A force and torque tensegrity sensor

Cornel Sultan, Robert Skelton

Harvard University, Boston, MA 02115, USA

Department of Mechanical and Aerospace Engineering, University of California, La Jolla, CA 92031-0411, USA

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Abstract

Tensegrity structures represent a special class of flexible structures, whose members can simultaneously perform the functions of strength, sensing, actuating, and feedback control. In this article we show how these structures intrinsic properties can be exploited to construct a smart sensor for simultaneous measurement of six different quantities: three orthogonal forces and three orthogonal torques. The static and dynamic characteristics of the sensor are computed and the influence of friction and prestress upon these characteristics is analyzed. An optimal estimator design is presented and its performance is evaluated through numerical simulations. These simulations indicate that the tensegrity sensor is capable of simultaneously providing correct estimates of the six quantities of interest.

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Keywords: Tensegrity; Sensor; Optical fiber

1. Introduction

A major challenge in the field of sensors technology is the development of smart sensors. These should be sensing devices, easy to calibrate and adapt to new operating conditions and requirements, and capable of accommodating complex algorithms for local processing to remote controllers and computers. A smart sensor should also be able to compensate algorithmically for the inherent noise. Another important demand for smart sensors is to be capable of simultaneously providing redundant, highly reliable information about different physical signals.

This article presents a smart sensor which enables the simultaneous measurement of six quantities: three orthogonal forces and three orthogonal torques. The device consists of a tensegrity structure (Fig. 1) composed of 18 elastic tendons, six rigid bars, and a rigid top, whose intrinsic properties are exploited to make it a smart sensor.

One of the most important advantages of tensegrity structures is that the elastic members (e.g. tendons) provide excellent opportunities for sensing functions while also acting as structural elements. Measurements of the geometry of these tendons are easy to obtain through embedded optical fibers and can be used to estimate the quantities of interest (here three external forces and three external torques). Also, because the number of candidate sensing elements is large, various combinations can be used to provide more exact as well as fault tolerant and redundant approximations of the measured quantities.

The article is organized as follows. First, a detailed description of the tensegrity sensor is given. Next, the static and dynamic characteristics of the sensor are analyzed. The article shows how these characteristics can be easily calibrated through pretension adjustment. Then an optimal estimator is designed, based on the linearized mathematical model of the structure’s dynamics and assumed models of the external quantities to be estimated. Finally the design is evaluated through numerical simulations.

2. Tensegrity sensor description

A perspective view of a tensegrity sensor is given in Fig. 1. Its components are: a triangular fixed base (A1A2A3); a triangular rigid top (B1B2B3); three bars attached through ball and socket joints to the base (A1B1); three bars similarly attached to the rigid top (A2B2); 18 tendons connecting the end points of the bars. The tendons are classified as saddle: S, (A1B1); vertical: V, (A1B1 and A2B2); diagonal: D, (A1A2 and B1B2) tendons, respectively.

For the tensegrity sensor, the rigid top plays the role of the proof element. The sensing task consists in estimating

E-mail address: cornel.sultan@yahoo.com (C. Sultan).

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the three orthogonal forces and three orthogonal torques acting on the rigid top. In order to accomplish this task, some of the tendons will act as sensing elements and provide real time information about their lengths. These individual measurements are the inputs to a dynamic state estimator which produces the desired estimates. In this design the sensing tendons are not only structural elements, but members of the intelligent system which accomplishes the estimating task. The tensegrity structure, with the estimator designed to meet certain performance specifications, represents a smart, integrated sensor, capable of providing simultaneous information about six different quantities.

2.1. Tensegrity structures

Tensegrity structures represent a class of space structures composed of a set of soft members and a set of hard members. The soft members cannot carry other significant loads except for tensile ones. The representative example is an elastic tendon which cannot be compressed for all practical purposes but can carry significant tension. Because of this property we shall also refer to these members as tensile members. On the other hand, the hard members are characterized by the fact that they can carry any type of load. The representative example is a bar which can carry significant and comparable tension, compression forces, bending torques, etc.

A structure composed of soft and hard elements as described above is a tensegrity structure if it has the property of prestressability. This property consists of the structure’s ability to maintain an equilibrium shape with all tensile members in tension and in the absence of external forces or torques. Tensegrity structures integrity is guaranteed by the tensile members in tension, hence their denomination, tensegrity, an acronym of tension-integrity coined by R.B. Fuller.

The origins of tensegrity structures can be pinpointed to a constructivist artist’s (Loganson) sculpture in 1921 (see [12]). Tensegrity structures research was initiated by Fuller [4] and Pugh [23]. Rigorous investigations have been carried out rather recently, starting, interestingly enough, with the fine contribution of the celebrated literary critic of last century’s modernism, Kenner [9]. Calladine [1], Motro et al. [11], Pellegrino and Calladine [22], Sultan et al. [31], made important contributions to the general theory of these structures statics. Sultan et al. [31], Murakami and Nishimura [13,14] and Nishimura and Murakami [18] published a series of results regarding analytical solutions of the statics problem, while Hanaor [5], Kebiche et al. [8], Vassart and Motro [38], developed numerical methods of large generality. The field of tensegrity structures dynamics research was pioneered by Motro et al. [11]. Significant theoretical advances have been later made by Murakami [15] and Sultan et al. [32,33] who developed nonlinear and linearized dynamics models for tensegrity structures. Using simpler models, Connelly and Whiteley [2] proved some important results regarding the stability and rigidity of equilibrium configurations, while Oppenheim and Williams [20,21] discovered interesting properties of their vibration and damping characteristics. Control design studies have been pioneered by Skelton and Sultan [26], followed by contributions of Sultan and Skelton [28], Djouadi et al. [3], and Kanchanasaratool and Williamson [7]. Applications of tensegrity structures have been proposed, ranging from tensegrity domes [6,39], to antennas [3], space telescopes [27], flight simulators [30], deployable structures [35,36].

2.2. Sensing mechanisms

The problem of choosing the sensing mechanism of the sensing tendons deserves some discussion. The small size, large dynamic bandwidth and ease of embedding make fiber optic sensors extremely attractive for use as sensing elements in a tensegrity sensor. Fiber optic sensors also present several inherent advantages over conventional (resistive) sen-
sors which include: electro-magnetic and hazardous environment insensitivity, the absence of Joule heating effect, high response bandwidth and low cost and weight per unit length. Optical fiber sensors are classified as being extrinsic or intrinsic. In intrinsic sensors the optical energy in the fiber is affected directly by the phenomenon being measured and changes in the output intensity give an indication of the magnitude of the disturbance. Typical intrinsic fiber sensors are interferometric sensors such as Fabry–Perot (see e.g. [40,41]), Mach–Zehnder (see [16]), Michelson (see [37]), polarimetric sensors (see [10]) and modal interferometers such as two-mode elliptic core [17] and twin core [37] ones. In extrinsic sensors the fiber only serves to carry optical power to, and sensing information from, the region of interest. The sensing mechanism of the tensegrity sensing tendons should be intrinsic in nature since the fiber must be embedded in the tendon whose characteristics (e.g. lengths) variations are to be measured.

Optical fiber sensors are also classified according to the transduction mechanism which brings about a change in some property of optical power passing through the fiber, such as intensity, phase, modal content, polarization, etc., the corresponding sensors being known as intensity, phase or interferometric based, modal interferometric and polarimetric sensors. Intensity sensors consist of an optical power source, an optical fiber, and a photo-detector or spectrum analyzer. They are simple and keep the signal processing complexity at a low level. The main disadvantage of intensity sensors is their relatively low sensitivity compared to interferometric or polarimetric sensors.

Interferometric sensors are more complicated, consisting of a reference fiber which is isolated from the perturbation being measured and a sensing fiber, extremely susceptible to this perturbation. The outputs from the two fibers interfere at the photo-detector, the magnitude of the interference being dependent on the perturbation. The Mach–Zehnder, Michelson, Fabry–Perot interferometric sensors, employ the classical corresponding interferometer schemes. Their main advantage is an excellent strain sensitivity, while the main disadvantage is that the coherence length of the optical source limits the maximum sensor length (typically 1 m or less). For most of the interferometric sensors the input/output curve is sinusoidal, requiring further post-processing electronics which increases the signal processing complexity. Of the interferometric sensors mentioned here, the Mach–Zehnder and Michelson interferometers are usually larger in size (since they are two-fiber sensors), which could make them difficult to embed in the sensing tendon. Fabry–Perot, on the other hand is single fiber, has excellent remote operability properties with insensitive leads and, in principle, sensors of virtually any length can be manufactured.

Modal interferometers are generally single fibers, lending themselves to embedment. They can, in principle measure strain and temperature simultaneously providing the opportunity for compensation during strain measurement. Disadvantages are mainly related to the signal processing complexity and to the difficult remote operability and lead insensitivity. The fabrication of modal interferometers is also somehow difficult. Their sensitivity is two or three orders of magnitude lower than for standard interferometers.

Polarimetric sensors use specialized high birefringence fiber, containing two well-defined “eigenaxes” (birefringent axes) which are orthogonal and have significantly different effective refractive indices. Light energy entering either “eigenmode” will remain in that mode as the fiber undergoes temperature or strain variations, but the relative propagation speed of energy in these two modes will be affected. Changes in relative delay between components of the light in these two modes have been shown to be proportional to longitudinal strain [19]. This relative phase delay can be measured by observing the output light through a correctly oriented polarizer. Polarimetric sensors are single fiber—thus easy to embed—exhibit excellent lead-in insensitivity, remote operability, are relatively easy to manufacture. On the other hand, they are expensive, since they require specialized fiber, fiber polarizers, etc. The signal processing complexity associated with polarimetric sensors is also significant. Each of the sensing mechanisms previously described represent viable options for tensegrity sensing tendons. The purpose of this paper is to discuss the mechanics and computational features of the sensor and to present a feasible design, which is done next.

3. Modeling assumptions for the tensegrity sensor

The assumptions made for the mathematical modeling of the tensegrity sensor are: the top is a rigid, homogeneous plate, the bars are rigid, axially symmetric, of negligible longitudinal inertia, the tendons are massless and linear elastic. The bars are attached to the fixed base or top through ball and socket joints. Damping torques which are proportional to the relative angular velocity between the bars and the base or top act at the joints. The influence of the gravitational opportunity for compensation during strain measurement. Disadvantages are mainly related to the signal processing complexity and to the difficult remote operability and lead insensitivity. The fabrication of modal interferometers is also somehow difficult. Their sensitivity is two or three orders of magnitude lower than for standard interferometers.

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The inertial system of reference, $b_{1}, b_{2}, b_{3}$, is a dextral set of unit vectors, whose center coincides with the geometric center of the base triangle, $A_{1}A_{2}A_{3}$. Axis $b_{1}$ is orthogonal to plane $A_{1}A_{2}A_{3}$ pointing upward while $b_{1}$ is parallel to $A_{1}A_{2}A_{3}$. The top reference frame, $t_{1}, t_{2}, t_{3}$, is a central principal system of the homogeneous top plane $B_{1}2B_{2}B_{3}$.

The 18 independent generalized coordinates necessary to describe the configuration of this system are $\psi, \phi, \theta$, the Euler angles for a 3–1–2 sequence to characterize the orientation of the top reference frame in the inertial one, $X, Y, Z$, the top mass center inertial Cartesian coordinates, $\theta_{p}, \phi_{p}, \psi_{p}$, the declination and the azimuth of the axis of symmetry of bar $A_{i}B_{j}$, measured with respect to the inertial frame.
(Fig. 1). Hence the vector of generalized coordinates is

\[
q = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \delta_{11} & \delta_{12} & \alpha_{21} & \alpha_{22} & \delta_{21} & \delta_{22} & \phi & \psi & X & Y & Z \end{bmatrix}^T
\]  

(1)

4. Statics: prestressable configurations

Under no external forces and torques the structure can attain equilibrium configurations with all tendons in tension, called prestressable configurations. The corresponding mathematical conditions are (see [31])

\[
A(q)T = 0, \quad T_j > 0
\]

where \(A(q)\) is the equilibrium matrix given by \(A[n,j] = \delta_{ij}/\delta_{qn}\), \(n = 1, \ldots, 18\), \(j = 1, \ldots, 18\), \(T_j\) are the length and tension of tendon \(j\), respectively. Sultan et al. [31] published closed form solutions of Eq. (2) for a particular class of prestressable configurations called symmetrical prestressable configurations (see Fig. 2). These configurations are characterized as follows: all bars are identical of mass \(m\) and length \(l\); the top and base are equal equilateral triangles of side \(b\); all bars have the same declination \(\delta\); the vertical projections of points \(A_i, B_i, i = 1, 2, 3\), onto the base make a regular hexagon. For simplicity, it is assumed that all saddle tendons are identical (of rest-length \(S_0\) and stiffness \(k_s\)), all verticals are identical (of rest-length \(V_0\) and stiffness \(k_v\)), all diagonals are identical (of rest-length \(D_0\) and stiffness \(k_d\)).

Sultan et al. [31] showed that these configurations properties can be described using three parameters, \(\alpha\), the azimuth of bar \(A_{11}B_{11}, \delta\), and \(P\), a positive scalar called the pretension coefficient. In this paper we shall consider that under no applied forces or torques the sensor yields a symmetrical prestressable configuration characterized by \(\alpha = 50^\circ\), \(\delta = 30^\circ\), \(J = 0.4\) m, \(b = 0.27\) m. The corresponding generalized coordinates values are

\[
q = [30 \, 50 \, 30 \, 290 \, 30 \, 170 \, 30 \, 170 \, 30 \, 50 \, 30 \, 290 \, 30 \, 0 \, 0 \, 0 \, 0 \, 0.58]^T
\]

(3)

where the angles are given in degrees and the lengths in meters. The tensions in all saddle, vertical, and diagonal tendons are, respectively, equal to \(T_S, T_V, T_D\). The compression force in all bars is the same, \(C_0\). The state of tension and compression of the structure is characterized by

\[
\begin{bmatrix} T_S & T_V & T_D \end{bmatrix} = [0.29 \, 0.09 \, 0.27]P, \quad C_0 = 0.37P
\]

(4)

The bars are designed for buckling and the tendons for maximum stress as shown in [27,34], yielding:

\[
m = \rho_{nt}l \left( R^2 - \left( \frac{4\pi^2\sigma_s}{E_s} \right)^2 \right)
\]

(5)

\[
k_s = \frac{E_sT_s\gamma}{\sigma_{max}}
\]

(6)

Here \(k_s\) is the stiffness of tendon \(s\), \(\lambda\) and \(\gamma\) are the safety coefficients, \(\rho_{nt}\), \(E_s\), and \(R\) the density, Young’s modulus, and exterior radius of a bar (assumed pipe), respectively, whereas \(E_s\) and \(\sigma_{max}\) are the Young’s modulus and maximum allowed stress in tendon \(s\), respectively.

5. Static characteristics

In order to gain further insight into this sensor’s properties, we shall investigate its static characteristics when it is subjected to constant external forces or torques acting on the rigid top. For several values of the pretension coefficient, \(P\), the tendons are designed for maximum stress according to Eq. (6).

Consider that a force along the \(f_1\) axis, \(F_1\), acts on the rigid top. The plot of vertical displacement of the top mass center
vertical coordinate ($\Delta Z$) versus the amplitude of $F_3$ is called the load–deflection characteristic. If a torque, $M_3$, along the $\hat{t}_3$ axis acts on the rigid top, the plot of $\psi$ variation ($\Delta \psi$) versus $M_3$ is called the torsional characteristic. In order to compute these characteristics a tangent predictor–corrector continuation procedure has been applied to solve the non-linear equations of equilibrium which result from the application of the virtual work principle (see \[27,29,31\] for details).

The results obtained for $\alpha = 50^\circ$, $\delta = 30^\circ$, $\lambda = \gamma = 4$, $l = 0.4\text{m}$, $b = 0.27\text{m}$, $E^* = 10^{10}\text{N/m}^2$, $\sigma_{\text{max}} = 10^9\text{N/m}^2$ are shown in Figs. 3 and 4.

At all equilibria on these characteristics, the tendons were tested for maximum stress (considering $\gamma = 1.5$) and slackness conditions and the bars for buckling (with $\lambda = 1.5$). None of these tests failed for the investigated ranges of $F_3$ and $M_3$.

We ascertain that the static characteristics vary with the pretension coefficient, $P$. As $P$ varies, the shapes of these curves change from a very nonlinear, hardening spring-like characteristic, to a more linear one. Also the slopes of these curves at the origin (sensor’s sensitivity) can be modified through pretension adjustment.

6. Dynamic characteristics

The nonlinear equations of motion derivation has been presented in \[27,32\]. Their linearization around a member of the symmetrical prestressable configurations class leads to the following equations of motion (see \[33\]):

$$\sum_{i=1}^{6} m_i M_i \ddot{\hat{q}} + \sum_{i=1}^{4} k_i K_i \dot{\hat{q}} + D f + w = 0$$

(7)

where $m_i \in \{M_t, J_1, J_2, J_3, m, J, \}$, $k_i \in \{P, k_S, k_V, k_D\}$ and $\sum_{i=1}^{6} m_i M_i$, $\sum_{i=1}^{4} k_i K_i$ are the mass, damping, and stiffness matrices, respectively. Detailed formulas for these matrices are given in \[33\]. In the above, $\hat{q} = q - q_e$, $M_t$ is the mass of the top, $J_1$, $J_2$, $J_3$ the principal moments of inertia of the top, $J$ the longitudinal moment of inertia of a bar, $d > 0$ is the friction coefficient at all joints, $f = [M_1, M_2, M_3, F_1, F_2, F_3]^T$ the vector of external forces and torques acting on the rigid top, given on $\hat{t}_i$, and $w$ accounts for the inherent plant noise.

The linearized equations of motion are used to evaluate the influence of friction and pretension upon the dynamics of the structure, evaluated by plotting the minimum and maximum natural frequencies ($\omega_{\text{min}}$ and $\omega_{\text{max}}$, respectively) versus $P$ and $d$. Static design constraints, Eqs. (5) and (6), are employed and the same numerical values as before are used. In addition the following characteristics of the bars and rigid top are specified: $E_b = 7 \times 10^{10}\text{N/m}^2$; $\rho_b = 2800\text{kg/m}^3$; $R = 0.01\text{m}$; $M_t = 0.1\text{kg}$; $J_1 = 0.06\text{kg m}^2$; $J_2 = 0.06\text{kg m}^2$; $J_3 = 0.1\text{kg m}^2$.

The results are shown in Figs. 5 and 6.

For the investigated range of $P$ and $d$, we ascertain that slightly damped structures exhibit only oscillatory modes, which, as the friction coefficient increases, transform into
pure exponentially decaying ones (Fig. 5). However, the oscillatory nature of the linearized motions does not completely disappear ($\omega_{\text{max}} \neq 0$, see Fig. 6). The pretension influence is opposite: natural frequencies increase with $P$. Fig. 6 shows that for small $P$ the maximum natural frequency is small; the range of $P$ over which this parameter remains small increases with $d$.

7. Estimator design

In order to estimate the external forces acting on the rigid top, we cast Eq. (7) in first-order form and consider the linearized measurement equations:

$$x_f = A_p x_f + D_p f + u_p, \quad z = M_p x_f + v$$  \hspace{1cm} (8)
Here $x_p = \begin{bmatrix} \tilde{q}^T \dot{\tilde{q}}^T \end{bmatrix}^T$, $f$ represents the vector of forces and torques acting on the rigid top, to be estimated using the measurements $z$, which consist of some of the tendons lengths, $w_p = [0 \ -w^T]^T$ and $v$ represent plant and measurement noise, respectively.

For the purpose of $f$ estimation we choose to represent its elements as some arbitrary collection of prescribed functions of time, which are orthogonal over an interval of length $\tau$.

The coefficients of the orthogonal functions are to be automatically updated by the use of the real time measurements. One possibility is to approximate each element of $f$ by specifying a system of the form:

$$
\dot{s}_i = Q_i s_i + B_i \Delta(t), \quad f_i = N_i^T s_i
$$

(9)

where the choices for $Q_i$ depend upon the particular orthogonal functions. Here $\Delta(t)$ is the Dirac impulse. The response of the system is equivalent to that of the unforced system, with initial conditions $s_i^0 = s_0 + B_i$.

Various $Q_i$ matrices can be constructed to generate as eigenfunctions of Eq. (9) different orthogonal polynomials. For example, if

$$
Q_i = \frac{2}{\tau} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}_{h \times h}
$$

(10)

then any solution of Eq. (9) (arbitrary initial conditions $s_0$) yields $f_i$ as a sum of Chebyshev polynomials $f_i = N_i^T H_j$.

The Chebyshev polynomial of first kind and degree $j - 1$, $H_j$ are orthogonal over $t \in [0, \tau]$ with weight $w(t) = \tau/2(t - \tau)/\tau^2$:

$$
\int_0^\tau H_j(t)g(t)\Delta_j(t)\,dt = \Delta_j\frac{\pi \tau}{4} \left\{ \begin{array}{ll} 2 & \text{if } j = 1 \\ 1 & \text{if } j \neq 1 \end{array} \right.
$$

(11)

$\Delta_j$ being the Kronecker delta symbol.

Of particular interest is the case when the Chebyshev system can generate arbitrary linear combinations of $H_j$, $j = 1, \ldots, h$. The next result is a direct consequence of the properties of Chebyshev systems proved by Skelton and Likins [24].

**Theorem 1.** The Chebyshev system in Eq. (9) can generate an arbitrary sum of Chebyshev polynomials of degree $h - 1$ if and only if the last element of $N_i$ is nonzero.

It follows that there exists an initial conditions vector ($s_0$) such that $h$ of the coefficients of the Chebyshev polynomial representation of $f_j$ can take on arbitrary values if and only if $N_i(h) \neq 0$. On the strength of this fact we can claim that any particular coefficients which are appropriate for fitting the actual $f_i$ can be automatically determined by employing state estimation techniques.

Following these observations, for each of the six elements of $f$ we construct a Chebyshev system of degree $h - 1$:

$$
\dot{s}_j = A_j s_f + B_j \Delta(t), \quad f = C_j s_f, \quad A_j = \text{diag}(Q_j),
$$

$$
B_j = \text{vec}(B_i), \quad C_j = I(h \times i), \quad B_i \neq 0
$$

(12)

with dim($Q_j$) = $h$, $\forall i = 1, \ldots, 6$. 

![Fig. 6. Maximum natural frequency ($\omega_{\text{max}}$) variation.](image-url)
The augmented system which results using the linear models of the structure, Eq. (8), and of the external forces and torques, Eq. (12), is

\[
\dot{x}_a = A_a x_a + B_a w_a, \quad f = C_a x_a + v, \quad a = 2 \to 31
\]

is minimized. Here \( e(t) \) is the response of the system given by Eq. (16) when the disturbance (impulse or initial state) is applied only in the \( i \)th channel (\( n_i \) is the dimension of the \( s \) vector). Under fairly general conditions, this deterministic treatment is mathematically equivalent with the stochastic one in which the disturbances are zero mean, white noise stochastic processes (see [25]).

The solution to this optimal estimation problem is given by

\[
G = E_0 M_a^T V_a^{-1}
\]

where \( W_a = \text{diag}(w_1, \ldots, w_{n_i}) \), \( V_a = \text{diag}(v_1, \ldots, v_{n_i}) \), \( E_0 = \text{diag}(e_{i0}, \ldots, e_{i0}) \), \( w_i \) and \( v_i \) being the intensity of the impulsive disturbance applied in the \( i \)th channel, and \( e_{i0} \) the \( i \)th initial condition.

8. Numerical example

Ideally a sensors selection algorithm should be integrated with the system design in order to choose an appropriate set of sensing tendons out of all possible combinations (see [28] for an example of integrated structure design, control system design, and actuators selection). In this article we assume that previous considerations (e.g. technological ones) have led to the decision that the sensing tendons are the six saddle ones.
The structure and estimator design procedures are applied assuming that the bars, made of Al (for which $E_b = 7 \times 10^{10}$ N/m$^2$, $\rho_b = 2800$ kg/m$^3$), have an exterior radius $R = 0.01$ m, are designed for buckling (with $\gamma = 4$) according to Eq. (5), and the tendons (assumed to have all the same mechanical properties: $E_\ast = 10^{10}$ N/m$^2$; $\sigma_{\text{max}} = 10^9$ N/m$^2$) are designed for maximum stress using $\lambda = 4$ in Eq. (6).

The inertial properties of the proof element (rigid top) were assumed to be: $M_t = 0.1$ kg; $J_1 = 0.06$ kg m$^2$; $J_2 = 0.06$ kg m$^2$; $J_3 = 0.1$ kg m$^2$. These values correspond to those used to generate the static and dynamic characteristics.

The pretension and friction coefficients are chosen from
static and dynamic considerations. We choose \( P = 200 \) such that the static characteristics in Figs. 3 and 4 look linear for the investigated ranges of \( F_3 \) and \( M_3 \) and their slopes are not very large, since large slopes result in low sensitivities. The friction coefficient is given a value \( d = 0.6 \) which assures that the minimum natural frequency is zero and the maximum natural frequency has a small value (see Figs. 5 and 6). For estimator design we choose \( e_0 = 0, \omega_{\infty} = 10, \nu_1 = 10^{-3}, B^T_i = [10 \ 0 \ 0], C_f(i, j) = \delta_{i,j} \nu_1, (\delta_{i,j} \nu_1 \text{ being the Kronecker delta symbol}).

In order to choose the Chebyshev system parameters (\( h \) and \( \tau \)) we evaluate their influence upon the estimation error,
defined by $Y = \text{trace}(C_2E(C_1^2))$, and the decay rate, $\beta = \min(\text{Re} \{\text{eig}(A_0 - GM)\})$. These variations are plotted in Fig. 7. Let us first analyze the influence of $h$. For fixed $\tau$, fast estimation (high decay rate) is obtained for small $h$. The estimation error also decreases with decreasing $h$. Apparently, a small $h$ would be desirable. On the other hand a small $h$ results in complicated functions being approximated by simple functions, which works if the time interval over which the approximation is computed is small. For fixed $h$, small $\tau$ results in fast estimation. But the estimation error increases with decreasing $\tau$. Thus, a trade-off must be made between the speed of the estimator and accuracy. For our design and the following simulations, we choose $h = 3$ and $\tau = 1$.

Comparisons between exact values of the quantities of interest and those provided by the estimator, when all six external forces and torques act on the rigid top, are shown in Figs. 8–10. The tensegrity sensor simultaneously provides fast and accurate estimations of all six quantities. Fig. 11 shows the numerical simulations of the estimations of $F_s$ compared with the exact values, when only a sinusoidal excitation, $F_s$, acts. It is easily seen that, as the frequency of external excitation increases, the estimator looses its capability to provide accurate information. The problem can be fixed by changing the estimator poles locations and making it fast enough. One way to do this is, as we saw, by changing the Chebyshev system parameters (e.g. decreasing $h$ and $\tau$). On the other hand, if more knowledge about the excitations to be estimated is available, the Chebyshev system might be replaced with other types of orthogonal filters. For example, if it is known that the excitations have a certain frequency, a harmonic system may be used instead of the Chebyshev one. We remark that our linear and nonlinear simulations of this tensegrity sensor mathematical models indicated that there is practically no difference between the linear and nonlinear models responses for the investigated range of parameters and disturbances. This is generally the case when the tensegrity structure is highly prestressed (see [33] for details).

9. Conclusions

Through the design and performance analysis of a tensegrity sensor for simultaneous measurement of three orthogonal forces and three orthogonal torques, we illustrate the opportunities tensegrity structures present for sensor technology.

The static properties reveal the fact that the pretension coefficient can be used as a tuning factor of the static characteristics; their shape can change from a very nonlinear, hardening spring-like curve, to an approximately linear one as the pretension increases. Also the slopes at the origin (sensitivities) can be modified through pretension adjustment.

Dynamic characteristics can be tuned through friction and pretension coefficients adjustment. The natural frequency decreases with increasing friction and the low frequency oscillatory modes transform into pure exponential decaying ones. The pretension has an opposite effect, its increase leading to higher natural frequencies.

The estimating task is accomplished by an optimal state estimator whose inputs are the measurements provided by some of the tendons. For estimator design the excitations are represented as some arbitrary collections of Chebyshev polynomials. The influence of the maximum degree of the Chebyshev polynomials and of the length of the interval over which these are orthogonal upon the estimator performance is evaluated by plotting the estimation error and the decay rate variations. The decay rate increases with decreasing the maximum degree and with decreasing the interval of orthogonality. The estimation error also decreases with the decreasing of the maximum degree but increases with decreasing the length of the interval of orthogonality.

Numerical simulations of the exact applied forces and torques and those estimated by the tensegrity sensor show the validity of the proposed approach. The estimation is fast enough (settling time less than 1.5 s) and accurate for all six quantities. As with all estimators, previous knowledge of the maximum possible frequency of the estimated quantity can help in the decision regarding poles allocation.

References
