Integral Theorems and Second Order Operators
1st Order Integral Theorems

- Gradient theorem
- Divergence theorem
- Curl theorem
- Stokes’ theorem
The Gradient Theorem

Begin with the definition of grad:

\[ \text{grad} \phi \equiv \lim_{\delta \tau \to 0} \frac{1}{\delta \tau} \int_{\Delta S} \phi \, n \, dS \]

Sum over all the \( d\tau \) in \( R \):
Assumptions in Gradient Theorem
Flow over a finite wing

\[ \int_{R} \nabla p \, d\tau = \int_{S} p \, n \, dS \]

\( R \) is the volume of fluid enclosed between \( S_1 \) and \( S_2 \). \( p \) is not defined inside the wing so the wing itself must be excluded from the integral.
Alternative Definition of the Curl

\[ \mathbf{e}. \text{curl} \mathbf{A} \equiv \lim_{\delta \sigma \to 0} \frac{1}{\delta \sigma} \int_{C_e} \mathbf{A} \cdot d\mathbf{s} = \lim_{\delta \sigma \to 0} \frac{\Gamma_{C_e}}{\delta \sigma} \]
Stokes’ Theorem

Begin with the alternative definition of curl, choosing the direction $e$ to be the outward normal to the surface $n$:

$$n \cdot \nabla \times A \equiv \lim_{\delta \sigma \to 0} \frac{1}{\delta \sigma} \oint_{C_e} A \cdot ds$$

Sum over all the $d\sigma$ in $S$: 
Stokes’ Theorem and Velocity

• Apply Stokes’ Theorem to a velocity field

\[ \int_S \nabla \times \mathbf{V} \cdot \mathbf{n} \, dS = \oint_C \mathbf{V} \cdot d\mathbf{s} \]

• Or, in terms of vorticity and circulation

\[ \int_S \mathbf{\Omega} \cdot \mathbf{n} \, dS = \oint_C \mathbf{V} \cdot d\mathbf{s} = \Gamma_C \]

• What about a closed surface?

\[ \int_S \mathbf{\Omega} \cdot \mathbf{n} \, dS = 0 \]
Assumptions of Stokes’ Theorem
Flow over a finite wing

\[ \int_S \nabla \times \mathbf{V} \cdot \mathbf{n} \, dS = \oint_C \mathbf{V} \cdot d\mathbf{s} \]

Wing with circulation must trail vorticity. *Always.*
Vector Operators of Vector Products

\[ \nabla(\psi \Phi) = \psi \nabla \Phi + \Phi \nabla \psi \]

\[ \nabla.(\Phi \vec{A}) = \Phi \nabla \cdot \vec{A} + \nabla \Phi \cdot \vec{A} \]

\[ \nabla \times (\Phi \vec{A}) = \Phi \nabla \times \vec{A} + \nabla \Phi \times \vec{A} \]

\[ \nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) \]

\[ \nabla.(\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B} \]

\[ \nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) + (\vec{B} \cdot \nabla)\vec{A} - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{B} \]
Convective Operator

\((\vec{A}.\nabla)\Phi\)  
\[= \left( A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) \Phi \]
\[= \vec{A}.(\nabla \Phi) \]

\(\nabla . \rho\)  = change in density in direction of \(\mathbf{V}\), multiplied by magnitude of \(\mathbf{V}\)

\((\vec{A}.\nabla) \vec{B}\)  
\[= \left( A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) \vec{B} \]
\[= \frac{1}{2} \left[ \nabla(\vec{A}.\vec{B}) - \vec{A} \times (\nabla \times \vec{B}) - \vec{B} \times (\nabla \times \vec{A}) \right. \]
\[\left. - \nabla \times (\vec{A} \times \vec{B}) + \vec{A}(\nabla . \vec{B}) - \vec{B}(\nabla . \vec{A}) \right] \]
Second Order Operators

\[ \nabla \cdot \nabla \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \]

The Laplacian, may also be applied to a vector field.

\[ \nabla (\nabla \cdot \mathbf{A}) \]

\[ \nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]

- So, any vector differential equation of the form \( \nabla \times \mathbf{B} = 0 \) can be solved identically by writing \( \mathbf{B} = \nabla \phi \).
  - We say \( \mathbf{B} \) is \textit{irrotational}.
  - We refer to \( \phi \) as the \textit{scalar potential}.

\[ \nabla \times \nabla \phi \equiv 0 \]

\[ \nabla \cdot \nabla \times \mathbf{A} \equiv 0 \]

- So, any vector differential equation of the form \( \nabla \cdot \mathbf{B} = 0 \) can be solved identically by writing \( \mathbf{B} = \nabla \times \mathbf{A} \).
  - We say \( \mathbf{B} \) is \textit{solenoidal} or \textit{incompressible}.
  - We refer to \( \mathbf{A} \) as the \textit{vector potential}.  

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2\textsuperscript{nd} Order Integral Theorems

- Green’s theorem (1\textsuperscript{st} form)
  \[
  \int_R \psi \nabla^2 \phi + \nabla \psi \nabla \phi \, d\tau = \oint_S \psi \frac{\partial \phi}{\partial n} \, dS
  \]

- Green’s theorem (2\textsuperscript{nd} form)
  \[
  \int_R \psi \nabla^2 \phi - \phi \nabla^2 \psi \, d\tau = \oint_S \psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \, dS
  \]

These are both re-expressions of the divergence theorem.
Helmholz Decomposition Theorem

• Any vector field may be expressed as the sum of a gradient vector field and a curl vector field.

\[ \mathbf{B} = \nabla \times \mathbf{A} + \nabla \phi \]

Vector Potential

Scalar Potential