9. Orbital Mechanics

Once the vehicle is launched into space, one must now consider the conditions that are necessary to keep the vehicle in orbit. In addition we are interested in size and shape of the orbits and in how they are oriented in space. Some of these issues will be discussed here. Historically, orbital mechanics was one of the first sciences that received a large amount of attention. Even before Newton, Galileo and Copernicus had their idea about the solar system, with the sun being the center of the system and the planets rotating around it. A fellow by the name of Kepler observed the planets through a telescope and eventually formulated three "laws" that described the motion of the planets about the sun. These "laws" were later verified by theory developed by Newton. Newton postulated the inverse square gravitational attractive force and subsequently calculated what the orbits must be for two bodies acting under this gravitational law. The result was precisely the shape observed by Kepler. The astronomer Halley (of comet fame) asked Newton if he knew the shape of an orbit that would occur in an inverse gravitational field, "An ellipse of course" replied Newton, and Halley was astounded. In any case all of Kepler's laws can be verified if one assumes an inverse square gravitational force.

The Two Body Problem

The two body problem can be stated as follows: Given the initial position and velocities of two bodies in space, that are attracted to each other by Newton's law of an inverse gravitational force, find the subsequent position and velocity in time. It turns out we can solve this problem although it is impossible to get an analytic expression for position and velocity as functions of time. We end up with a parametric solution for an orbit, and determine time as a function of the parameter -kind of the reverse of what we want! However, a partial solution can be determined in terms of what the orbit looks like and with Kepler's laws applying to the orbit.

Kepler's Laws

Kepler stated three laws based on his observations. All of these laws can be shown to be valid. The laws are stated here in a modified form that is a slight extension of the original laws:

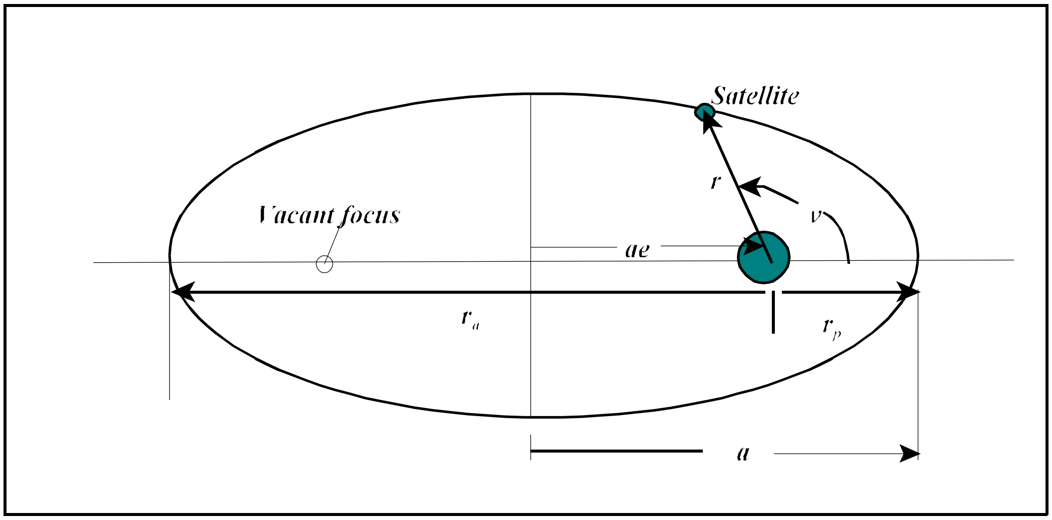
1. The orbits of one body about the other are conic sections (ellipse, parabola, hyperbola) with the central body at a focus.

2. The radius vector from the center of one mass to the center of the other, sweeps out equal areas in equal times - the areal rate is a constant

3. For an elliptic orbit (the only one of the group that is periodic) the square of the period of the orbit is proportional to the cube of the size of the orbit (the semi-major axis).

The Elliptic Orbit

The only orbit type we will discuss here is the elliptic orbit. All satellites that orbit the Earth are in a elliptic orbit. A parabolic and hyperbolic orbit are nonperiodic, and hence represent escape orbits, that is, the satellite in these orbits leaves the Earth. The parabolic orbit is the minimum energy escape orbit.



Elliptic Orbit

In the figure we can identify the following items:

Occupied focus		location of the Earth or central attracting body
a	=	semi-major axis
r	=	radius from center of Earth to satellite
r_p	=	periapsis distance (for Earth perigee distance)
r_a	=	apoapsis distance (for Earth apogee distance)
e	=	orbit eccentricity ($0<e<1$ for elliptic orbits, $e = 0$ is circular orbit)
ν	=	true anomaly

Note that r is measured from the center of the Earth, hence we can note that:

$$r = R_e + h \qquad (1)$$

where h = height ***above*** Earth's surface

Often times the perigee and apogee distances are given in terms of the height above the Earth's surface. This is not correct, and the radius of the Earth must be added to the height to get the perigee and apogee radii.

The polar equation for the elliptic orbit, with the origin at one focus is given by:

Polar Equation of Elliptic Orbit

$$r = \frac{a(1 - e^2)}{1 + e\cos\nu} \tag{2}$$

From the figure, and from the equation of the orbit, Eq. (2), we can extract the following information:

$$a = \frac{r_p + r_a}{2}$$

$$r_p = a(1 - e)$$

$$r_a = a(1 + e)$$

$$e = \frac{r_a - r_p}{r_a + r_p}$$

Example The international space station is in a "372 x 381 km orbit", what is the eccentricity of the orbit?

The two numbers refer to the perigee height (372 km) and the apogee height (381 km). However, these are the heights above the Earth's surface and must be converted to perigee and apogee radii by adding the Earth's radius.

$$r_p = R_e + h_p = 6378.1363 + 372 - 6750.1363 \text{ km}$$

$$r_a = R_e + h_a = 6378.1363 + 381 = 6759.1363 \text{ km}$$

Then the eccentricity is given by:

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{6759.1363 - 6750.1363}{6759.1363 + 6750.1363} = 0.00067$$

Hence the international space station is in a near circular orbit.

Energy in Orbit

Since there is no atmosphere in space, the are no forces to dissipate energy, and the ***energy in any given orbit is constant***. It turns out that the size of an orbit (the semimajor axis) is the only parameter the affects the energy. All orbits with the same semimajor axis, regardless of the eccentricity, have the same energy. That energy (per unit mass) is given by:

Energy of an Elliptic Orbit

$$En = -\frac{\mu}{2a} \tag{3}$$

We can now use our energy expression to determine the velocity at any point of the orbit

$$\frac{1}{2}V^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \tag{4}$$

Special Case - Circular Orbit - (e = 0)

A large number of orbits are intentionally made circular. If we set e = 0 in the equation of the orbit, Eq. (2), we see that in a circular orbit the radius is a constant, equal to the semi-major axis,

$$r = a \qquad \text{Circular orbit} \tag{5}$$

Then if we substitute into the energy equation, Eq. (4) we have:

Circular Orbit Speed

$$V_c = \sqrt{\frac{\mu}{r}} \tag{6}$$

If we examine the energy equation, Eq.(4) we can see that as the semi-major axis gets bigger, the energy in the orbit will increase (become less negative). Then in the limit, as the ***semi-major axis goes to infinity***, the total energy of the orbit will be ***zero***. This case is the special case of a parabolic orbit. We see that if the total energy is zero, when the satellite reaches infinity, $r - = \infty$, then the velocity goes to zero. So that a zero energy orbit is the least amount of energy that an orbit can have and allow the satellite to go to infinity. When it gets there, it will have zero velocity. Consequently this is the boundary between repeating orbits, elliptical, and escape orbits. Therefore the minimum velocity at any radius that will allow the vehicle to escape is given by:

Parabolic Orbit Energy and Escape Speed

$$\frac{1}{2} V^2 - \frac{\mu}{r} = 0$$

$$V_{esc} = \sqrt{\frac{2\mu}{r}} \tag{7}$$

Example: Find the orbital speed of the international space station

We can approximate the orbital speed of the international space station by assuming it is in a circular orbit. We can obtain the semi-major axis which we consider to be the circular orbit radius. We will assume:

$$r_c = a = \frac{r_a + r_p}{2} = \frac{6759.1363 + 6750.1363}{2} = 6754.64 \text{ km}$$

$$V_c = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{3.9860x10^5}{6754.64}} = 7.68 \text{ km/s}$$

What would be the period (time for one orbit) of the ISS?

We can calculate the period by taking the distance traveled and dividing it by the time. Hence the period for a circular orbit can be determined from:

$$T_p = \frac{2\pi r_c}{\sqrt{\frac{\mu}{r_c}}} = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{6754.64^3}{3.960x10^5}} = 5524.77 \text{ s} = 92.08 \text{ min} = 1.535 \text{ hours}$$

Period in Any Elliptic Orbit

It turns out that the expression we obtained for the period of the circular orbit in the example problem is true for any elliptic orbit!

Period of Elliptic Orbit

$$T_p = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{8}$$

We can make a couple of observations:

1) The energy of an elliptic orbit depends only on the semi-major axis
2) The period of an elliptic orbit depends only on the semi-major axis

Consequently, if two orbits with the same size major axis intersect, and a satellite is launched in each orbit from one intersection point, the two satellites will collide one orbit later, regardless of the eccentricities of the two orbits!

The Orbit in Space

From the previous discussions, it is implied that the orbit lies in a plane. This fact is indeed true. For the two-body problem, all orbits occur in a fixed plane in space. We now need to locate that plane in space, and at the same time orient the orbit in that plane. We can fully describe the orbit in space by means of six quantities called orbital elements. These orient the orbital plane and the orbit in that plane with respect to an inertial coordinate system. We call the inertial coordinate system the Earth-Centered-Inertial (ECI) system.

Coordinate Systems

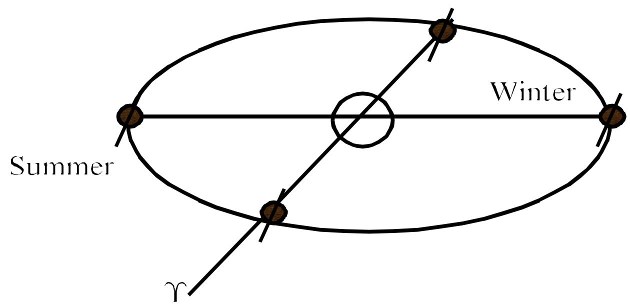
All coordinate systems have a reference plane (usually the x-y plane) and a reference direction in that plane (usually the x axis). We wish to study the motion of artificial satellites about the Earth and real and artificial satellites about the Sun and possibly other planets. We will restrict ourselves to Earth-centered (geocentric) and Sun-centered (heliocentric) satellites. For each of these cases, the origin of the coordinate system of interest is at the center of the Earth or Sun. ***The reference plane*** for each of these coordinate systems is as follows:

1. Heliocentric Orbits - The reference plane for heliocentric orbits is the plane of the Earth's orbit about the Sun (or the apparent path of the Sun about the Earth from our point of view). This plane is called the ***plane of the ecliptic***, and for all practical purposes is considered an inertial plane.

2. Geocentric Orbits - The reference plane for geocentric orbits is the equatorial plane of the Earth. Since the Earth's spin axis is tipped approximately 23.5 degrees from the normal to its orbit, the two reference planes do not coincide.

The two reference planes intersect along a line that is obviously common to both planes. When the Sun (as observed from the Earth) is along one of these lines, the day and night are of equal length. Hence these points are called equinoxes and are designated the spring (or vernal) and autumnal equinox. If we look from the Earth to the Sun at the vernal equinox we are looking in the ***common reference direction*** for both the heliocentric and geocentric coordinate systems.

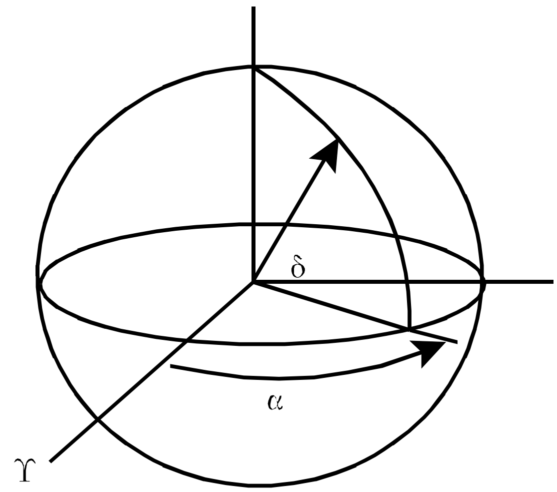
Definition - Vernal Equinox - The Vernal Equinox (also called the "First Point of Aires") is designated by ♈ and is a "fixed" direction in space used as the reference direction. It is the direction from the Earth to the Sun when the Sun appears to pass from the Southern hemisphere to the Northern hemisphere.



Definition - Celestial Sphere - The celestial sphere is an imaginary sphere of infinite radius that is centered at the origin of the coordinate system of interest. The position of the satellites and stars are projected onto the celestial sphere and can be located by specifying two angles: one measured in the reference plane from the reference direction, and the other in a plane perpendicular to the reference plane. The two angles are designated as:

$$\alpha \equiv \text{right ascencion}$$

$$\delta \equiv \text{declination}$$



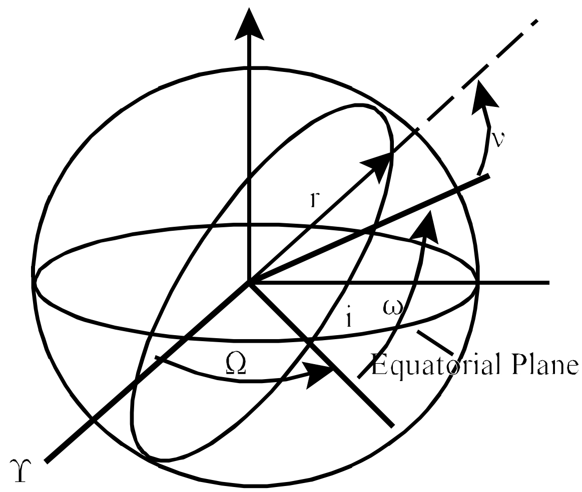
This coordinate system is often used by astronomers to locate stars in the heavens.

Orbital Elements

Recall that we started our two-body problem with a vector, second order, ordinary differential equation that requires six constants of integration to obtain a complete solution to the problem. We extracted these six constants in our previous work. By combining these constants in certain ways, we can arrive at a different set of six independent constants. There are many ways to do these combinations and permutations and hence there are a large number of different groups of six constants. Whatever the final selection is, these six constants are called the orbital elements. Here we will look at the "classical" set of orbital elements.

The classic orbital elements include two elements to locate the plane of the orbit in space, two to describe the size and shape of the orbit, one to orient the orbit in the plane, and the last to locate the satellite in the orbit. ***The elements*** are:

a Semi-major axis
e Orbit eccentricity
i the orbit inclination
Ω the right ascension of the ascending node
(Or longitude of the ascending node)
ω argument of periapsis
τ time of periapsis passage
or
ν true anomaly



The orbit plane and the reference plane (equatorial plane) intersect in a line called the "line of nodes". The end of this line where the satellite moves from the southern hemisphere to the northern hemisphere is called the ascending node. It takes two angles to orient the orbit plane in space, the angle between the vernal equinox and the ascending node, Ω, called the longitude of the ascending node (or right ascension of the ascending node), and the angle between the normal to the orbit plane (along the angular momentum vector), and the normal to the reference plane, the z or north-pointing axis, called the inclination angle, *i*. As will be shown later, these two angles are directly related to the angular momentum vector as might be expected. We should also note that we define direct and retrograde orbits in the following manner:

$0 \le i < 90$ deg	Direct orbit
$90 < i \le 180$deg	Retrograde orbit
$i = 90$ deg	Polar orbit

The energy of the orbit gives us the size of the orbit, or the semi-major axis, *a*, and the magnitude of the angular momentum (combined with the energy) give us the shape or eccentricity of the orbit. Finally the orbit is oriented in the plane by defining the angle, ω, between the line of the ascending node, and the line passing through the periapsis of the orbit. This angle is called the argument of periapsis and is measured in the plane of the orbit.

The final constant is the time of periapsis passage, τ, that allows us to determine where the satellite is in orbit at some given time, t.

These six elements are called the "classic" orbital elements and fully describe the orbit and the position of the satellite in orbit. Hence if we know the elements and time, a, e, i, Ω, ω, τ, and t, we can locate the orbit and the satellite in space. These elements are convenient to use because they bring the physical picture to mind and allow one to visualize the orbit and its orientation. It is convenient to designate the true anomaly, or more properly, the true anomaly at some specified time or epoch as an "element." We can completely describe the orbit and satellite position is space if we know the "elements" a, e, i, Ω, ω, and ν_0.

Example

A satellite is in an orbit hose eccentricity is 0.9. It perigee height is 300 km. Find the satellite velocity at perigee and apogee, and determine the period of the orbit.

We need to find the properties of the orbit, mainly, the semi-major axis. Since we know the perigee height, and the eccentricity, we can determine the semi-major axis:

$$r_p = a(1 - e) = R_e + h_p = 6378{,}1363 + 300 = 6678.1363 = a(1 - 0.9)$$

$$a = 66{,}781.363\text{km}$$

We can determine the period from:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{66{,}781.363^3}{3.9860x10^5}} = 171{,}749 \text{ s} = 2862.48 \text{ min} = 47 \text{ hours}$$

We can determine speeds from the energy equation:

$$\frac{1}{2}V^2 - \frac{\mu}{r} = En = -\frac{\mu}{2a} = -\frac{3.9860X10^5}{2(66781.363} = -2.9844 \frac{\text{km}^2}{\text{s}^2}$$

At perigee:

$$\frac{1}{2}V_p^2 - \frac{\mu}{r_p} = \frac{1}{2}V_p^2 - \frac{3.9860x10^5}{6678.1313} = -2.9844 \quad \Rightarrow \quad V_p = 10.6492 \text{ km/s}$$

Angular rate is:

$$\dot{\nu} = \frac{V_p}{r_p} = \frac{10.6492}{6678.1363} = 0.001595\text{rad/s} = 0.09 \text{ deg/s}$$

At apogee:

$$r_a = a(1 + e) = 66{,}781.363\,(1 + 0.9) = 126{,}884.590 \text{ km}$$

$$\frac{1}{2}V_a^2 - \frac{\mu}{r_a} = \frac{1}{2}V_a^2 - \frac{3.9860x10^5}{126{,}884.590} = -2.9844 \quad \Rightarrow \quad V_p = 0.5605 \text{ km/s}$$

Angular rate is:

$$\dot{\nu} = \frac{V_a}{r_a} = \frac{0.5605}{126{,}884.59} = 4.4174x10^{-6} \text{ rad/s} = 2.531x10^{-4} \text{ deg/s}$$