\gamma - 1}{\gamma}} - 1 \right] \\ &= 5 \left[\left(\frac{4.24 \times 10^4}{2.642 \times 10^4} \right)^{0.2857} - 1 \right] \\ &= 0.7235 \\ M &= 0.85 \end{aligned}$$

Since it is not a standard atmosphere, (temperature is given), we must calculate the speed of sound using the given temperature:

$$a = \sqrt{\gamma R T} = \sqrt{(1.4) 287.05 (230)} = 304.02 \text{ m/s}$$

$$V = a M = 304.02(0.85) = 258.4 \text{ m/s}$$

Calibrated airspeed

In order to compute the calibrated airspeed, we need the sea-level speed of sound, and pressure. These are $a = \sqrt{\gamma R T} = \sqrt{1.4 (287.05) 288.16} = 340.3$ m/s, and 1.01325×10^5 N/m².

Then:

$$\begin{aligned}
 V_{cal_{comp}}^2 &= \frac{2 a_{SL}^2}{\gamma - 1} \left[\left(\frac{P_0 - P}{P_{SL}} + 1 \right)^{\frac{\gamma - 1}{\gamma}} - 1 \right] \\
 &= 5 (340.36)^2 \left[\left(\frac{4.25 \times 10^4 - 2.642 \times 10^4}{1.01325 \times 10^5} + 1 \right)^{0.2857} - 1 \right] \\
 &= 24739.63 \\
 V_{cal_{comp}} &= 157.3 \text{ m/s}
 \end{aligned}$$

Equivalent airspeed is given by $V_{eq} = \sqrt{\sigma} V$. Hence we need to know the density. Since it is not a standard atmospheric condition, we must calculate it from the pressure and temperature:

$$\begin{aligned}
 \rho &= \frac{P}{R T} = \frac{2.642 \times 10^4}{(287.05) 230} = 0.4002 \text{ kg/m}^3 \\
 V_{eq} &= \sqrt{\frac{0.4002}{1.2250}} (258.4) = 147.7 \text{ m/s}
 \end{aligned}$$

Calibrated_{comp} airspeed gives a reading closely related to (by not exactly equal to) equivalent airspeed (it is equal to equivalent airspeed for an incompressibly calibrated airspeed indicator).

Supersonic Airspeed Indicator

Although the above equations hold for supersonic flow, $M > 1$, they cannot be used to measure airspeed. The reason is that as the supersonic flow comes to rest at the tip of the pitot tube, a shock wave forms so that the assumptions used to derive the high speed flow equations above are no longer valid. We can, however develop equations that can be used across a shock wave, after which the above equations can work. If we combine these two sets of equations, we can come up with the a relation between the total pressure after the shock, P_{02} , the static pressure, P , and the Mach number, M . This relation is called the Rayleigh Pitot tube formula:

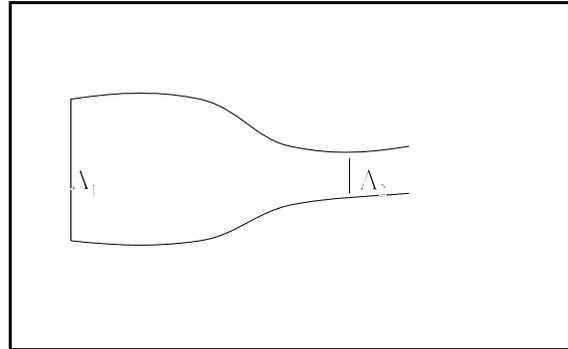
Rayleigh Pitot Tube Formula (Supersonic flow)

$$\frac{P_{02}}{P} = \left[\frac{(\gamma + 1)^2 M^2}{4 \gamma M^2 - 2(\gamma - 1)} \right]^{\frac{\gamma}{\gamma - 1}} \left(\frac{1 - \gamma + 2 \gamma M^2}{\gamma + 1} \right) \quad (23)$$

where P_{02} = stagnation pressure after the shock wave
 P = static pressure (same before or after shock wave)
 M = Mach number before the shock wave

Measuring Airspeed in a Subsonic Low-Speed Wind Tunnel

We can measure the airspeed in a low speed wind tunnel in several different ways. One way is to insert a pitot-static tube directly into the airstream and apply the incompressible form (for low speed, and the compressible form (for high subsonic speed) of Bernoulli's equation. However, another method is often used to measure speed in **low-speed** wind tunnels that does not require any instrument to be inserted into the flow. This method makes use of the incompressible form of Bernoulli's equation, and the incompressible form of the continuity equation:



$$P_1 + \frac{1}{2} \rho V_1^2 = P_2 + \frac{1}{2} \rho V_2^2 \quad (24)$$

$$A_1 V_1 = A_2 V_2$$

Here the points 1, and 2 refer to two different locations in the wind tunnel.

The location of point 1 is up stream of the test section in the settling or plenum chamber of the wind tunnel, and the location of point 2 is in the test section of the wind tunnel where the airspeed is to be determined. We can measure the static pressure at each of these locations by putting a pressure tap in the wall of the wind tunnel. Then by knowing the pressure difference, the area ratios and properties of the fluid, we can determine the speed in the test section. From the continuity equation, we have:

$$V_1 = \frac{A_2}{A_1} V_2$$

If we substitute this expression for V_1 back into Bernoulli's equation we obtain:

$$P_1 - P_2 = \frac{1}{2} \rho [V_2^2 - V_1^2] = \frac{1}{2} \rho V_2^2 \left[1 - \left(\frac{A_2}{A_1} \right)^2 \right] \quad (25)$$

or equivalently:

$$V_2 = \sqrt{\frac{2(P_1 - P_2)}{\rho \left[1 - \left(\frac{A_2}{A_1} \right)^2 \right]}} \quad (26)$$

Example

In a low-speed wind tunnel, the contraction ratio (area of plenum over area of test section) is 3. What would be the pressure difference ($P_1 - P_2$) in inches of water between the pressure in the plenum and the pressure in the test section, that is required to maintain a dynamic pressure of 4 inches of water in the wind tunnel?

From Bernoulli's equation we have for the dynamic pressure:

$$\begin{aligned} P_0 - P_2 &= \frac{1}{2} \rho V_2^2 = \rho_w g (h_0 - h_2) \\ &= 1.94 \frac{\text{slugs}}{\text{ft}^3} (32.174 \text{ ft/sec}^2) 4 \text{ in} \frac{1}{(12 \text{ ft/in})} \\ &= 20.80 \text{ lbs/ft}^2 \end{aligned}$$

Then for the wind tunnel we have:

$$\begin{aligned} P_1 - P_2 &= \frac{1}{2} \rho V_2^2 \left[1 - \left(\frac{A_2}{A_1} \right)^2 \right] \\ &= 20.80 [1 - (1/3)^2] \\ &= 18.49 \text{ lbs/ft}^2 \end{aligned}$$

The manometer reading then is given by:

$$|P_1 - P_2| = \rho_w g |\Delta h| \quad \Rightarrow \quad \Delta h = \frac{|P_1 - P_2|}{\rho_w} g = \frac{18.49}{1.94 (32.174)} = 0.296 \text{ ft}$$

$$\Delta h = 0.269 \text{ ft} \times 12 \frac{\text{in}}{\text{ft}} = 3.55 \text{ in } H_2O$$

Hence if we set the tunnel manometer attached to the plenum and the test section to 3.55 inches of water, we will achieve a dynamic pressure in the wind tunnel of "4 inches of water" or 20.80 lb/ft²