APPENDIX A

PROOFS

Proof 1: No Generalized Inverse Will Allocate $u \in \Omega \ \forall \ m \in \Phi$

Denote by $\Psi$ the subset of $\partial(\Omega)$ which maps to $\partial(\Phi)$, and by $u'$ the vectors in $\Psi$. The span($\Psi$) = $\mathbb{R}^m$ unless there are columns of $B$ which are all zeros. It would be impractical for a column of $B$ to be all zeros because that would mean that a “control” had no effect.

If a generalized inverse allocates admissible controls for all the attainable moments, $P$ must satisfy $u' = PBu' \ \forall \ u'$, or $[PB - I_m]u' = 0 \ \forall \ u'$. The constrained control vectors that map to $\partial(\Phi)$ must lie in $\mathbb{N} [PB - I_m]$, the null-space of $[PB - I_m]$:

$$u' = \{ u \in \Omega \cap \mathbb{N} [PB - I_m] \mid Bu \in \Phi \}$$  \hspace{1cm} (A-1)

If the controls $u'$ span $\mathbb{R}^m$, they cannot all lie in $\mathbb{N} [PB - I_m]$, because span($\mathbb{N} [PB - I_m]$) = $\mathbb{R}^n$. ($n < m$) Therefore, there is no generalized inverse that yields solutions that are admissible everywhere on the boundary of $\Phi$, and no single generalized inverse allocates admissible controls for all the attainable moments.

Proof 2: Convexity is Preserved Under Linear Transformations

If $W$ is a convex set and $T$ is a linear transformation, then $F = TW$ is a convex set.

The transformation of a vector by a matrix multiplication is a linear transformation, and the following is true:

$$A(\alpha x + \beta y) = A(\alpha x) + A(\beta y) = \alpha (Ax) + \beta (Ay)$$  \hspace{1cm} (A-2)

$W$ is a subset of the vector space $\mathbb{R}^m$;
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\[ W \subset R^m \]  

(T is a matrix which maps vectors in \( R^m \) to some space \( R^n \),

\[ T:R^m \rightarrow R^n \]  

\( F \) is the image of \( W \) in \( R^n \),

\[ F = TW, \ F \subset R^n \]  

\( A \) is a vector in \( W \) to some point \( a \), \( A \in W \), and \( B \) is a vector in \( W \) to some point \( b \), \( B \in W \), such that \( B \neq A \). See Figure A-1. A convex set is a set of points which contains, for any two points \( a,b \) in the set, the entire segment \( \overline{ab} \). \( C \) is any vector to a point \( c \) which lies on the line \( \overline{ab} \). \( C \) can be expressed in terms of \( A \) and \( B \):

\[ C = A + \alpha (B - A) = (1 - \alpha)A + \alpha B, \ 0 \leq \alpha \leq 1 \]  

Because \( W \) is convex, \( C \in W \). \( A' \) and \( B' \) are the images of \( A \) and \( B \):

\[ A' = TA, \ A' \in F \]  

\[ B' = TB, \ B' \in F \]  

\( D \) is a vector to any point \( d \), which lies on \( \overline{a'b'} \). \( D \) can be expressed in terms of the vectors \( A' \) and \( B' \):

\[ D = A' + \alpha (B' - A') = (1 - \alpha)A' + \alpha B', \ 0 \leq \alpha \leq 1 \]  

Substituting for \( A' \) and \( B' \) from Equations A-7 and A-8:

\[ D = (1 - \alpha)TA + \alpha TB \]
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Because $T$ is a linear transformation, this can be written

$$\mathbf{D} = T((1- \alpha)\mathbf{A} + \alpha \mathbf{B}) = TC$$

(Equation A-11)

Equation A-11 shows that a point on $a'b'$ is the image of a point on $ab$.

All points in $W$ map to points in $F$. If all points on $\overline{ab}$ are in $W$, and all points on $\overline{a'b'}$ are images of points in $\overline{ab}$, then all points on $\overline{a'b'}$ are in $F$:

$$\mathbf{D} \in F$$

If any two points $a$ and $b$ are in $W$, then they map to points $a'$ and $b'$ which are in $F$. If the entire line $\overline{ab}$ is contained in $W$, then the entire line $\overline{a'b'}$ is contained in $F$. Thus, if $W$ is convex, under the linear mapping $T$, $F$ is also convex.
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Figure A-1: Convexity Proof
Proof 3: Objects Remain Parallel Under Linear Transformations

If the vectors \( \mathbf{A} \) and \( \mathbf{B} \) are parallel, one can be expressed as a scalar multiple of the other:

\[
a \mathbf{A} = \mathbf{B}
\]  

Under the linear transformation, \( T \), they will remain parallel.

\[
\mathbf{A}' = T \mathbf{A}
\]

\[
\mathbf{B}' = T \mathbf{B} = T(a \mathbf{A}) = a T \mathbf{A} = a \mathbf{A}'
\]

Objects in \( \mathbb{R}^m \) are defined by vectors in \( \mathbb{R}^m \). If parallel vectors in \( \mathbb{R}^m \) map to parallel vectors in \( \mathbb{R}^n \), the objects defined by the parallel vectors in \( \mathbb{R}^m \) will map to parallel objects in \( \mathbb{R}^n \).
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**Proof 4: Points on \( \partial(\Phi) \) Map to Unique Points on \( \partial(\Omega) \)**

To prove that if \( B \) contains no singular \( nxn \) submatrices, points on the boundary of \( \Phi \) are images of unique points on the boundary of \( \Omega \), it will first be shown that points on \( \partial(\Phi) \) must be images of points on \( \partial(\Omega) \). Next, two solutions on \( \partial(\Omega) \) which map to the same point on \( \partial(\Phi) \) are examined. It will be shown that this can only occur if there is a \( nxn \) submatrix of \( B \) which is singular. An \( nxn \) submatrix of \( B \) is a matrix which contains any \( n \) of the columns of \( B \).

Consider a moment on the boundary of \( \Phi \):

\[
m^*_1 \in \partial(\Phi) \tag{A-16}
\]

Assume that there is a point in \( R^m, u_1 \), which is admissible, but not on the boundary of \( \Omega \) which maps to \( m^*_1 \):

\[
u_1 \in \Omega, \ u_1 \notin \partial(\Omega) \tag{A-17}
\]

\[
Bu_1 = m^*_1 \in \partial(\Phi) \tag{A-18}
\]

Because \( u_1 \) is not on the boundary of \( \Omega \), there exists some \( a > 1 \) for which \( au_1 \) is on the boundary of \( \Omega \):

\[
a u_1 \in \partial(\Omega) \text{ for some } a > 1 \tag{A-19}
\]

However, because \( m^*_1 \) is on the boundary of \( \Phi \) and \( \Phi \) is convex, there can be no moment greater than \( m^*_1 \) which is in \( \Phi \) (See Figure A-2):

\[
a > 1 \Rightarrow B(au_1) = am^*_1 \notin \Phi \tag{A-20}
\]

By definition, all controls in \( \Omega \) map to points in \( \Phi \).
Figure A-2: Moment on Boundary of $\Phi$
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\[ au_1 \in \Omega \Rightarrow B(au_1) \in \Phi \]  

(A-21)

Equations A-20 and A-21 contradict each other. Thus, the assumption that a point not on the boundary of \( \Omega \) can map to a point on the boundary of \( \Phi \) is false, and all points on the boundary of \( \Phi \) must be images of points on the boundary of \( \Omega \).

If there are two solutions in \( R^m \) which are not identical, they will have one or more of the components of \( u \) set to different values. In general, there will be between 1 and \( m \) controls which have different values. This general problem is divided into two categories. The first involves cases where the number of controls which are have different values is fewer than \( n \). The second involves cases where \( n \) or more controls have different values. It will be shown for each category that if two points in \( \partial(\Omega) \) map to the same point in \( \partial(\Phi) \), then there must be a singular \( nxn \) submatrix of \( B \).

**Category 1:** Two points in \( R^m \) on \( \partial(\Omega) \) map to the same point in \( R^n \) on \( \partial(\Phi) \) and the two points in \( R^m \) have \( n-1 \) or fewer controls which are set to different values.

Consider two points, \( u^*_1 \) and \( u^*_2 \), which map to \( m^*_1 \). By definition,

\[ u^*_1 \in \partial(\Omega) \]  

(A-22)

\[ u^*_2 \in \partial(\Omega), \; u^*_2 \neq u^*_1 \]  

(A-23)

Therefore,

\[ Bu^*_{\text{1}} = m^*_1 \]  

(A-24)

\[ Bu^*_{\text{2}} = m^*_1 \]  

(A-25)

Subtracting Equation A-25 from A-24, yields:
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\[ Bu^*_1 - Bu^*_2 = B(u^*_1 - u^*_2) = 0 \quad (A-26) \]

So that,

\[ u^*_1 \neq u^*_2 \Rightarrow (u^*_1 - u^*_2) \neq 0 \quad (A-27) \]

Because there are \( n\)-1 or fewer controls which have different values in the vectors \( u^*_1 \) and \( u^*_2 \), there will be \( n\)-1 or fewer non-zero elements in \( (u^*_1 - u^*_2) \). Equation A-26 can be re-written, removing the columns of \( B \) which correspond to the elements of \( (u^*_1 - u^*_2) \) which are zero.

Define \( B' \) and \( u_b \):

\[ B' \equiv \text{a submatrix of } B \text{. The columns of } B' \text{ are the columns of } B \text{ which correspond to the non-zero elements of } (u^*_1 - u^*_2) \quad (A-28) \]

\[ u_b \equiv \text{a vector containing the non-zero elements of } (u^*_1 - u^*_2) \quad (A-29) \]

Then,

\[ B' \in \mathbb{R}^{nxn}, \ u_b \in \mathbb{R}^y, \ y \leq (n-1) \quad (A-30) \]

\[ B'u_b = 0 \quad (A-31) \]

Equation A-31 shows that a set of \( (n-1) \) or fewer columns of \( B \) are linearly dependent.

If columns of an \( nxn \) matrix are linearly dependent, the matrix is singular. The \( nxn \) submatrices of \( B \) containing the linearly dependent columns in \( B' \) will be singular.

**Category 2:** Two points in \( R^m \) on \( \partial(\Omega) \) map to the same point in \( R^n \) on \( \partial(\Phi) \) and the two points in \( R^m \) have \( n \) or more controls which are set to different values.
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1) If 2 points in \( \partial(\Omega) \), \( \mathbf{u}^*_1 \) and \( \mathbf{u}^*_2 \), map to the same point in \( \partial(\Phi) \), \( \mathbf{m}^*_1 \), and have \( n \) controls set to different values, there is some point in \( \mathbb{R}^m \) which maps to \( \mathbf{m}^*_1 \) which has those \( n \) controls set to values within their constraints, \( u_i \text{ Min} < u_i < u_i \text{ Max} \).

If \( \mathbf{u}^*_1 \) and \( \mathbf{u}^*_2 \) map to a point in \( \mathbb{R}^n \), then all points on a line between them also map to the same point in \( \mathbb{R}^n \). This can be seen from Equation A-6, which shows that any point on a line between two vectors can be expressed as a linear combination of those vectors. Consider a point, \( \mathbf{u}^*_3 \), which lies on a line between \( \mathbf{u}^*_1 \) and \( \mathbf{u}^*_2 \).

\[
\mathbf{u}^*_3 = (1-\alpha)\mathbf{u}^*_1 + \alpha \mathbf{u}^*_2, \quad 0 \leq \alpha \leq 1
\]  \hspace{1cm} (A-32)

If \( \mathbf{u}^*_1 \) and \( \mathbf{u}^*_2 \) map to \( \mathbf{m}^*_1 \), so does \( \mathbf{u}^*_3 \).

\[
B\mathbf{u}^*_3 = (1-\alpha)B\mathbf{u}^*_1 + \alpha B\mathbf{u}^*_2 = (1-\alpha)\mathbf{m}^*_1 + \alpha \mathbf{m}^*_1 = \mathbf{m}^*_1
\]  \hspace{1cm} (A-33)

Consider Figure A-3 to be a box on the boundary of \( \Omega \). At \( \mathbf{u}^*_1 \) the three controls which define the box \( (u_i, u_j, u_k) \) are at their positive limits. At \( \mathbf{u}^*_2 \) these controls are at different admissible values. The line connecting them contains points which have all three controls at non-limited values.

2) Controls which are not at limiting values may be moved in either direction to generate moments.

3) The columns of \( B \) define directions in \( \mathbb{R}^n \).

A column of \( B \), \( B_i \), is a vector in \( \mathbb{R}^n \). Changing the value of the control, \( u_i \), will move the moment produced in the direction defined by \( B_i \). See Figure A-4.

4) From a point on \( \partial(\Phi) \), the value of a control can be changed, changing the moment produced in a direction along the boundary or towards the interior.
Figure A-3: A 3-D Object on $\partial(\Omega)$

$Bu$

Figure A-4: Columns of $B$ Are Directions in $\mathbb{R}^n$
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If a control is saturated, it can move in only one direction. Changing a saturated control may move the moment produced to a point interior to the boundary without being able to move the moment produced to a point exterior to the boundary. See Figure A-5.

5) If a control is not saturated, it can move the moment produced only along the boundary.

If changing an unsaturated control could move the moment produced interior to the boundary, it could also move the moment produced exterior to the boundary, which is not possible. See Figure A-6.

6) The $n$-D polytope $\Phi$ is bounded by $(n-1)$-D polytopes.

From a point on the boundary, there are at most $(n-1)$ independent directions to move which are along the boundary. If there are $n$ or greater non-saturated controls at some point on the boundary, then there are $n$ or greater directions defined by these controls. However, these $n$ or greater directions span a space of dimension $n-1$ or less. See Figure A-7.

If $n$ columns of $B$ span an $n$-1 dimensional space, the rank of the nxn matrix containing those columns will be $n-1$. Thus, an nxn submatrix of $B$ will be singular.
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Figure A-5: Saturated Controls Can Move Either Along or Interior to $\partial(\Phi)$

Figure A-6: Unsaturated Controls Can’t Move Interior to $\partial(\Phi)$

Figure A-7: 3 Columns of $B$ Contained in a 2-D Bounding Object of $\Phi$