Control of distributed systems over graphs

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Abstract—This paper focuses on designing distributed controllers for interconnected systems in situations where the controller sensing and actuation topology is inherited from that of the plant. The main contribution of this work is new results on general graph interconnection structures. This is accomplished by extending our previous machinery developed for systems with spatial dynamics on \( \mathbb{Z}^{n} \). We derive analysis and convex synthesis conditions for design in this setting. Furthermore, the methodology developed here provides a unifying viewpoint for our previous and related work on distributed control.

I. INTRODUCTION

Distributed control is of considerable interest to the control community because of its importance in large-scale systems, and pursuit of the associated design methods has a long history. Because of the recent emergence of many new applications (e.g., formation flight, antenna arrays, or fluid control) that would benefit from systematic methods for designing distributed controllers, this class of problems has become even more important.

Our goal here is to generalize the work in [2], which was developed for distributed models posed on the standard spatial grid \( \mathbb{Z}^{n} \), to the more general object of graphs. The motivation for this generalization stems from two applications. The first is communication over networks where the plant subsystems have an a priori prescribed communication topology that is not a grid. The second is in connection with control models that arise from finite element approximations, where in general meshes can be arbitrary graphs.

We build on our previous work in [1], [2]. The development here is closely connected with [9] and [7]. In the former, distributed control is considered over groups that are not necessarily abelian, and the latter paper considers distributed control over finite graphs. The framework here is able to capture arbitrary infinite graphs and provides a unifying framework to view previous work. In general, the results obtained here are given in terms of infinite operator inequalities, but can be reduced to semidefinite programs in the cases of infinite graphs with periodic space and time structure or finite graphs with temporal periodicity. One further case in which the results can be stated as semidefinite programming problems is when the spatial and temporal variation is eventually periodic as depicted in Figure 1. Work on the purely time-varying case, where the term eventually periodic was introduced, appears in [4]. In this paper, for space considerations, we will only consider the general case, leaving the aforesaid special cases for the journal version.

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Fig. 1. An eventually periodic graph.

II. PRELIMINARIES

The natural numbers (including zero), integers, integers modulo \( m \) and real numbers are denoted by \( \mathbb{N}_{0}, \mathbb{Z}, \mathbb{Z}_{m} \), and \( \mathbb{R} \) respectively. We use \( G(V,E) \) to denote a digraph with set of vertices \( V \) and set of directed edges \( E \); throughout we will assume that the number of vertices is countable and that the vertex degree is uniformly bounded. We use the ordered-pair \( (i,j) \) to denote the element of \( E \) corresponding to an edge directed from vertex \( i \in V \) to vertex \( j \in V \). We will use \( s(G) \) to denote the maximum over the indegrees and outdegrees of the graph; namely,

\[ s(G) := \max_{k \in V} \{m(k), p(k)\}, \]

where \( m(k) \) and \( p(k) \) denote the indegree and outdegree of a vertex \( k \in V \) respectively. We will call a bijection \( \sigma : V \rightarrow V \) a permutation of \( V \). A subset \( C \) of \( V \) is a cycle of the permutation \( \sigma \) if there exist an integer \( 1 \leq m \leq \infty \) and a bijection \( \beta : C \rightarrow \mathbb{Z}_{m} \) such that

\[ \beta \circ \sigma(v) = (1 + \beta(v)) \mod m. \]

Note that here by \( m = \infty \) we mean \( \mathbb{Z}_{m} \) is simply the integers \( \mathbb{Z} \). It is easy to show that the family of cycles of \( \sigma \), denoted \( \mathcal{C}_{\sigma} \), are countable in number, disjoint, and that \( V = \bigcup_{\sigma} \mathcal{C}_{\sigma} \).

We also define the set \( \mathbb{G} \) as

\[ \mathbb{G} = \mathbb{Z} \times V = \{ \bar{k} = (t,k) \mid t \in \mathbb{Z} \text{ and } k \in V \}. \]

If we have two vectors of real numbers \( x \) and \( y \), the notation \( x \leq y \) will be used to mean that the inequalities \( x_{j} \leq y_{j} \) hold for all indices \( j \); if \( x \) and \( y \) are infinite sequences, we adopt a similar pointwise meaning. The maximum singular value of a matrix \( M \) will be denoted by \( \sigma(M) \). Given a symmetric matrix \( H \), its inertia \( \text{in}(H) \) is the triplet \( (\text{in}_{+}(H), \text{in}_{0}(H), \text{in}_{-}(H)) \) giving the number of positive,
III. DISTRIBUTED SYSTEM FORMULATION

This paper deals with distributed systems, whose interconnection structures are defined by directed graphs. Specifically, the different linear time-varying systems comprising the distributed model in question correspond to the vertices of the graph, and the interconnections between these systems are described by the directed edges of the graph. Given such a distributed system, say $G$, we will consider in this paper a controller synthesis problem, illustrated in Figure 2, where the controller $K$ has a similar structure as $G$.

Consider a digraph $G(V,E)$, which defines the information structure of the distributed system under consideration, and so, for instance, graph (a) in Figure 3 gives rise to the interconnected system in (b). Henceforth, we will conveniently use the term vertex to also refer to a constituent system in a distributed model. Our goal now is to present general state-space equations which describe each of the vertices of the distributed model. For convenience of notation, we set $s = s(G)$. Now, let $\sigma_1, \sigma_2, \ldots, \sigma_s$ be permutations on the vertices $V$. We choose these with the property that if $(i,j) \in E$, then there exists $r$ such that $\sigma(i) = j$, and hence $\sigma^{-1}(j) = i$ because of the bijective correspondence.

Example: Consider the distributed system in Figure 4(a). We can use the graph in Figure 4(b) to represent this system. Here, $s = \max_k \{ m(k), p(k) \} = \max \{ 2, 2 \} = 2$, and we can define the permutations $\sigma_1$ and $\sigma_2$ for each vertex $k$ as

\[ \sigma_1(1) = 2 \quad \sigma_1(2) = 3 \quad \sigma_1(3) = 4 \quad \sigma_1(4) = 1 \]

\[ \sigma_2(1) = 3 \quad \sigma_2(2) = 1 \quad \sigma_2(3) = 4 \quad \sigma_2(4) = 2 \]

We will regard the interconnections between the systems $G^{(k)}$ as states when formulating the system equations, and accordingly we define our state-space model as follows:

\[
\begin{bmatrix}
    x_0(t+1,k) \\
    x_1(t,\sigma_1(k)) \\
    \vdots \\
    x_s(t,\sigma_s(k))
\end{bmatrix} = A(t,k)
\begin{bmatrix}
    x_0(t,k) \\
    x_1(t,k) \\
    \vdots \\
    x_s(t,k)
\end{bmatrix} + B(t,k)
\begin{bmatrix}
    w(t,k) \\
    u(t,k)
\end{bmatrix}
\]

\[
\begin{bmatrix}
    z(t,k) \\
    y(t,k)
\end{bmatrix} =
\begin{bmatrix}
    C_{11}(t,k) & C_{12}(t,k) \\
    C_{21}(t,k) & C_{22}(t,k)
\end{bmatrix}
\begin{bmatrix}
    x_0(t,k) \\
    x_1(t,k) \\
    \vdots \\
    x_s(t,k)
\end{bmatrix}
\]

\[
\begin{bmatrix}
    w(t,k) \\
    u(t,k)
\end{bmatrix}
\]

where $t \in \mathbb{Z}$ is discrete-time and $k$ is any vertex in $V$. The input signals $w$ and $u$ denote the exogenous disturbances and applied control, respectively, whereas the output signal $z$ denotes the exogenous disturbances and the measurements. Notice that, in the previous equations, the state vector $x(t,k)$ is partitioned into $s+1$ separate vector-valued channels, namely $x(t,k) = (x_0(t,k), \ldots, x_s(t,k))$. Then, comfortably with these channels, we partition the state-space matrices as
dimensions that depend on \( t, k \), and thus define the sequences \( e(t, k), n_r(t, k), n_c(t, k), \) and \( b(t, k) \) so that, for each \((t, k) \in \mathbb{Z} \times V'\), we have that \( C(t, k) = e(t, k) \times n_r(t, k) \), \( A(t, k) = n_r(t, k) \times n_c(t, k) \), and \( B(t, k) = n_r(t, k) \times b(t, k) \). For now, we will assume that this model admits a unique solution \( x \) in \( \ell_2 e \) given inputs \( w, u \in \ell_2 e \), and that the associated mapping \( (w, u) \mapsto x \) is causal; in the next section, we will give precise conditions to ensure these properties.

It is important to point out that Figures 3 and 4 represent the same system; notice however that there are two extra edges represented by dashed arrows in Figure 4. These virtual edges are added because, for our notation to work, the indegree and outdegree of each vertex has to be equal to \( s \). This is necessary so that it is possible to define the permutations \( \sigma_1, \ldots, \sigma_s \) to recall that each \( \sigma_i \) has to be defined over the entire vertex set. Now, since the states \( x_2(t, 4) \) and \( x_2(t, 1) \) correspond to non-existent interconnections, we set the row dimensions of these states equal zero; also the associated blocks in the state-space matrices will appropriately have zero row or column dimensions. Although a slight abuse of notation, this will allow for a succinct representation, and all the manipulations and proofs in the sequel will follow directly, namely, when we encounter matrix blocks of zero dimensions, say in some operator inequality corresponding to some vertex, this just means that the rows and columns of which these blocks are elements are not present, and the inequalities remain valid. It is worth noting, furthermore, that in some instances, especially those involving semi-infinite vertex arrays, it might be necessary to add virtual vertices along with virtual edges for the notation to work.

Notice that if we define \( \sigma_i(i) = j \) corresponding to edge \( (i, j) \), then the associated state will be denoted by \( x_i(t, j) \). Last, as an example, we write the system equation of \( G^{(3)} \):

\[
\begin{bmatrix}
x_0(t+1, 3) \\
x_1(t, \sigma_1(3) = 4) \\
x_2(t, \sigma_2(3) = 4) \\
z(t, 3)
\end{bmatrix} =
\begin{bmatrix}
A(t, 3) & B(t, 3) \\
C(t, 3) & D(t, 3)
\end{bmatrix}
\begin{bmatrix}
x_0(t, 3) \\
x_1(t, 3) \\
x_2(t, 3) \\
w(t, 3)
\end{bmatrix},
\]

where \( x_2(t, 4) \) has zero row dimension.

\[\blacksquare\]

A. Graph-diagonal operators

The next step is to develop operator theoretic machinery along the lines of [2], which will allow us to represent the system equations of (1) in a compact operator form.

**Definition 2**: Let \( v \) and \( n \) be sequences mapping \( G \) to \( \mathbb{N}_0 \), and \( Q \) be a linear mapping from \( \ell_2(\mathbb{R}^{v(k)}) \) to \( \ell_2(\mathbb{R}^{n(k)}) \). Then \( Q \) is said to be a graph-diagonal operator if there exists a uniformly bounded sequence of matrices \( Q(\bar{k}) \in \mathbb{R}^{n(k) \times v(k)} \) such that the equality \( (Qw)(\bar{k}) = Q(\bar{k})w(\bar{k}) \) holds for each \( \bar{k} \in G \).

Graph-diagonal operators are defined similarly to hyper-diagonal operators, considered in [2], and both are generalizations of block-diagonal operators.

As in the case of hyperdiagonal operators in [2] and block-diagonal operators in [3], we generalize the concept of inertia to graph-diagonal operators. Given a self-adjoint graph-diagonal operator \( Q \), we define its inertia to be the mapping

\[
\text{In}(Q) : G \to \mathbb{N}_0^3 \text{ defined by } \text{In}(Q)(\bar{k}) := \text{in}_{+}(Q(\bar{k})).
\]

Similarly, we define \( \text{In}_{-}(Q)(\bar{k}) := \text{in}_{-}(Q(\bar{k})) \) and \( \text{In}_{0}(Q)(\bar{k}) := \text{in}_{0}(Q(\bar{k})) \). Then, Proposition 1 generalizes immediately to the following congruence result for graph-diagonal operators.

**Proposition 3**: Suppose \( H \) and \( M \) are graph-diagonal operators, with \( H \) self-adjoint. Then \( \text{In}_{+}(H) \geq \text{In}_{+}(M \ast HM) \) and \( \text{In}_{-}(H) \geq \text{In}_{-}(M \ast HM) \). Furthermore, if \( M \) is nonsingular, then \( \text{In}(H) = \text{In}_{0}(M \ast HM) \).

We say that \( W \) is a partitioned graph-diagonal operator if it has the form

\[
\begin{bmatrix}
H & P \\
R & J
\end{bmatrix},
\]

where \( H, P, R \) and \( J \) are graph-diagonal operators. We then define the **graph-diagonal representation** of \( W \) as the graph-diagonal operator \( [W] \) given by:

\[
([W] \vec{x})(\bar{k}) := \begin{bmatrix}
H(\bar{k}) & P(\bar{k}) \\
R(\bar{k}) & J(\bar{k})
\end{bmatrix} \vec{x}(\bar{k}).
\]

Clearly these concepts generalize to arbitrary partitions. Given vector-valued sequences \( \vec{q} = (q_1, \ldots, q_r) : G \to \mathbb{N}^r \) and \( \vec{v} = (v_1, \ldots, v_c) : G \to \mathbb{N}^c \), we denote by \( P(\vec{q}, \vec{v}) \) the set of partitioned graph-diagonal operators of the form

\[
\begin{bmatrix}
J_{11} & \cdots & J_{1c} \\
\vdots & \ddots & \vdots \\
J_{r1} & \cdots & J_{rc}
\end{bmatrix},
\]

where each \( J_{ij} \) is a graph-diagonal operator mapping \( \ell_2(\mathbb{R}^{q_i(k)}) \) to \( \ell_2(\mathbb{R}^{n_j(k)}) \). The following notation will be convenient: given a partitioned graph-diagonal operator \( J \) in \( P(\vec{q}, \vec{v}) \), we define \( p(J) := (\vec{q}, \vec{v}) \). Furthermore, if \( \vec{q} \) and \( \vec{v} \) are dependent on each other (e.g., \( \vec{q} = \vec{v} \)), we will simply set \( p(J) = \vec{q} \) and write \( P(\vec{q}) \) in place of \( P(\vec{q}, \vec{v}) \). Also, when the partition dimensions \( (\vec{q}, \vec{v}) \) are not important, we will use the abbreviation \( P \) to denote the set of partitioned graph-diagonal operators.

It is not difficult to see that \([\cdot]\) is a homomorphism from \( P \) into the space of graph-diagonal operators, which is isometric, and preserves products, addition, and ordering. We define the inertia of a self-adjoint, partitioned graph-diagonal
operator $W$ by $\text{In}(W) := \text{In}(\mathbb{W})$. Clearly the definition of graph-diagonal operators is extendable to $\ell$ and $\ell_{2e}$, and in the sequel, we will not distinguish between these objects. We now define a number of shift operators. For some sequence $n(k)$, we define the time shift or delay operator $S_0$ by

$$S_0: \ell(\mathbb{R}^{n(k)}) \to \ell(\mathbb{R}^{q(k)}),$$

where $q(t + 1, k) = n(t, k)$,

$$(S_0w)(t, k) = v(t - 1, k),$$

and further define, for $i = 1, 2, \ldots, s$, the spatial shift operators $S_i$ by

$$S_i: \ell(\mathbb{R}^{n(k)}) \to \ell(\mathbb{R}^{q(k)}),$$

where $q(t, \sigma_i(k)) = n(t, k)$,

$$(S_iw)(t, k) = v(t, \sigma_i^{-1}(k)).$$

Clearly, these shifts are invertible, and it is easy to verify that ($S_0^{-1}v)(t, k) = v(t + 1, k)$ and ($S_i^{-1}v)(t, k) = v(t, \sigma_i(k))$ for $i = 1, 2, \ldots, s$. In fact, each of these shifts is also unitary, and so the inverse is equal to the adjoint. We also define the composite shift operator $S := \text{diag}(S_0, S_1, \ldots, S_s)$.

Before stating the desired equations, notice that the matrix sequences $A_{ij}(k), B_{ij}(k), C_{ij}(k)$ and $D_{ij}(k)$ in (1) define partitioned graph-diagonal operators $A, B, C$ and $D$. Then, we can equivalently write the state-space equations (1) in compact operator form as

$$x = SAX + SB \begin{bmatrix} w \\ u \end{bmatrix}, \quad \begin{bmatrix} z \\ y \end{bmatrix} = CX + D \begin{bmatrix} w \\ u \end{bmatrix}. \quad (3)$$

Then, assuming the relevant algebraic inverse exists, we can rewrite (3) in the linear fractional form

$$G = C(I - SA)^{-1}SB + D; \quad (4)$$

here $G$ designates the input-output mapping of the model (1).

Before concluding this section, it is convenient to partition the composite shift and state-space operators in (3) along temporal-spatial lines. Namely, let us partition the shift as $S = \text{diag}(S_0, S)$, where $S = \text{diag}(S_1, \ldots, S_s)$, and compatibly partition the state-space operators so that

$$A = \begin{bmatrix} A_{00} & A_{0*} \\ A_{*0} & \bar{A} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_0 \\ B \end{bmatrix}. \quad (5)$$

IV. ANALYSIS RESULTS

This section defines the basic concepts of well-posedness and stability for distributed systems in the form of (1). Also, a version of the Kalman-Yakubovich-Popov (KYP) Lemma is given for such systems, which gives sufficient conditions for stability and contractiveness of open-loop systems.

**Definition 4:** A distributed model in the form of (1) is well-posed if, given inputs in $\ell_{2e}$, the equations (1) admit unique solutions $x_i \in \ell_{2e}$ for $i = 0, 1, \ldots, s$, and furthermore define a linear causal mapping on $\ell_{2e}$.

Regarding the unique solutions in $\ell_{2e}$, this can always be guaranteed if the operator $I - SA$ has an algebraic inverse on $\ell_{2e}$, as evident from (4). Moreover, causality is ensured if this inverse is causal; this also follows from (4) since the other operators in this equation are causal and the space of causal operators is an algebra.

**Lemma 5:** The operator $I - SA$ has an algebraic causal inverse on $\ell_{2e}$, and hence the system in (1) is well-posed, if (i) $A(t, k)$, and all the other state-space matrices, are equal to zero when $t < 0$; and (ii) the linear mapping $I - SA$ has an algebraic inverse on $\ell_{2e}$, where $S$ and $A$ correspond to the partition in (5).

The proof is basically the same as that of [2, Lemma 8].

**Definition 6:** We say the system in (1) is stable if it is well-posed and, given inputs in $\ell_{2e}$, the equations (1) admit unique solutions $x_i \in \ell_{2e}$ for $i = 0, 1, \ldots, s$, and further define a linear causal mapping on $\ell_{2e}$.

In other words, the system is stable if $I - SA$ has a causal inverse on $\ell_{2e}$. This ensures that $G$ from (4) is a linear causal map on $\ell_{2e}$. We will formulate next Lyapunov-based tests for stability.

At this point, it is important to state some important properties about shifts and graph-diagonal operators. To begin, suppose that $W$ is a graph-diagonal operator on $\ell_{2e}$. Then both $S_iW$ and $S_iW^*$ are also graph-diagonal, and given by the relationships

$$(S_iW)_{ij}(t, k) = W(t + 1, k),$$

$$(S_iW^*)_{ij}(t, k) = W(t - 1, k),$$

$$(S_iW^*)_{ik}(t, k) = W(t, \sigma_i(k)),$$

$$(S_iW^{**})_{ik}(t, k) = W(t, \sigma_i^{-1}(k)),$$

for $j = 1, \ldots, s$. If we define $X \in \mathcal{P}$ to be block-diagonal with respect to its partition (also that of $S$), i.e. $X = \text{diag}(X_0, X_1, \ldots, X_s)$, then $S^*X$ and $X^*S^*$ are in $\mathcal{P}$ as well. We now define the subset $\mathcal{X}$ of $\mathcal{P}$ as

$$\mathcal{X} = \{X \in \mathcal{P} : X = \text{diag}(X_0, X_1, \ldots, X_s),$$

$$X^{-1} \in \mathcal{L}(\ell_{2e}), \text{ and } X_0 > 0\}.$$ 

**Lemma 7:** Given $A \in \mathcal{P}$, if there exists an operator $X \in \mathcal{X}$ satisfying $\text{In}_{-,}(S^*XS) = \text{In}_{-,}(X)$ and $S^*AXSA - X \prec 0$, then $I - SA$ has a causal inverse on $\ell_{2e}$, and hence the system in (1) is stable.

The proof is formally identical to that of [2, Lemma 13].

The above result provides a sufficient condition for stability, which in general is not necessary and so some conservatism is introduced into our analysis.

**Lemma 8:** Suppose $X$ is in $\mathcal{X}$ and satisfies the inertia condition $\text{In}_{-,}(S^*XS) = \text{In}_{-,}(X)$. If the inequality

$$\begin{bmatrix} SA & SB \\ C & D \end{bmatrix} \succ 0, \quad \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \succ 0 \quad (6)$$

holds, then $I - SA$ has a causal inverse on $\ell_{2e}$ and

$$\|C(I - SA)^{-1}SB + D\| < 1.$$ 

The proof is formally identical to that of [2, Lemma 14].

V. SYNTHESIS

This section tackles the controller synthesis problem for the distributed system in (1).

**Definition 9:** A controller $K$ is an admissible synthesis for plant $G$ in Figure 2 if it ensures a stable closed-loop system and achieves the inequality $\|u \mapsto z\| < 1$. To start, the system realization $(A, B, C, D)$ from (3) represents the distributed system in question. To ensure well-posedness, we assume that this realization satisfies the conditions (1) and (ii) in Lemma 5. We also assume for convenience that $D_{22} = 0$. Suppose this system is being controlled by a controller $K$ with realization $(A_K, B_K, C_K, D_K)$, and state dimensions given by the sequence $n_{K}, i.e. n_K = p(A_K)$. The closed-loop system in Figure 2 can then have the representation $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$, where

$$A_{cl} = \begin{bmatrix} A + B_3D_{21}C_2 + B_2C_K & B_2C_K \\ B_KC_2 & A_K \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} B_1 + B_3D_{21}D_2 \\ B_KD_{23} \end{bmatrix},$$

$$C_{cl} = \begin{bmatrix} C_1 + D_{12}D_{21}C_2 + D_{12}C_K \end{bmatrix}, \quad D_{cl} = D_{11} + D_{12}D_{21}.$$
Clearly, the state dimensions of this closed-loop realization are given by \( \tilde{x}_{cl} := (p(A), \tilde{n}_K) \). Defining \( \bar{S}_{cl} = \text{diag}(S, S) \), we can write the closed-loop equations in the fractional form

\[
\begin{bmatrix}
X \\
X_GK \\
X_K
\end{bmatrix} = \bar{S}_{cl}^{-1} \begin{bmatrix} A & B & 0 \\ C & D & 0 \end{bmatrix} + \begin{bmatrix} X_GK \\
X_K0 \\
X_K0
\end{bmatrix}.
\]

Let \( X_{cl} \) be a partitioned graph-diagonal operator of the form

\[
X_{cl} = \begin{bmatrix}
X & X_GK \\
X_GK & X_K
\end{bmatrix} \in \mathcal{P}(\tilde{n}_{cl}),
\]

where \( X \in \mathcal{X}, X_GK = \text{diag}(X_{GK0}, \ldots, X_{GK_k}) \in \mathcal{P}(p(A) ; \tilde{n}_K), \) and \( X_K = \text{diag}(X_{K0}, \ldots, X_{K_k}) \in \mathcal{P}(\tilde{n}_K) \).

We define the set \( \{X_{cl} \in \mathcal{P}(\tilde{n}_{cl}) : X_{cl} = X_{cl}^* \} \) partitioned as in (7),

\[
X_{cl}^{-1} \quad \text{exists, and} \quad \begin{bmatrix} X_{cl}^0 & X_{GK0} \\
X_{GK0} & X_{K0}
\end{bmatrix} \succ 0.
\]

**Lemma 10:** A controller \( K \) is an admissible synthesis for the configuration in Fig 2 if there exists an operator \( X_{cl} \in \mathcal{X}_{cl} \) such that \( \text{In}_- (X_{cl}) = \text{In}_- (S_{cl}^* X_{cl} S_{cl}) \) and

\[
\begin{bmatrix} A_{cl} & B_{cl} \\
C_{cl} & D_{cl} \end{bmatrix} = R + U^* QV,
\]

where \( Q = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \). \( \quad \Box \)

Note that the partitioned graph-diagonal operators \( R, U, \) and \( V \) depend only on the plant. We next exploit this parametrization to rid the conditions of Lemma 10 from the controller data. But first, we need to make the following definition. Given symbols \( E_1, \ldots, E_s, Q_1, Q_2, \) and \( N \), we define the notation \( L \left( \begin{bmatrix} E_1 & E_2 \\
E_3 & E_4 \end{bmatrix} ; Q_1, Q_2, N \right) := \begin{bmatrix} E_1 & E_2 \\
E_3 & E_4 \end{bmatrix}^* Q_1 0 0 0 I 0 0 0 \left( \begin{bmatrix} Q_2 & 0 \\ 0 & 1 \end{bmatrix} \right) N \).

**Proposition 11:** Suppose that \( X_{cl} \in \mathcal{X}_{cl} \) and satisfies \( \text{In}_- (X_{cl}) = \text{In}_- (S_{cl}^* X_{cl} S_{cl}) \), and consider the partitioned graph-diagonal operators \( X, Y \in \mathcal{X} \) defined by

\[
X_{cl} = \begin{bmatrix} X & X_GK \\
X_GK & X_K
\end{bmatrix} \quad \text{and} \quad X_{cl}^{-1} = \begin{bmatrix} Y & Y_GK \\
Y_GK & Y_K
\end{bmatrix}.
\]

Then, inequality (8) holds if and only if

\[
L \left( \begin{bmatrix} A & B \\
C & D \end{bmatrix} ; S^* X S, X, N \right) \prec 0
\]

and

\[
L \left( \begin{bmatrix} A & B \\
C & D \end{bmatrix} ; Y, S^* Y S, N \right) \prec 0
\]

where \( N_X \) and \( N_Y \) are coercive operators with

\[
\text{Im} N_X = \text{Ker} \begin{bmatrix} C_2 & D_{21} \end{bmatrix} \quad \text{and} \quad \text{Im} N_Y = \text{Ker} \begin{bmatrix} B_2 & D_{12} \end{bmatrix}.
\]

The proof is very similar to that of [2, Proposition 20].

**Lemma 12:** Suppose \( n \) and \( h \) are sequences of nonnegative integers, and that \( X \) and \( Y \) are positive definite graph-diagonal operators in \( \mathcal{P}(n) \). Then, there exist operators \( X_2, Y_2 \in \mathcal{P}(n, h) \), and self-adjoint operators \( X_3, Y_3 \in \mathcal{P}(h) \) satisfying

\[
\begin{bmatrix} X & X_2 \\
X_2 & X_3 \end{bmatrix} = \begin{bmatrix} Y & Y_2 \\
Y_2 & Y_3 \end{bmatrix} \quad \text{and}
\]

\[
\text{In}_- \left( \begin{bmatrix} X & X_2 \\
X_2 & X_3 \end{bmatrix} \right) = \text{In}_- \left( \begin{bmatrix} X & X_2 \\
X_2 & X_3 \end{bmatrix} \right) \text{S} \right)
\]

if and only if, for each \( C_{ij} \in C(j) \) and \( t \in \mathbb{Z}, \)

\[
\text{In}_- \left( \begin{bmatrix} X & X_2 \\
X_2 & X_3 \end{bmatrix} \right) \right) \text{S} \right)
\]

\[
\text{In}_- \left( \begin{bmatrix} X & X_2 \\
X_2 & X_3 \end{bmatrix} \right) \right) \text{S} \right)
\]

\[
\text{In}_- \left( \begin{bmatrix} X & X_2 \\
X_2 & X_3 \end{bmatrix} \right) \right) \text{S} \right)
\]

\[
\text{In}_- \left( \begin{bmatrix} X & X_2 \\
X_2 & X_3 \end{bmatrix} \right) \right) \text{S} \right)
\]

\[
\text{In}_- \left( \begin{bmatrix} X & X_2 \\
X_2 & X_3 \end{bmatrix} \right) \right) \text{S} \right)
\]
In\( t^i \) (\( \begin{bmatrix} X & I \\ Y & Y \end{bmatrix} \)) (t, k) \leq n(t, k) + h(t, k)
 holds for all \( k \in C_i^{(j)} \). (16)

**Proof:** Observe that condition (15) is equivalent to
\[
\operatorname{in}_{- \langle} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} (t, k) = \operatorname{in}_{- \langle} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} (t, \sigma_j (k)) \quad (17)
\]
f for all \( (t, k) \in Z \times V \).
(Only if): From equation (17), it is immediate that, for every \( t \in Z \) and \( C_i^{(j)} \in C^{(j)} \), the equation
\[
\operatorname{in}_{- \langle} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} (t, C_i^{(j)}) = \operatorname{in}_{- \langle} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} (t, k)
\]
holds for all \( k \in C_i^{(j)} \), and so, we have \( n(t, k) + h(t, k) = \)
\[
\operatorname{in}_{+} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} (t, k) + \operatorname{in}_{- \langle} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} (t, C_i^{(j)}) \quad (18)
\]
for all \( (t, k) \in Z \times C_i^{(j)} \). We can write the factorization
\[
\begin{bmatrix} I & 0 \\ Y & Y_2 \end{bmatrix} = \begin{bmatrix} X & I \\ Y & Y_2 \end{bmatrix}.
\]
Then, by Proposition 3, we get
\[
\operatorname{in}_{- \langle} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} (t, C_i^{(j)}) \geq \operatorname{in}_{- \langle} \begin{bmatrix} X & I \\ Y & Y_2 \end{bmatrix} (t, C_i^{(j)})
\]
\[
\operatorname{in}_{+} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} \geq \operatorname{in}_{+} \begin{bmatrix} X & I \\ Y & Y_2 \end{bmatrix}
\]
for all \( (t, k) \in Z \times C_i^{(j)} \), and so the result follows from (18).
(If): Fix \( t \in Z \), \( C_i^{(j)} \in C^{(j)} \). Then, from (16), there exist a nonnegative integer \( q_- \) and a sequence \( q_+(k) \) such that, for all \( k \in C_i^{(j)} \), we have \( n(t, k) + h(t, k) = q_+(k) + q_- \), and
\[
\operatorname{in}_{- \langle} \begin{bmatrix} X & I \\ Y & Y_2 \end{bmatrix} (t, k) \leq q_- \quad \operatorname{in}_{+} \begin{bmatrix} X & I \\ Y & Y_2 \end{bmatrix} (t, k) \leq q_+(k).
\]
Then, by [2, Lemma 21], there exist matrix sequences with \( X_2 (k), Y_2 (k) \in \mathbb{R}^{n \times h} \), and symmetric matrices \( X_3 (k), Y_3 (k) \in \mathbb{R}^{n \times h} \), such that
\[
\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix}
\]
\[
in_{- \langle} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} = q_- \quad \text{for all } k \in C_i^{(j)}.
\]
We can repeat this procedure for all the other values of \( t \in Z \) and \( C_i^{(j)} \in C^{(j)} \), and then following the same argument as that at the end of the proof of Lemma 12, we can show that these matrix sequences can be uniformly bounded from below and above. This ensures that the graph-diagonal operators specified from these sequences would satisfy the desired invertibility and boundedness conditions. Finally, (19) ensures equal negative inertia over each permutation cycle, and hence condition (15) is also met.

**Theorem 14:** Given \( G \) as in (1), a vector-valued sequence \( (n_{K_0}, \ldots, n_{K_s}) \), and \( (n_0, \ldots, n_s) := p(A) \), then there exists an admissible synthesis \( K \) for \( G \) with realization dimensions satisfying \( p(A_K) \leq (n_{K_1}, \ldots, n_{K_s}) \) if there exist \( X \) and \( Y \) in \( \mathcal{X} \) such that, for each \( j \in \{1, \ldots, s\} \), \( C_i^{(j)} \in C^{(j)} \) and \( t \in Z \), the inequality \( n_j (t, k) + n_{K_j} (t, k) \geq \)
\[
\operatorname{in}_{+} \begin{bmatrix} X_j & I \\ I & Y_j \end{bmatrix} (t, k) + \operatorname{in}_{- \langle} \begin{bmatrix} X & I \\ Y & Y_2 \end{bmatrix} (t, C_i^{(j)})
\]
holds for all \( k \in C_i^{(j)} \), and furthermore,
\[
\begin{bmatrix} X & I \\ Y & Y_2 \end{bmatrix} \leq n_0 + n_{K_0}.
\]

**References**


