Equations of Motion
Supersonic Turbulent Jet Flow and Near Acoustic Field

Freund *et al.* (1997)
Stanford Univ.
DNS
Conservation Laws

• Conservation of mass

• Conservation of momentum

• Conservation of energy
Perspectives

**Eulerian Perspective** – the flow as as seen at fixed locations in space, or over fixed volumes of space. (The perspective of most analysis.)

**Lagrangian Perspective** – the flow as seen by the a specific piece of the gas. (The perspective of the laws of motion.)

Control volume: finite fixed region of space (Eulerian)  System: finite piece of the gas (Lagrangian)
Strategy

• Write down equations of motion from the Lagrangian perspective of a system
• Derive relationship between Lagrangian and Eulerian perspectives
• Substitute to get Eulerian equations of motion
Conservation Laws

- Conservation of mass

- Conservation of momentum

- Conservation of energy
Pressure Force and Pressure Work

• Pressure force

\[ p \] System of gas molecules
\[ n \] Surface area \( S \)
\[ U \]

• Pressure work

\[ \text{Work} = \text{Force} \times \text{distance moved in direction of force} \]
\[ \text{Rate of Work} = \text{Force} \times \text{in direction of force} \]
Relation between Lagrangian and Eulerian time Derivatives

- All the conservation laws have the form

\[ \frac{\partial N}{\partial t} \bigg|_{\text{sys}} = \ldots \]

where \( N \) is a quantity that depends on mass

\[
\begin{align*}
N &= M & \vec{P} & E_{\text{tot}} \\
\eta &= 1 & \vec{U} & e_{\text{tot}} = e + \frac{1}{2} U^2
\end{align*}
\]

amount of \( N \) per unit mass

\[ \eta = \int \eta \rho dV \]
System moving through space

Control volume (fixed)
System (moving)

Time $t$

Time $t+dt$
\[
\frac{\partial N}{\partial t} \bigg|_{sys} = \frac{\partial}{\partial t} \int \eta \rho dV + \text{Net outflow of } N \text{ from control volume}
\]

Control volume, surface \(CS\) 

\[\int \eta \rho dV\]

Mass flow rate out of \(dS\) = Velocity normal to \(dS\) \(\times\) Area of \(dS\) \(\times\) density

Flow rate of \(N\) out of \(dS\) = Mass flow rate \(\times\) \(\eta\)
\[
\frac{\partial N}{\partial t} \bigg|_{\text{sys}} = \frac{\partial}{\partial t} \int_{V} \eta \rho dV + \oint_{S} \eta \rho \vec{U} \cdot \vec{n} dS
\]

\[
N = \begin{bmatrix} M \\ \eta \vec{U} \end{bmatrix}, \quad e_{\text{tot}} = e + \frac{1}{2} U^2
\]

**Conservation Laws**

- Conservation of mass

\[
\frac{\partial M}{\partial t} \bigg|_{\text{sys}} = 0
\]

- Conservation of momentum

\[
\frac{\partial \vec{P}}{\partial t} \bigg|_{\text{sys}} = -\oint_{S} \vec{p} \vec{n} dS
\]

- Conservation of energy

\[
\frac{\partial E_{\text{tot}}}{\partial t} \bigg|_{\text{sys}} = \frac{\partial Q}{\partial t} - \oint_{S} \vec{p} \vec{n} \cdot \vec{U} dS
\]
Conservation Laws

• Conservation of mass

\[ \frac{\partial}{\partial t} \int_{CV} \rho dV + \oint_{CS} \rho \vec{U} \cdot \hat{n} dS = 0 \]

• Conservation of momentum

\[ \frac{\partial}{\partial t} \int_{CV} \rho \vec{U} dV + \oint_{CS} \rho \vec{U} (\vec{U} \cdot \hat{n}) dS = - \oint_{CS} p \hat{n} dS \]

• Conservation of energy

\[ \frac{\partial}{\partial t} \int_{CV} \rho (e + \frac{1}{2} U^2) dV + \oint_{CS} (e + \frac{1}{2} U^2) \rho (\vec{U} \cdot \hat{n}) dS = \frac{\partial Q}{\partial t} - \oint_{CS} p \hat{n} \vec{U} dS \]
Application: The Speed of Sound

Undisturbed air (still)

\[ p, \rho, T, u=0 \]

Sound speed \( a \)

Disturbed air (moving slowly to right)

\[ p+dp, \rho+d\rho, T+dT \]

\[ du \rightarrow \]

Sound speed \( a+da=a+du \)

Moving with the wave....

\[ p, \rho, T \]

\[ \vec{i} \rightarrow \]

\[ p+dp, \rho+d\rho, T+dT \]

Control volume thin enough for sides to be ignored
Conservation of Mass

\[ \oint_{CS} \rho \vec{U} \cdot \vec{n} dS = 0 \]
Conservation of Momentum

\[ \oint_{CS} \rho \vec{U} (\vec{U} \cdot \vec{n}) dS = -\oint_{CS} p \vec{n} dS \]

\( \vec{i} \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad a \quad \rightarrow \quad a + da \)

\( p, \rho, T \quad p + dp, \rho + d\rho, T + dT \)
\[ a^2 = \frac{\partial p}{\partial \rho} \]

**Speed of sound**

To do this derivative we need to decide what kind of process sound is.