Performance

7. Aircraft Performance -Basics

In general we are interested in finding out certain performance characteristics of a vehicle. These typically include: how fast and how slow an aircraft can fly, rate-of-climb, how high (ceiling), its range, endurance, take-off and landing distances, etc. In addition, we are interested in how changes in the aircraft parameters would affect these performance measures. We can get some of these answers by using a mathematical model of the vehicle of interest, and applying various analysis techniques. The first item of business is to get some mathematical model at some (possibly crude) level. Previously we developed a point-mass model of an aircraft because we were primarily interested in the effects of the forces on the vehicle. These differential equations of motion took the form:

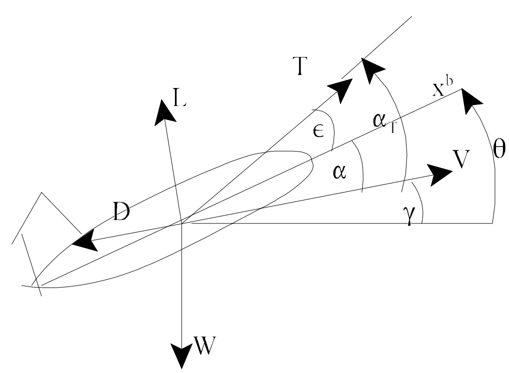
$$\begin{aligned} T\cos\alpha_T - D - W\sin\gamma &= m\dot{V} \\ T\sin\alpha_T + L - W\cos\gamma &= m\frac{V^2}{r} = mV\dot{\gamma} \end{aligned} \qquad (1)$$

and

$$\begin{aligned} V\sin\gamma &= \dot{h}_G \\ V\cos\gamma &= \dot{x}^g = \dot{R} \end{aligned} \qquad (2)$$

The first set of equations are an application of Newton's Laws, while the second set are called kinematic relations or trajectory equations. These variables and others are noted in the following figure:

D	=	drag
L	=	lift
T	=	thrust
V	=	Velocity
W	=	weight
α	=	angle-of-attack (AOA)
α_T	=	thrust AOA
ϵ	=	thrust angle relative to reference line
γ	=	flight-path angle
θ	=	aircraft pitch angle



In order to simplify the problem greatly, we usually assume (at least to start) that the thrust is aligned with the velocity vector, i.e. $\alpha_T = 0$, or is small and can be neglected. Under these assumptions, the equations-of-motion can be simplified to:

$$\begin{aligned} T - D - m g \sin\gamma &= m \dot{V} \\ L - m g \cos\gamma &= m V \dot{\gamma} \\ V \cos\gamma &= \dot{x} \\ V \sin\gamma &= \dot{h} \end{aligned} \tag{3}$$

We can note another subtle change here in that $\dot{h}_G = \dot{h}$. That is we are neglecting the variation of gravity with altitude and assuming geopotential and geometric altitude are the same. In order to deal with these equations and put in the form useful for simulation on a computer we put all the derivatives on one side to get:

$$\begin{aligned} \dot{V} &= \frac{T - D}{m} - g \sin\gamma \\ \dot{\gamma} &= \frac{L}{m V} - \frac{g \cos\gamma}{V} \\ \dot{h} &= V \sin\gamma \\ \dot{x} &= V \cos\gamma \end{aligned} \tag{4}$$

Equations (4) represent a set of ordinary differential equations that are nonlinear, or what are usually referred to as nonlinear ODEs. In order to obtain a solution to these equations, we must only have the four variables *V, γ, h,* and *x* in them. However we observe a T, a D, and an L, in addition to a γ and a V that appear explicitly. To solve these we need to establish how these functions (T, D, and L, depend on *V, γ, h* and *x*, and/or time.

By a solution to Eq. (4), we want to establish how the dependent variables, *V, γ, h* and *x* depend on the independent variable, time (t) and are called ***state variables***. Hence a solution is $V = f_1(t),\ \gamma = f_2(t),\ h = f_3(t),\ \text{and}\ x = f_4(t)$. That is, a solution consists of determining the time histories of these four variables. In general, we are unable to find a solution to these equations for most realistic functions for T, D, and L. We can, however obtain numerical solutions to these equations in either a sophisticated or in a crude way. Here we will opt for crude. Before we do that, we will introduce a definition and some functions for drag and thrust.

Definition : Load Factor (n) - The load factor is defined as the lift to weight ratio

$$\boxed{n = \frac{L}{W}} \quad \text{Usually designated in "g"} \tag{5}$$

Hence we usually fly level in 1g flight, or do a 2g pull up.

With this definition we can rewrite the Eqs. (4) as:

$$\begin{aligned} \dot{V} &= \frac{T - D}{m} - g\sin\gamma \\ \dot{\gamma} &= \frac{g}{V}(n - \cos\gamma) \\ \dot{h} &= V\sin\gamma \\ \dot{x} &= V\cos\gamma \end{aligned} \tag{6}$$

From our previous studies, if we know lift, we know drag. Hence to get a unique solution to the above equations, we need to specify T(t) and L(t), that in turn will give us n(t).

Drag Model

In order to make some headway here, we will consider the drag term first:

$$\begin{aligned} D &= C_D \frac{1}{2}\rho V^2 S = (C_{D_{0L}} + K C_L^2)\frac{1}{2}\rho V^2 S \\ &= C_{D_{0L}} \frac{1}{2}\rho V^2 S + K\left(\frac{L^2}{(\frac{1}{2}\rho V^2 S)^2}\right)\frac{1}{2}\rho V^2 S \\ &= C_{D_{0L}} \frac{1}{2}\rho V^2 S + \frac{K W^2 n^2}{\frac{1}{2}\rho V^2 S} \qquad = f(V, n) \end{aligned} \tag{7}$$

If we assume ***low speed flight***, we can assume $C_{D_{0L}}$ and, K are constant. If in addition we assume that the ***altitude doesn't change much*** ($\rho = \text{const}$), we can write the drag as:

$$D = A V^2 + B \frac{n^2}{V^2} \tag{8}$$

where: $A = C_{D_{0L}} \frac{1}{2} \rho S$ and $B = \frac{K W^2}{1/2 \rho S}$.

Thrust model

For the thrust model we will assume that the magnitude of the thrust is independent of speed. However, we can set the magnitude by selecting a throttle setting, δ_T. Hence the thrust model takes the form:

$$T = \delta_T T_{\max} \qquad\qquad 0 \le \delta_T \le 1 \tag{9}$$

Numerically Integrating the equations of motion and kinematic equations

It is assumed that we know the initial conditions: At t = 0, $V(t_0) = V_0$, $\gamma(t_0) = \gamma_0$, $h(t_0) = h_0$, and $x(t_0) = x_0 = 0$. In addition we will also assume that we are given δ(t) and n(t). With this information, we will can determine a unique time history for the variables *V*, γ, *h*, and *x*.

Euler Integration

A crude method of numerical integration that can be used to "solve" the above equations is called Euler's method for integration. It is what is called a first order method and consequently is not extremely accurate. It approximates the solutions by straight lines and thus for accuracy it is necessary to take small steps. Regardless, we can use it because it is simple and provides solutions that are close to the actual solutions. The idea is to approximate the derivatives in the equations of motion by finite differences:

$$\frac{\Delta V}{\Delta t} = \frac{T(\delta_T(t)) - D(V, n(t))}{m} - g \sin\gamma$$

$$\frac{\Delta \gamma}{\Delta t} = \frac{g}{V} [\, n(t) - \cos\gamma]$$

$$\frac{\Delta h}{\Delta t} = V \sin\gamma$$

$$\frac{\Delta x}{\Delta t} = V \cos\gamma$$

We can now select a time increment, Δt, and let (for example) $\Delta V = V_{k+1} - V_k$ for the generic k^{th} integration step. In this manner we can determine the time history of each variable as time progresses from the initial time $t_0 = 0$ to the present. Hence for the generic k^{th} step, we have:

$$
\begin{aligned}
V_{k+1} &= V_k + \left[\frac{T(\delta_{T_k}) - D(V_k, n_k))}{m} - g \sin\gamma_k \right] \Delta t \\
\gamma_{k+1} &= \gamma_k + \left[\frac{g}{V_k} [\, n_k - \cos\gamma_k] \right] \Delta t \\
h_{k+1} &= h_k + \left[V_k \sin\gamma_k \right] \Delta t \\
x_{k+1} &= x_k + \left[V_k \cos\gamma_k \right] \Delta t \\
t_{k+1} &= t_k + \Delta t
\end{aligned}
\qquad (9)
$$

Where $D(V_k, n(t_k)) = C_{D_{0L}} \frac{1}{2} \rho S V_k^2 + \frac{K W^2 n_k^2}{\frac{1}{2} \rho S V_k^2} = A V_k^2 + B \frac{n_k^2}{V_k^2}$, and $n_k = n(t_k)$,

and $\delta_{T_k} = \delta_T(t_k)$

For any accuracy, the time increment must be small, say 0.01 to 0.05 seconds. (Note that there are several issues that govern accuracy. Too small a step will cause inaccuracies because of an excess number of calculations with approximate (round off errors) numbers, while too big a step will cause an inaccuracy because $\Delta t \neq dt$).

Example

Here we will do the calculations for one integration step using the Euler method. We ahve an aircraft the weighs 10,000 lbs, flying at sea-level. It has a drag polar of $C_D = 0.02$~+~$0.05\ C_L^2$ It is equipped with a jet engine that is delivering 3000 lb thrust and maintains that thrust level over a wide range of airspeed. In addition, the wing area is 200 ft. Under these circumstances, we would like to determine the maximum speed in level flight.

One way to approach this problem is to numerically integrate the equations of motion for the conditions given until there is no change in the airspeed. We are given T(t), (it equals a constant 3000 lbs.), so we need to determine n(t) such that the flight remains level ($\gamma = 0$). Lets start the problem with some arbitrary speed and the following initial conditions:

$$V_0 = 300 \text{ ft/sec}$$

$$\gamma_0 = \gamma(t) \equiv 0$$

$$h_0 = 0$$

$$x_0 = 0$$

We can compute the constants:

$$A = C_{D_{0L}} \frac{1}{2} \rho S = 0.02 \left(\frac{1}{2} \right) (0.002377)(200) = 0.00476 \frac{\text{lb sec}^2}{\text{ft}^2}$$

$$B = \frac{K W^2}{(1/2)\rho S} = \frac{0.05\,(10{,}000)^2}{(1/2)\,0.002377\,(200)} = 2.10084 x 10^7 \frac{\text{lb ft}^2}{\text{sec}^2}$$

Hence $D_k = 0.00476\, V_k^2 + 2.10084\, x\, 10^7 \dfrac{n_k^2}{V_k^2}$ lbs

We need to select n_k to insure that $\dot{\gamma} = 0$:

$$\gamma_{k+1} = \gamma_k + \left[\frac{g}{V_k} \left(n_k - \cos\gamma_k \right) \right] \Delta t$$

We want $\gamma_{k+1} = \gamma_k = 0$ for level flight, so clearly $n_k = 1$, for all k. Furthermore since T = T_{max} = constant, then $\delta_{T_k} = 1$ for all k. at We can now numerically integrate the equations. It turns out that both the γ and h equations are trivial h = γ = 0 for all time, so we need only be concerned with the airspeed and range equations. All the following is done in the computer, but the details are shown here. For demonstration purposes, we will pick a step size of 0.10 sec. The first step will be (for the airspeed equation)

$$V_1 = V_0 + \left[\frac{T_k - D(V_0, n_0)}{m} - g \sin\gamma_0 \right] \Delta t$$

where

$$D(V_0, n_0) = 0.00476\, V_0^2 + \left.\frac{2.10084\, x\, 10^7\, n_0^2}{V_0^2}\right|_{n_0 = 1,\, V_0 = 300}$$

$$= 0.00476\,(300^2) + \frac{2.10084\, x\, 10^7\,(1^2)}{300^2}$$

$$= 428.4 + 233.4$$

$$= 661.83 \text{ lbs}$$

Then

$$V_1 = 300 + \left[\frac{3000 - 661.83}{10{,}000/32.174} - (32.174)\,(0)\right](0.10) = 300.75 \text{ ft/sec}$$

Similarly, for x, we have:

$$x_1 = x_0 + V_0 \cos\gamma_0 = 0 + [\,300 \cos 0\,]\,(0.10) = 30 \text{ ft}$$

Hence after 1 time step of 0.1 sec, the new airspeed is 300.75 ft/sec, and it has moved 30 ft. We can repeat for the next step.

$$x_2 = x_1 + V_1 \cos\gamma_1 = 30 + [\,300.75 \cos 0\,]\,(0.10) = 60.075 \text{ ft}$$

$$V_2 = V_1 + \left[\frac{3000 - D(300.75, 1)}{310.810}\right](0.10) = ? \qquad \text{Etc.}$$