Mapping of Ideal Flows
Summary / Crib

1. Uniform flow of $V_\infty$ at angle $\alpha$ to the $x$ axis

\[ W(z) = V_\infty e^{-i\alpha} \quad F(z) = V_\infty z e^{-i\alpha} \]

2. Source at $z_1$ flow producing volume flowrate $q$

\[ W(z) = \frac{q}{2\pi(z-z_1)} \quad F(z) = \frac{q}{2\pi} \log_e(z-z_1) \]

3. Vortex at $z_1$ producing circulation $\Gamma$

\[ W(z) = \frac{i\Gamma}{2\pi(z-z_1)} \quad F(z) = -\frac{i\Gamma}{2\pi} \log_e(z-z_1) \]

4. Doublet at $z_1$, strength $\mu$ aligned at angle $\beta$ to the $x$ axis

\[ W(z) = \frac{\mu e^{i\beta}}{2\pi(z-z_1)^2} \quad F(z) = -\frac{\mu e^{i\beta}}{2\pi(z-z_1)} \]

5. Flow of velocity $V_\infty$ at angle $\alpha$ past a circular cylinder of radius $a$ at $z_1$ with circulation $\Gamma$

\[ F(z) = V_\infty e^{-i\alpha} z + \frac{V_\infty a^2 e^{i\alpha}}{z-z_1} - \frac{i\Gamma}{2\pi} \log_e(z-z_1) \]

\[ W(z) = V_\infty e^{-i\alpha} - \frac{V_\infty a^2 e^{i\alpha}}{(z-z_1)^2} - \frac{i\Gamma}{2\pi(z-z_1)} \]
Mapping Functions

\[ z = x + iy \] represents coordinate \((x,y)\)

\[ \zeta = \xi + i\eta \] represents coordinate \((\xi,\eta)\)
Mapping a Flow

Imagine the simple mapping $\zeta = 2z$. We would want the mapping to transfer the flow at $1+i$, say, to the point $2+2i$. 
When is a mapped flow valid?
Mapping complex potential or velocity?

We usually map the complex potential because it is easy to envisage moving or distorting the streamlines with the mapping function. One can map the complex velocity instead (hodograph mapping) but this is harder to visualize and in general produces a different result.
Effects of Mapping on Microscopic Geometry

\[ \zeta = \zeta(z) \]

\[ z = z(\zeta) \]
1. Rotation and scaling $\zeta = Az$
   $A = \text{const.} = ce^{i\beta}$

2. Power $\zeta = z^a$ $a = \text{real} > 0$

3. Inversion in a circle $\zeta = 1/z$

4. Logarithm $\zeta = \log_e z$

5. Joukowski $C = \text{real} > 0$
   $z = \zeta + C^2 / \zeta$

Effects of Mapping on Macroscopic Geometry

“To understand what a mapping does to a flow, one must first understand what it does to the space containing that flow”
1. Magnification and Rotation

\[ \zeta = (1 + i)z \]

\[ z = \zeta / (1 + i) \]

Flow past a circular cylinder
2. Power

\[ \zeta = z^{0.8} \]

\[ z = \zeta^{1.25} \]

Flow past a circular cylinder
3. Inversion in a unit circle

\[ \zeta = 1/z \]

\[ z = 1/\zeta \]

Uniform flow
4. Logarithm Mapping

\[ \zeta = \log_e z \]

\[ z = e^\zeta \]

What happens to the flow under the x axis?
Example: Flow past a 90° Corner

Determine the complex potential and complex velocity for flow past an external 90 deg corner by mapping a uniform flow.

What happens at the critical point?

What happens if we consider the flow below the x axis?
Equopotentials

Velocity goes to infinity at critical point

Streamlines
Branches ...
Example: Flow through a channel

Determine the complex potential and complex velocity for flow past a Rankine halfbody in a channel.