Lifting Line Theory

• Applies to *large aspect ratio unswept* wings at small angle of attack.
• Developed by Prandtl and Lanchester during the early 20th century.
• Relevance
  – Analytic results for simple wings
  – Basis of much of modern wing theory (e.g. helicopter rotor aerodynamic analysis, extends to vortex lattice method,)
  – Basis of much of the qualitative understanding of induced drag and aspect ratio

\[ \Gamma = \frac{h}{4\pi} \]

Biot Savart Law:
Velocity produced by a semi-infinite segment of a vortex filament

\[ V = \frac{\Gamma}{4\pi h} \]

Thin-airfoil theory
\[ C_f = 2\pi(\alpha - \alpha_o) \]

1868-1946
1875-1953
Physics of an Unswept Wing

\[ p_u < p_l \]

\[ p_u \approx p_l \]

Lift varies across span

Circulation is shed (Helmholz thm)

Vortical wake

Vortical wake induces downwash on wing…

…changing angle of attack just enough to produce variation of lift across span
Simplest Possible Model

Wake model

Section model

Induced drag $d_i$
LLT – The Section Model
LLT – The Wake Model
The Monoplane Equation

Wake model

\[ w(y) = \int_{-s}^{s} \frac{d\Gamma/\,dy \bigg|_{y_1}}{4\pi(y-y_1)} \, dy_1 \]

Section model

\[ \Gamma = \pi V_\infty (\alpha - \alpha_0)c - \pi wc \]

\[ \Gamma = \pi V_\infty (\alpha - \alpha_0)c - \frac{c}{4} \int_{-s}^{s} \frac{d\Gamma/\,dy \bigg|_{y_1}}{y-y_1} \, dy_1 \]

Substitute for \( \theta \), and express \( \Gamma \) as a sine series in \( \theta \)

\[ \Gamma = 4U_\infty s \sum_{n=1, \text{odd}}^{\infty} A_n \sin(n\theta) \]

\[ \frac{\pi c}{4s} (\alpha - \alpha_0) \sin \theta = \sum_{n=1, \text{odd}}^{\infty} A_n \sin(n\theta) \left[ \frac{\pi cn}{4s} + \sin \theta \right] \]

The Monoplane Eqn.
Results

Substituting \( \Gamma = 4U_\infty s \sum_{n=1, \text{odd}}^{\infty} A_n \sin(n \theta) \) into

\[
C_L = \frac{2}{V_\infty S} \int_{-s}^{s} \Gamma dy \\
C_{D_i} = \frac{2}{V_\infty^2 S} \int_{-s}^{s} w \Gamma dy \\
w(y) = \int_{-s}^{s} \frac{d\Gamma}{dy} \bigg|_{y_1} dy_1
\]


\begin{align*}
C_L &= AR \pi A_1 \\
C_{D_i} &= \frac{C_L^2}{\pi AR} (1 + \delta) \\
w &= \sum_{n=1, \text{odd}}^{\infty} nA_n \sin(n \theta) \\
\frac{w}{V_\infty} &= \sum_{n=1, \text{odd}}^{\infty} nA_n \sin(n \theta) \\
\delta &= \sum_{n=3, \text{odd}}^{\infty} n(A_n / A_1)^2
\end{align*}

\[
\frac{\pi c}{4s} (\alpha - \alpha_0) \sin \theta = \sum_{n=1, \text{odd}}^{\infty} A_n \sin(n \theta) \left[ \frac{\pi cn}{4s} + \sin \theta \right]
\]
Solution of monoplane equation

\[
\frac{\pi c}{4s} (\alpha - \alpha_0) \sin \theta = \sum_{n=1, \text{odd}}^{\infty} A_n \sin(n \theta) \left[ \frac{\pi c n}{4s} + \sin \theta \right]
\]

\[ y / s = -\cos \theta \]
s=2.8; \hspace{1cm} \%Half span (distances normalized on root chord)
alpha=5*pi/180; \hspace{1cm} \%5 degrees angle of attack
alpha0=-5.4*pi/180; \hspace{1cm} \%Zero lift AoA=-5.4 deg. for Clark Y
N=20; \hspace{1cm} \%N=20 points across half span
th=[1:N]'/N*pi/2; \hspace{1cm} \%Column vector of theta's
y=-cos(th)*s; \hspace{1cm} \%Spanwise position
c=ones(size(th)); \hspace{1cm} \%Rectangular wing, so c = c_r everywhere
n=1:2:2*N-1; \hspace{1cm} \%Row vector of odd indices
res=pi*c/4/s.*(alpha-alpha0).*sin(th); \hspace{1cm} \%N by 1 result vector
coeff=sin(th*n).*(pi*c*n/4/s+repmat(sin(th),1,N)); \hspace{1cm} \%N by N coefficient matrix
a=coeff\res; \hspace{1cm} \%N by 1 solution vector

\[ \Gamma = 4U_\infty S \sum_{n=1,\, odd}^{\infty} A_n \sin(n \theta) \]

\[ C_L = AR \pi A_1 \]

\[ C_{Di} = \frac{C_L^2}{\pi AR} (1 + \delta) \]

\[ \frac{\pi c}{4s} (\alpha - \alpha_0) \sin \theta = \sum_{n=1,\, odd}^{\infty} A_n \sin(n \theta) \left[ \frac{\pi c n}{4s} + \sin \theta \right] \]
Example

Our AR=5.6 Rectangular Clark Y Wing

\( \alpha_0 \approx -5.4^\circ \)

Determine aerodynamic characteristics of our rectangular Clark Y wing
Drag Polar

\[ C_D = \frac{C_L^2}{\pi AR} \]

\[ C_D = C_L (\tan \alpha - \alpha_{0li}) \]

Curve for minimum drag (elliptical wing)

Curve assuming wing only generates force normal to chord

Note that friction drag coefficient of 0.01 added to \( C_{Di} \)
The Elliptic Wing

The minimum drag occurs for a wing for which \( A_n = 0 \) for \( n \geq 3 \). For this wing:

\[
\cos \theta = -\frac{y}{s}
\]

1. \[
\Gamma = 4U_\infty s \sum_{n=1, odd}^{\infty} A_n \sin(n\theta)
\]

2. \[
\frac{w}{V_\infty} = \sum_{n=1, odd}^{\infty} \frac{nA_n \sin(n\theta)}{\sin \theta}
\]

3. \[
\Gamma = \pi V_\infty (\alpha - \alpha_0)c - \pi wc
\]
Further results

\[ C_L = AR \pi A_1 \]
\[ C_{D_i} = \frac{C_L^2}{\pi AR} \]
\[ \frac{w}{V_\infty} = A_1 \]
Spitfire

Note that the chordlengths are all lined up along the quarter chord line so the actual wing shape is not an ellipse.
Not done yet…

\[ C_L = \frac{2\pi AR(\alpha - \alpha_0)}{AR + 2} \]

\[ C_{D_l} = \frac{C_L^2}{\pi AR} \]
Geometrically Similar Wings

These results work quite well even for non-elliptical wings:

\[
\alpha_A - \alpha_B = \frac{C_L}{\pi} \left( \frac{1}{AR_A} - \frac{1}{AR_B} \right)
\]

\[
C_{D_{iA}} - C_{D_{iB}} = \frac{C_L^2}{\pi} \left( \frac{1}{AR_A} - \frac{1}{AR_B} \right)
\]