Classical Element Feedback Control for
Spacecraft Orbital Maneuvers
Bo J. Naasz* and Christopher D. Hall†
Aerospace and Ocean Engineering
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

Abstract

In this paper, we develop a nonlinear Lyapunov-based control law and a novel mean-motion control strategy. This control is specifically intended for spacecraft orbital maneuvers and formation flying applications. We apply the control to formation-establishment and formation-keeping maneuvers for a proposed NASA mission. To provide target orbit states for feedback control, we develop and apply an algorithm to calculate a formation master orbit representing the geometric center of the formation. We also define a new technique for choosing nonlinear feedback control gains which appropriately scales the gains for orbital maneuvers. The orbital element feedback control law, augmented by mean motion control, and applied with appropriate gains, forces asymptotic convergence to a spacecraft target orbit, for a large variety of spacecraft maneuvers.

Introduction

Spacecraft flying in close proximity is not a new concept. Gemini VI accomplished the first space rendezvous on December 15, 1965, and performed station-keeping with Gemini VII over three orbits at distances of 0.3 to 90 meters. Gemini VIII followed with the first docking of two spacecraft in orbit on March 16, 1966. The first automated rendezvous was performed by Soviet spacecraft Cosmos 186 and Cosmos 188 on October 27, 1967, when the two unmanned

*Graduate Research Assistant. Currently with NASA Goddard Space Flight Center, Flight Dynamics Analysis Branch, Code 572 Greenbelt, MD 20771, (301) 2863819, bo.naasz@gsfc.nasa.gov
†Associate Professor. Associate Fellow AIAA. (540) 231-2314, cdhall@vt.edu
spacecraft performed a preprogrammed closure and docking maneuver. Since the 1960s, numerous manned and unmanned spacecraft have performed operations in close proximity, mostly for the purpose of docking for re-supply or crew transfer, and observation, as in the case of Shuttle fly-aways of the International Space Station. Autonomous formation flight became a reality for spacecraft on May 17, 2001, when Earth Observing-1 (EO-1) performed its first autonomous maneuver to maintain a one minute in-track separation with Landsat-7. These close proximity spacecraft encounters have paved the way for spacecraft formation flight for use in a number of varied scientific, military, and satellite service operations.

Motivation

Unlike previous close proximity operations, which were brief, high-thrust encounters, formation flying spacecraft must maintain relative separations for extended lengths of time (their entire lifetimes), while expending minimal amounts of fuel. The need to conserve fuel in formation flying spacecraft has motivated a great deal of research directed at improving our understanding of the long term relative motion of spacecraft formations. Equipped with a superior model of differential perturbation effects, we can design relative formation geometries which minimize the need for formation-keeping maneuvers.

Obviously, the most effective method of conserving fuel on formation flying spacecraft is to design relative trajectories which do not degrade in the presence of differential perturbations. Unfortunately, it is not possible or feasible to design every formation with only fuel conservation in mind. At some point, the requirements of formation flying missions will conflict with the requirements of invariant formation design, making orbital maneuvers inevitable.

Problem Definition

Traditionally, spacecraft orbits are controlled by engineers on the ground, who command the spacecraft to perform thrust profiles designed to force optimal maneuver trajectories.
Linear feedback control is used primarily to counter unexpected perturbations, and maintain precalculated optimal trajectories. We develop nonlinear feedback control algorithms to autonomously maneuver and maintain spacecraft formations, without the use of computationally expensive optimization techniques. These control laws should be applicable to real satellites, and therefore must work in the presence of constraints such as thrust magnitude and direction limitations. They should also seek to minimize fuel consumption, both of the individual formation flyers, and of the formation as a whole. The formation flying strategies should cover issues including the definition of reference, or target trajectories, and the definition of appropriate nonlinear feedback control gains. Intuitively, the use of nonlinear control in place of linear control should result in significant fuel savings. This savings should be particularly evident when feedback control is used to perform large maneuvers, where the nonlinear dynamics effects become pronounced.

This research is motivated by Leonardo, a proposed NASA mission where formation flying is used to enhance the scientific mission of measuring the bi-directional reflectance distribution function.\textsuperscript{5}

One proposed Leonardo formation includes six spacecraft in low Earth orbits with inclinations varying from about 0.5 to 5.0 degrees, with right ascension and argument of latitude variations resulting in satellite separation distances on the order of hundreds of kilometers. Additional information on the Leonardo mission may be found in Ref. 6.

**Formation Control Literature Review**

Control of satellite formations can be considered in terms of two operational modes: formation-keeping, and formation-maneuvering. The difference between the two modes is, at first, subtle. It is not difficult to think of formation-keeping as a subcategory of formation-maneuvering. However, when we consider the application of orbit control to establish large formations, or to reconfigure them (formation-maneuvering), as compared to the mitigation of differential perturbations to maintain formation geometries (formation-keeping), the distinction becomes more clear.
Formation-keeping controllers have been developed by Vassar and Sherwood,\textsuperscript{7} for two satellites in a circular orbit, Leonard, Hollister, and Bergmann,\textsuperscript{8} for two satellites with drag panels to generate differential drag, and Middour,\textsuperscript{9} for along-track formation-keeping of low eccentricity formation flyers.

A more recent formation-keeping control, the Folta-Quinn (FQ) Formation Flying Algorithm\textsuperscript{10} uses Lambert’s two point boundary value problem and the ‘C*’ guidance and navigation matrix to maintain a formation of two spacecraft. Carpenter, Folta and Quinn\textsuperscript{10,11} use the FQ algorithm, in conjunction with a linear-quadratic-Gaussian (LQG) control\textsuperscript{12} to develop a decentralized control for autonomous formation flying for use in the New Millennium Program and the Earth Observing-1 (EO-1) mission. The formation model includes two spacecraft, one with real-time GPS and autonomous control (EO-1), and one with orbit determination and maneuver generation performed on the ground (Landsat-7). The authors conclude that autonomous, decentralized control is not only feasible, but beneficial over traditional orbit control procedures. The preliminary results\textsuperscript{2} of the EO-1 experiment show that the FQ formation flying algorithm has succeeded in providing robust, closed-loop, autonomous control of NASA’s first formation flying spacecraft.

Formation-maneuvering is especially important in establishing a formation, but may also be used to modify a formation based on changing mission requirements. Schaub and Alfriend\textsuperscript{13,14} use mean orbital element differences to control the relative orbits of formation flying spacecraft with identical ballistic coefficients. The authors’ use of mean orbital element differences stresses long-term behavior over short-term deviation, by forcing the control to compensate for secular and long period drifts, while ignoring short period oscillations. A modified form of Gauss’ variational equations of motion is used to develop a sequential, impulsive algorithm to correct orbit element errors. The control law establishes the desired formation in a nearly fuel-optimal sense by requesting impulses in the polar and equatorial regions of the orbit, only. This impulse timing allows correction of each element, with minimal impact on the other elements. The authors also develop a cartesian, Lyapunov-
control law, and apply it using relative position and velocity tracking vectors calculated from mean orbital elements. A comparison of the two nonlinear feedback control laws shows that the cartesian form is effective for short time-frame, high thrust maneuvers, while the orbital element form performs best in multiple-orbit or low-thrust maneuvers.

Ilgen\textsuperscript{15} develops Lyapunov-optimal feedback control laws using Gauss’s form of Lagrange’s planetary equations (LPE) in classical and equinoctial orbital-element forms. The author presents equations of motion for the equinoctial elements in terms of thrust components in the equinoctial coordinate axes. While the author does not discuss formation-flying control directly, the paper provides an excellent starting point for the development of an elemental, Lyapunov-optimal, formation flying feedback control law.

In Ref. 16 the authors develop non-impulsive control for transfers between elliptic orbits, using feedback of the angular momentum vector, $L$, and the Laplace (or eccentricity) vector, $A$. The authors suggest the algebraic structure of these vectors leads to superior control as compared with elemental feedback control. The authors do not discuss gain selection, nor do they comment on controlling the angular position within an orbit. Future work suggestions include investigation of $J_2$ effects, and optimization through gain selection. While the authors do not address formation-flying control directly, it may be possible to expand their approach to include an in-orbit angular position parameter, and thus control the full six state feedback control problem.

A more thorough review of formation flying literature may be found in Ref. 17.

**Approach**

We approach formation flying control in two steps: target orbit definition; and target orbit rendezvous. We assume desired relative formation geometries are known, and most likely chosen to optimize the performance of mission operations other than formation control. In the case of the Leonardo mission, the relative formation geometries have been designed by NASA scientists to optimize the mission performance.\textsuperscript{2} If possible, these relative geometries should also be designed to minimize the relative drift of the formation.
To maintain and maneuver a formation of spacecraft, we first develop a method for defining the absolute state of the formation, and the relative states of the spacecraft within the formation. We define the absolute state of the formation in terms of the osculating elements of some “master” orbit, positioned in the geometric center of the desired formation. From this master orbit, we define differential orbital element sets, used to calculate the desired spacecraft states at any given time. During simulation, we integrate the individual spacecraft states, as well as the master orbit state, and calculate the target orbits at each time step by adding constant differential orbital element sets to the formation master orbital elements.

To force convergence of the formation flying spacecraft to the desired orbit states, we develop a nonlinear orbit control law which uses classical orbital element error feedback, and mean motion control to force global, asymptotic convergence to the desired spacecraft states. We present the results of control law application in formation-establishment and formation-keeping maneuvers for the Leonardo formation.

**Model and Equations of Motion**

The dynamics of space objects is most often described by two-body motion in a Keplerian gravitational field, with corrections to compensate for forces from perturbations such as nonspherical central body effects, atmospheric drag, solar and lunar gravitational effects, and solar radiation pressure. In this section, we present a brief description of Keplerian orbits, and the effect of some perturbations on formation dynamics. We conclude the section with a discussion of some formation flying concepts, including formation models and formation master orbit definitions.

In cartesian form, the equations of motion for a point-mass satellite take the form

\[ \ddot{\mathbf{r}} = -\frac{\mu}{\|\mathbf{r}\|^3} \mathbf{r} + \mathbf{a}_p \]

where \( \mathbf{r} \) is the position vector of the satellite measured from the center of mass of the
primary body, \( \mu \) is the gravitational parameter of that body, and \( \mathbf{a}_p \) includes perturbation accelerations caused by non-spheroidal gravitational effects of the central body, atmospheric drag, third body effects and so on. If \( \mathbf{a}_p = \mathbf{0} \) then the equations of motion reduce to the ideal Keplerian situation where the central body is a perfect sphere and all other disturbances are zero.

We can also express the motion of a point-mass satellite in terms of orbital elements. We define the classical orbital elements\(^{18} \) as the semi-major axis, \( a \), the eccentricity, \( e \), the inclination, \( i \), the right ascension of the ascending node, \( \Omega \), the argument of periapse, \( \omega \), and the mean anomaly, \( M \). We also need to refer to the true anomaly, \( \nu \), and the argument of latitude, \( \theta = \omega + \nu \). The equations of motion of a controlled spacecraft in terms of the classical orbital element set are given by Gauss’s form of Lagrange’s planetary equations:\(^{19} \)

\[
\begin{align*}
\frac{da}{dt} &= \frac{2a^2}{h} \left( e \sin \nu \ u_r + \frac{p}{r} \ u_\theta \right) \\
\frac{de}{dt} &= \frac{1}{h} \left\{ p \sin \nu \ u_r + \left[ (p + r) \cos \nu + re \right] \ u_\theta \right\} \\
\frac{di}{dt} &= \frac{r \cos \theta}{h} \ u_h \\
\frac{d\Omega}{dt} &= \frac{r \sin \theta}{h \sin i} \ u_h \\
\frac{d\omega}{dt} &= \frac{1}{he} \left[ -p \cos \nu \ u_r + (p + r) \sin \nu \ u_\theta \right] - \frac{r \sin \theta \cos i}{h \sin i} \ u_h \\
\frac{dM}{dt} &= n + \frac{b}{ahe} \left[ (p \cos \nu - 2re) \ u_r - (p + r) \sin \nu \ u_\theta \right]
\end{align*}
\]

where \( u_r \), \( u_\theta \), and \( u_h \) are the radial, transverse, and orbit normal control, or disturbance acceleration components, \( n \) is the mean motion, and \( p \), \( h \), and \( b \) are the semi-latus rectum, the angular momentum, and the semi-minor axis, respectively.

**Perturbation Effects**

Whereas orbital perturbations have a significant impact on the absolute motion of individual spacecraft, the effect of perturbations on the relative motion of formation flying spacecraft can be quite limited. For formations in Earth orbit, the two most significant perturbations
are Earth oblateness effects, and atmospheric drag effects.

For formations of identical spacecraft (equal size and mass), the differential drag effect is negligible, and the most significant perturbation to the formation geometry is the differential $J_2$ effect. Earth oblateness causes regression of the node, or rotation of the orbit about the inertial $\hat{i}_3$ vector, as well as secular rotation of the orbit about the orbit normal vector. The secular drifts of $\Omega$, $\omega$, and $M$ due to $J_2$ perturbations are:

\[
\frac{d\Omega}{dt} = -\frac{3nJ_2}{2} \left( \frac{r_\oplus}{p} \right)^2 \cos i \\
\frac{d\omega}{dt} = \frac{3nJ_2}{4} \left( \frac{r_\oplus}{p} \right)^2 (5 \cos^2 i - 1) \\
\frac{dM}{dt} = \frac{3nJ_2}{4} \left( \frac{r_\oplus}{p} \right)^2 \sqrt{1 - e^2} (3 \cos^2 i - 1)
\]

where $r_\oplus$ is the radius of the Earth. Notice that the secular drift due to $J_2$ effects varies with inclination. Formations of spacecraft with no inclination variation will drift as a group, experiencing no relative deformation. Formations with inclination variations encounter differential secular drift, which must be mitigated to maintain the formation geometry. A more detailed description of formation perturbation effects is available in Ref. 20.

**Formation Flying Concepts**

In this paper, we study the control of a formation of six identical spacecraft. We calculate the desired orbits of the maneuvering spacecraft from a central reference orbit, which we refer to as the master orbit. We refer to the desired orbits as target, or virtual orbits. We treat each maneuver as a rendezvous between a maneuvering spacecraft, and a passive, target orbit, or virtual spacecraft. We now introduce some basic formation geometries and models, and a method for calculating the formation master orbit and desired spacecraft orbits.

**Formation Models**

Figure 1 illustrates three models for controlling and simulating formation dynamics. The three models increase in complexity from a simple target-interceptor problem to cooperative,
multi-craft maneuvering problem. The target-interceptor problem simply involves maneuvering one spacecraft to rendezvous with another.

The intermediate model is a leader-follower (or master-slave) type formation where the desired orbit of the active spacecraft (the follower) is determined by a formation planner from the leader’s orbit. In the case of an anomaly shift formation, the active spacecraft targets a virtual orbit defined by a $\delta \nu$ shift of the leader’s orbit. This model is only required if differential perturbations, such as Earth oblateness or orbit control at the leader spacecraft, are applied, causing the virtual target orbit to drift with respect to leader’s orbit. If no perturbations are applied in orbit propagation, this model is identical to the target-interceptor problem, as the virtual target orbit follows a path identical to the leader’s.

The cooperative, multi-craft maneuvering problem involves the complicated task of determining target orbits for multiple, active spacecraft from the inertial orbits of the spacecraft, and a predetermined relative formation type. In Ref. 21, the authors show that the use of two actively controlled spacecraft as opposed to a leader-follower type formation architecture improves the efficiency of formation control by minimizing both the time, and the total number of thrust impulses required to establish the desired formation.

The desired relative orbit can be chosen in many ways, possibly to optimize the mission performance in some sense,$^{22}$ or to minimize the effect of gravitational perturbations on the formation geometry$^3$ in an effort to minimize formation-keeping propellant consumption. For this study, we assume that the formation geometry is already defined, and seek to find ways to model, maneuver, and maintain that formation geometry.

**The Master Orbit**

To implement the cooperative formation control problem we must develop a method for calculating the desired orbits of all of the spacecraft in the formation as a function of the current location of each spacecraft, and the required formation geometry. We would like to define a master orbit, close to the center of the cluster, for use as a reference point in the
calculation of the individual satellite target orbits. Starting with the position and velocity vectors of each of the formation flyers, and the required relative geometry, we can calculate the individual reference orbits in three steps: 1) initialize the formation; 2) calculate the orbit which remains closest to the center of the formation throughout the formation’s motion (the master orbit); 3) define orbital element offsets used to calculate the desired orbits from the master orbit.

**Initializing the Formation.** Formation geometries are often defined in relative terms, such as differential orbital elements, and leave the absolute positioning general. We study the control of a sample formation for the Leonardo mission which can be fully described in terms of some absolute orbital elements, and some relative, or differential, orbital elements. The sample Leonardo formation, defined by Table 1, consists of six spacecraft in 400 km, circular orbits, with varying inclinations, node vectors, and arguments of latitude. Note that in this formation, the inclination varies between spacecraft, but is still given in absolute terms.

This formation is defined in terms of both relative and absolute parameters. The values of $\Omega_j$ and $\theta_j$ are not important, as long as their relative values match those given in the table. To initialize the formation, we choose reference values of $\Omega_0$ and $\theta_0$ at random, and construct the individual orbits from the defined offset values.

**Defining the Master Orbit.** Once the formation has been initialized, it is convenient to define a master orbit which remains in the center of the formation as the spacecraft complete their orbits. For simple formations such as the anomaly shift formation, where the argument of latitude, $\theta$, is the only parameter that differs between spacecraft, this master orbit is easily found. Since five of the orbital parameters are shared by all spacecraft in the formation, the sixth, the argument of latitude of the master, $\theta_m$, can be found by averaging:

$$\theta_m = \frac{1}{N} \sum_{j=1}^{N} \theta_j$$  \hspace{1cm} (11)
For more complex formations such as the one described in Table 1, the definition of a central master orbit requires a more robust method of calculation. We derive this method by considering some conserved properties of the formation: the mean angular momentum vector, and the formation equivalents of the node and eccentricity vectors.

For a single spacecraft, the orbital elements are calculated from the position and velocity vectors by manipulation of these three vectors. The orbital elements of the master orbit can be found from the position and velocity vectors, $r_j$ and $v_j$, of individual formation flyers in a similar way, using the three fundamental vectors of the cluster: the mean angular momentum vector, $\bar{h}$, the mean node vector, $\bar{n}$, and the mean eccentricity vector, $\bar{e}$.

**Algorithm OrbitMaster**

Following the same basic steps used to calculate the orbital elements of a single satellite from position and velocity vectors, we calculate the master orbit elements for a formation of $N$ spacecraft as follows:

1. Calculate the mean angular momentum vector, $\bar{h}$,

$$\bar{h} = \frac{1}{N} \sum_{j=1}^{N} r_j \times v_j = \frac{1}{N} \sum_{j=1}^{N} h_j $$ (12)

2. Calculate the mean node vector, $\bar{n}$,

$$\bar{n} = \frac{1}{N} \sum_{j=1}^{N} K \times h_j = \frac{1}{N} \sum_{j=1}^{N} n_j $$ (13)

3. Calculate the mean eccentricity vector, $\bar{e}$,

$$\bar{e} = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\mu} \left[ \left( \frac{v_j^2}{r_j} - \frac{\mu}{r_j} \right) r_j - (r_j \cdot v_j)v_j \right] = \frac{1}{N} \sum_{j=1}^{N} e_j $$ (14)
4. Calculate the inclination of the master orbit, $i_m$, 
\[ \cos i_m = \frac{\hat{h} \cdot \hat{i}_3}{h} \quad i_m \in [0, 180^\circ] \] (15)

5. Calculate the right ascension of the ascending node of the master orbit, $\Omega_m$, 
\[ \cos \Omega_m = \frac{\hat{n} \cdot \hat{i}_1}{n} \quad \text{if } \hat{n} \cdot \hat{i}_2 > 0, \Omega_m < 180^\circ \] (16)

6. Calculate the argument of periapsis of the master, $\omega_m$ (assuming $\vec{e}$ is non-zero), 
\[ \cos \omega_m = \frac{\vec{e} \cdot \vec{e}}{n e} \quad \text{if } \vec{e} \cdot \hat{i}_3 > 0, \omega_m < 180^\circ \] (17)

7. Calculate the true anomaly of the master, $\nu_m$ (assuming $\vec{e}$ is non-zero), 
\[ \cos \nu_m = \frac{\vec{e} \cdot \vec{r}}{e r} \quad \text{if } \vec{r} \cdot \vec{v} > 0, \nu < 180^\circ \] (18)

where $\vec{r}$ is the component of the mean radius vector, $\vec{r}_{mean}$, perpendicular to the mean angular momentum vector, $\vec{h}$:
\[ \vec{r}_{mean} = \frac{1}{N} \sum_{j=1}^{N} \vec{r}_j \] (19)
\[ \vec{r} = \vec{r}_{mean} - (\vec{r}_{mean} \cdot \vec{h})\vec{h} \] (20)

8. Calculate the argument of latitude of the master, $\theta_m$, the angle between $\vec{n}$ and $\vec{r}$, 
\[ \cos \theta_m = \frac{\vec{n} \cdot \vec{r}}{n r} \quad \text{if } \vec{r} \cdot \hat{i}_3 > 0, \theta_m < 180^\circ \] (21)

Calculating the Desired Orbits. Given these definitions for the master orbit elements, we calculate the desired states of the formation flyers at any time by propagating the master orbit to the desired time, and adding orbital element offsets slightly modified from those
in Table 1. These new offsets, $\Delta \Omega_j$ and $\Delta \theta_j$, are calculated by subtracting the initialized spacecraft orbital element values from the master orbital elements:

$$
\Delta \Omega_j = \Omega_j - \Omega_m, \quad \Delta \theta_j = \theta_j - \theta_m
$$

(22)

We calculate the desired orbit of the formation flyers from these new offsets as in Table 1, except that we use the master orbit for the reference values of $\Omega$ and $\theta$.

**Example: Master Orbit Calculation.** To calculate the master orbit of a cluster of spacecraft in the Leonardo formation, we choose reference values of $\Omega_0 = 0$ and $\theta_0 = 0$. For these reference values, the orbital elements of the Leonardo spacecraft are given in Table 2, along with the corresponding master orbit elements.

As shown in the table, the orbit that best defines the center of the formation has an inclination of about $0.25^\circ$, and right ascension and argument of latitude of about $98^\circ$ and $266^\circ$, respectively. Notice that these values are far from the average values, and that the master inclination is nearly zero. This result is not surprising, as the orbital planes of the individual spacecraft are fairly evenly dispersed around the equator by varying values of $\Omega$.

With the master orbit identified, we can calculate the orbital element offsets in Eq. (22) by subtracting the initialized spacecraft orbital element values from the master orbital elements. The new offsets can then be used to calculate the desired state of any of the spacecraft in the formation at any time by propagating the master orbit, and shifting the $\Omega_j$ and $\theta_j$ values. Note that these new offsets are not the same as those given in Table 1, and apply only to the formation generated using the reference values $\Omega_0 = 0$ and $\theta_0 = 0$.

The definition of a central formation master orbit is essential to formation flying operations. We calculate the target spacecraft orbits as a function of the formation master orbital elements, and predefined formation geometries. We also use the master orbit calculation algorithm to re-center the formation after periods of differential drift due to differential perturbation effects. The master orbit defined in this section is one of many possible choices for
the formation master. In a subsequent section, we show how other formation master orbit definitions can result in improved fuel efficiency for the formation.

**Lyapunov Control Law**

Whereas linear control techniques can be highly effective in the control of nonlinear systems, nonlinear control strategies can be beneficial in a variety of ways. For example, nonlinear controllers may: a) increase the region of state space in which we can effectively control the system, b) improve control robustness to parametric uncertainty, c) obtain truer optimality results, d) enable control of systems which are not linearly controllable, and e) allow us to preserve and exploit physical insight.

In this section we develop nonlinear, Lyapunov-based classical element feedback control for spacecraft orbital maneuvers. The first part of this control law, based on previous work by Ilgen,\(^\text{15}\) and comparable to work by Schaub and Alfriend,\(^\text{23}\) establishes the desired orbital plane. The second part, which we call “mean motion control,” takes advantage of the natural relative dynamics to control spacecraft position within the orbit.

**Classical Orbital Element Control**

The equations of motion (2 – 7) can be written as:

\[
\dot{\mathbf{e}} = \mathbf{f}(\mathbf{e}) + \mathbf{G}(\mathbf{e})\mathbf{u}
\]  

where \(\mathbf{e}\) is the vector of orbital elements, \([a e i \Omega \omega M]^T\), and \(\mathbf{u}\) is the vector of controls, \([u_r u_g u_h]^T\).

We separate these equations of motion into two systems: one system consisting of the first five elements, which define the shape, size, and orientation of the orbit, and the second system consisting of the in-plane angular position of the spacecraft.
For the first system, we define the error $\eta$ as

$$
\eta = \begin{bmatrix}
    a - a^* \\
    e - e^* \\
    i - i^* \\
    \Omega - \Omega^* \\
    \omega - \omega^*
\end{bmatrix} = \begin{bmatrix}
    \delta a \\
    \delta e \\
    \delta i \\
    \delta \Omega \\
    \delta \omega
\end{bmatrix}
$$

(24)

where $(\cdot)^*$ is the target element.

The equations of motion for this system are:

$$
\dot{\eta} = Gu
$$

(25)

where the input matrix, $G$, from Eqs. (2–7), is

$$
G = \begin{bmatrix}
    \frac{2a^2 p \sin \nu}{h} & \frac{2a^2 p}{h} & 0 \\
    \frac{p \sin \nu}{h} & \frac{(p+r) \cos \nu + re}{h} & 0 \\
    0 & 0 & \frac{r \cos \omega}{h} \\
    0 & 0 & \frac{r \sin \omega}{h} \\
    -\frac{p \cos \nu}{h} & \frac{(p+r) \sin \nu}{h} & -\frac{r \sin \omega \cos \nu}{h}
\end{bmatrix}
$$

(26)

where the sine and cosine functions are abbreviated as $s(\cdot)$ and $c(\cdot)$.

A candidate Lyapunov function for this system is

$$
V(\eta) = \frac{1}{2} \eta^T \eta
$$

(27)

whose time derivative is given by

$$
\dot{V}(\eta) = \eta^T \dot{\eta} = \eta^T G(\eta) \ u
$$

(28)
Choosing the control, \( u \), as

\[
u = -G^T K \eta
\]  

(29)

where \( K \) is a positive definite gain matrix, results in

\[
\dot{V} = -\eta^T G G^T K \eta \leq 0
\]  

(30)

The time derivative of the Lyapunov function is negative semi-definite. To prove asymptotic stability, we apply LaSalle’s invariance theorem. The time derivative of the Lyapunov function is always zero when \( \eta = 0 \), and could be zero when the trigonometric functions of \( \theta = \omega + \nu \) are zero, which occurs when \( \theta = k\pi/2 \), where \( k \) is an integer. The set where \( \theta = k\pi/2 \) is not an invariant set, because \( \theta \) is time-varying and therefore trajectories that start in the set do not remain in the set. Therefore, the largest invariant set where \( \dot{V} = 0 \) is the set \( \eta = 0 \). So by LaSalle’s invariance principle, the system is asymptotically stable under the choice of control given in Eq. (29). Furthermore, the system is globally asymptotically stable because \( V \to \infty \) as \( |\eta| \to \infty \).

Choosing a diagonal, positive definite gain matrix, \( K \), the control law from Eq. (29) is:

\[
u_{cl} = -G(\eta)^T K \eta = -
\]

\[
\begin{bmatrix}
\frac{2\alpha^2 \sin \omega}{h} & \frac{2\alpha^2 \rho}{h \epsilon} & 0 \\
\frac{p \sin \omega}{\pi} & \frac{(p+r) \epsilon \cos \epsilon + re}{h} & 0 \\
0 & 0 & \frac{r \cos(\omega + \nu)}{h} \\
0 & 0 & \frac{r \sin(\omega + \nu)}{h s_i} \\
\frac{-p \cos \epsilon}{h e} & \frac{(p+r) s_i}{he} & -\frac{r \sin(\omega + \nu) \epsilon}{h s_i}
\end{bmatrix}
\begin{bmatrix}
K_\eta \delta \eta \\
K_e \delta e \\
K_\omega \delta \omega
\end{bmatrix}
\]  

\]  

(31)

The angle errors \( \delta \Omega, \delta \omega, \) and \( \delta M \) are measured from parameters defined between 0 and \( 2\pi \). To ensure proper error feedback, these angular errors should be “short-way” angle measurements, defined between \( -\pi \) and \( \pi \).

**Mean Motion Control**
To control the in-plane angular motion of a spacecraft, we take advantage of a useful natural component of the nonlinear dynamics: the differential mean motion, $\delta n$. Observe that an effective method for correcting in-track errors is to force a semi-major axis error, changing the mean motion such that the in-plane angle is corrected. For an uncontrolled spacecraft in two-body motion, the orbital element dynamics simplify to

$$\frac{dM}{dt} = n$$

and thus the relative dynamics can be written as

$$\frac{d}{dt}(\delta M) = \frac{dM}{dt} - \frac{dM^*}{dt} = n - n^* = \delta n$$

where

$$\delta n = \sqrt{\frac{\mu}{a^3}} - \sqrt{\frac{\mu}{a^*}}$$

To drive the mean anomaly error, $\delta M$, to zero, we use

$$\delta M = -K_n \delta M$$

where $K_n$ is a positive gain. Combining Eqs. (34) and (35), and using canonical units ($\mu = 1$), we obtain

$$-K_n \delta M = \sqrt{\frac{1}{a^3}} - \sqrt{\frac{1}{a^*}}$$

Solving for $a$, we define a new target semi-major axis, $a^{**}$, which forces the mean anomaly error to zero:

$$a^{**} = \left(-K_n \delta M + \frac{1}{a^{*3/2}}\right)^{-2/3}$$

Notice that as $\delta M$ goes to zero, the mean motion control target semi-major axis approaches the original target value, $a^*$. In application, we replace the mean anomaly error, $\delta M$, in Eq. (37) with argument of latitude error, $\delta \theta$, so that the mean motion control properly
positions the spacecraft within the orbital plane, even in the presence of argument of periapse error.

Using the orbit control law defined in Eq. (31), with the target semi-major axis $a^\prime\prime$ defined in Eq. (37), we can control the full, nonlinear motion of an orbiting spacecraft.

Control Application Issues

To apply the control laws developed in this paper, we must first address some control application issues. Real spacecraft have numerous constraints limiting the application of thrust. It is not always possible for thrusters to deliver the exact magnitude and direction of thrust requested by the control law. In this section, we discuss the impact of constrained thrust magnitude and direction on control law design, as well as a method for choosing appropriate gains for the nonlinear control.

**Constrained Thrust Magnitude.** Since most spacecraft have fixed magnitude thrusters, we must develop some way of discretizing the control acceleration requested by the control laws. For example, if the orbital element feedback control law requests acceleration of magnitude $A$ in the $\hat{\alpha}$ direction, we must determine whether or not to thrust, and if so, in what direction.

This decision is straightforward for this elemental control law, as we have full scaling freedom in the choice of gains. We simply calculate a desired thrust direction and magnitude from the control law, and thrust if and only if the desired thrust magnitude is greater than the available thrust magnitude. Mathematically, the thruster on/off logic requests thrust in the $\hat{\alpha}$ direction if and only if $A > T/m$ where $T$ is the available thrust magnitude, and $m$ is the spacecraft mass.

**Constrained Thrust Direction.** In the case of the HokieSat mission, we must not only scale the thrust magnitude, but also constrain the thrust direction. HokieSat’s orbit control consists of four pulsed plasma thrusters (PPTs$^7$) aligned in the local horizontal frame, with no thrust available in the radial direction.
The classical feedback control laws can be modified to exclude radial thrusting by setting the terms in the first column of the $G$ matrix in the control definition to zero, thus requesting only transverse and orbit normal thrust.

**Circular Orbits.** While the equinoctial control law presented in Ref. 17 avoids the singularities associated with circular and equatorial orbits, the lack of physical meaning makes the equinoctial elements difficult to visualize. In the case of circular orbits, the argument of periapse and the mean anomaly are undefined, and singularities appear in the classical element equations of motion of those variables. We can modify the classical orbital element Lyapunov control to deal with these issues by: 1) setting the gain $K_w$ to zero so that the errors associated with that gain do not contribute to the total requested thrust; 2) using the argument of latitude error, $\delta \theta$, in place of $\delta M$ in the mean motion control. These modifications are acceptable as long as the eccentricity error remains small, so that $\delta \theta$ is a good approximation of the in-plane-angle error.

**Gain Selection**

Intuitively, nonlinear control of formation flying spacecraft should provide improvements over more traditional, linear feedback control techniques such as LQR and $H_\infty$. One major obstacle in the use of nonlinear controls such as the Lyapunov-based control developed in this paper is the lack of a well-defined method for choosing gains. Whereas numerous tools exist for gain selection for linear feedback control methodologies, nonlinear gain selection techniques are often limited to trial-and-error type approaches. In the full state feedback problem, the orbital element feedback control laws require a $[6 \times 6]$ gain matrix. Even using only the diagonal terms, we are unlikely to find six effective gains using a trial-and-error approach.

To motivate a more effective gain selection process for the full-state feedback problem, we explore the use of the nonlinear control laws in more limited control problems. The orbital element Lyapunov control law can be used to control specific elements, by setting the gains corresponding to the uncontrolled elements to zero. For example, the Lyapunov orbit control
law can be used to perform a circle-to-circle orbit-raising maneuver using feedback of the semi-major axis error, $\delta a$, and the eccentricity error, $\delta e$, as in

$$u_{cl} = -\begin{bmatrix} \frac{2a^2eS\nu}{h} & \frac{2a^2p}{hr} \\ \frac{ggS\nu}{h} & \frac{(p+r)C\nu+re}{h} \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} Ka\delta a \\ Ke\delta e \\ Ki\delta i \end{bmatrix} \quad (38)$$

The selection of gains for this problem is simple, as there are only two gains to choose. For a sample orbit-raising maneuver from 7,000 km to 10,000 km, using 1 N thrusters fired continuously, trial-and-error tuning yields gains $Ka = 0.7$ and $Ke = 1.174$.

We can expand this problem to include inclination feedback by using

$$u_{cl} = -\begin{bmatrix} \frac{2a^2eS\nu}{h} & \frac{2a^2p}{hr} & 0 \\ \frac{ggS\nu}{h} & \frac{(p+r)C\nu+re}{h} & 0 \\ 0 & 0 & \frac{rC(\omega+\nu)}{h} \end{bmatrix}^T \begin{bmatrix} Ka\delta a \\ Ke\delta e \\ Ki\delta i \end{bmatrix} \quad (39)$$

Figure 2 shows an example of this type of maneuver, in which the same circle-to-circle orbit-raising maneuver is performed, but with an inclination shift from $28^\circ$ to $60^\circ$. Plot A of the figure shows the maneuver performed with gains $Ka = 1$, $Ke = 1$, and $Ki = 50$. Plot B shows the maneuver performed with gains $Ka = 1$, $Ke = 1$, and $Ki = 5$. Varying the gains has a predictable effect on the maneuver: because the inclination is weighted more heavily in plot A, the trend in the maneuver is to correct the out-of-plane error, and then complete the orbit-raising. In plot B, where the semi-major axis and inclination are more equally weighted, the control forces orbit-raising to occur first, and then completes the inclination maneuver. The maneuver illustrated in plot A requires about 13% more fuel than maneuver B. This fuel savings is easily explained by the fact that the $\Delta V$ required for simple plane change maneuvers is directly proportional to the initial speed. Maneuver B saves fuel by performing the majority of the plane change maneuver after it has already increased the semi-major axis (and thus decreased the speed).

To choose appropriate gains for the full orbital element feedback problem, we strive
to understand the response of the individual elements to thrust impulses. The effect of radial, transverse, and orbit-normal thrust impulses on the osculating orbital elements varies throughout the orbit as a function of the true anomaly, as given by Gauss’s form of Lagrange’s planetary equations. For example, the change in semi-major axis resulting from an impulsive \( \Delta V = (T/m)\Delta t \), in the radial and transverse directions, respectively is

\[
\Delta a_r = \frac{2a^2e \sin \nu T}{h \frac{m}{\Delta t}} \\
\Delta a_\theta = \frac{2a^2(1 + e \cos \nu) T}{h \frac{m}{\Delta t}}
\]

where \( T \), \( m \), and \( \Delta t \) are the thrust force, spacecraft mass, and thrust duration. The maximum change in the semi-major axis results from an impulse in the transverse direction at periapse (where \( \nu \) is 0):

\[
\Delta a_{\text{max}} = \frac{2a^2(1 + e)}{h \frac{m}{\Delta t}}
\]

We use this knowledge as a guide for selecting the gain, \( K_a \), associated with the semi-major axis error feedback. From Eq. (31), the transverse Lyapunov control acceleration resulting from a semi-major axis error, \( \delta a \), is

\[
u_{a_\theta}(\delta a) = -\left(\frac{2a^2(1 + e \cos \nu)}{h}\right) K_a \delta a
\]

We choose \( K_a \) such that the maximum value of \( u_{a_\theta} \) is equal to the available thrust acceleration magnitude, in which case the control on-off logic asks for transverse thrust only when the semi-major axis error \( \delta a \) is at least \( \Delta a_{\text{max}} \). Furthermore, the control should thrust in the presence of this error only once in the orbit, at the point where the control required to correct the error is minimized. Mathematically, we choose the minimum value of \( K_a \) such that the magnitude of the control given in Eq. (43) is equal to the available control
acceleration, $T/m$:

$$|u_{ao}(\Delta a_{max})| = \left| - \left( \frac{2a^2(1 + e \cos \nu)}{h} \right) K_a \frac{2a^2(1 + e) T}{m} \frac{\Delta t}{\Delta a_{max}} \right| = \frac{T}{m}$$ (44)

This minimum value of $K_a$ occurs at periapse, and is given by

$$K_a = \frac{h^2}{4a^4(1 + e)^2 \Delta t}$$ (45)

With this choice, the thruster on-off logic requests thrust when the semi-major axis error is greater than $\Delta a_{max}$, and does nothing when the error is less than $\Delta a_{max}$. Choosing all of the gains in this way properly weights the elemental errors, and also prevents chatter in the system, as the control law does not attempt to control elemental errors more precisely than is possible with the available impulse.

With the exception of $K_{\omega}$ and $K_M$, the remaining control gains are easy to calculate using the same method used to find $K_a$, and are given by:

$$K_e = \frac{h^2}{4p^2 \Delta t}$$ (46)

$$K_i = \left[ \frac{h + eh \cos (\omega + \arcsin(e \sin \omega))}{p(-1 + e^2 \sin^2 \omega)} \right]^2 \frac{1}{\Delta t}$$ (47)

$$K_\Omega = \left[ \frac{h \sin i(-1 + e \sin (\omega + \arcsin(e \cos \omega)))}{p(1 - e^2 \cos^2 \omega)} \right]^2 \frac{1}{\Delta t}$$ (48)

The maximum value of $\Delta \omega$ occurs when transverse thrust is applied near $\nu = \pi/2$. Therefore, the appropriate selection of $K_{\omega}$ comes from the transverse $\omega$ control equation:

$$u_{ao}(\delta \omega) = -\frac{p \sin \nu}{he} \left( 1 + \frac{1}{1 + e \cos \nu} \right) K_{\omega} \delta \omega$$ (49)

$$K_{\omega}(\nu) = \min \left[ \frac{e^2h^2 \csc^2 \nu}{p^2 \left( 1 + \frac{1}{1 + e \cos \nu} \right)^2} \frac{1}{\Delta t} \right]$$ (50)
Equation (50) is minimized on \( \nu \) when the function

\[
f(\nu) = \frac{\csc^2 \nu}{\left(1 + \frac{1}{1+e \cos \nu}\right)^2}
\]  

is minimized, or the first derivative of \( f(\nu) \) with respect to \( \nu \) is zero, and its second derivative is positive. The first derivative is

\[
\frac{df}{d\nu} = -\frac{2 (1 + e \cos \nu) \csc \nu (e + e (3 + e \cos \nu) \cot^2 \nu + 2 \cot \nu \csc \nu)}{(2 + e \cos \nu)^3} = 0
\]  

To find the roots of the first derivative, we expand \( g(\nu) \), the numerator of \( f(\nu) \), in a first-order Taylor series about \( e = 0 \), and solve for the approximate root:

\[
g(\nu, e) = 2 (1 + e \cos \nu) \csc \nu \left( e + e (3 + e \cos \nu) \cot^2 \nu + 2 \cot \nu \csc \nu \right)
\]  

\[
g(\nu, 0) = 4 \cot \nu \csc^2 \nu
\]

Equation (54) is zero when \( \nu = \pi/2 \). The first order approximation of the root is thus

\[
\nu \simeq \frac{\pi}{2} + \nu_1 e
\]

where \( \nu_1 \) is the root of the following equation

\[
\left. \frac{dg(\pi/2 + \nu_1 e, e)}{de} \right|_{e=0} = 4\nu_1 - 2 = 0
\]

The approximate root of \( df/d\nu \) is

\[
\nu \simeq \frac{\pi}{2} + \frac{e}{2}
\]
Substituting into Eq. (50), we obtain the approximate minimum value of $K_\omega$:

$$\min(K_\omega) \simeq \frac{e^2 h^2 \sec^2 \frac{\epsilon}{2}}{p^2 \left(1 + \frac{1}{1 - e \sin \frac{\pi}{2}}\right)^2} \frac{1}{\Delta t}$$  \hspace{1cm} (58)$$

Since the root is only valid to $O(e)$, we expand $K_\omega$ in a Taylor series expansion about $e = 0$, and arrive at the final selection of the gain $K_\omega$:

$$K_\omega = \frac{e^2 h^2}{4p^2} \left(1 - \frac{e^2}{4}\right) \frac{1}{\Delta t}$$  \hspace{1cm} (59)$$

The last of the six gains, $K_M$, found using the same Taylor series approximations, also occurs at $\nu \simeq \frac{\pi}{2} + e$, and is given by

$$K_M = \frac{a^2 e^2 h^2}{4b^2 p^2} \left(1 - \frac{e^2}{4}\right) \Delta t$$  \hspace{1cm} (60)$$

We have defined a positive definite, diagonal gain matrix for use in classical orbital element Lyapunov control that: 1) guarantees asymptotic convergence to an error envelope, 2) appropriately weights the elemental errors, and 3) eliminates the problem of chattering by properly defining the size of the error envelope. The gains are defined at Eqs. (45), (46), (47), (48), (59), and (60). These gains should be calculated as a function of the target orbit at the beginning of a maneuver, and held constant throughout the maneuver.

**Examples**

In this section we

**Sample Orbit Maneuvers**

In addition to controlling the full orbital motion of a spacecraft, the orbital element feedback control law can control individual elements of the orbit, as shown in subsequent examples below. In any maneuver performed using the orbital element feedback control law, the
control gains should be scheduled so that the spacecraft first acquires the desired orbit, and then establishes its angular position within that orbit. Gain scheduling is especially important for spacecraft in nearly circular or nearly equatorial orbits, where small changes in some orbital parameters (eccentricity and inclination) can result in large changes in other elements (argument of periapse and right ascension). For spacecraft in more moderate orbits, we can relax the gain scheduling requirement, but should never forget that often the angular errors in the feedback loop are calculated from varying reference parameters.

Leonardo Maneuvers

Two maneuvers of interest to the Leonardo formation are the formation-establishment, and formation-keeping maneuvers. In the formation-establishment maneuver, the spacecraft are initially positioned in a parking orbit, and must use orbit control laws to initialize the formation. In the formation-keeping maneuver, the spacecraft are initially in a perturbed arrangement, and must re-establish the formation geometry using orbit control.

Leonardo Formation Establishment. The Leonardo formation-establishment problem involves the maneuvering of six spacecraft from a post-launch parking orbit into the desired formation geometry. We assume the spacecraft are launched as a stack, and therefore share the same initial conditions. One choice for the initial parking orbit is the formation master orbit defined in Table 2. This master orbit is the geometric center of the formation and therefore a logical choice for the initial parking orbit.

Figure 3 shows a three-dimensional plot of the application of scaled classical orbital element Lyapunov control to the formation-establishment problem, for one of the spacecraft in the Leonardo formation. In the simulation, a 150-kg spacecraft is maneuvered from the formation master orbit to the desired orbit defined in Table 2 as spacecraft 1, with thrust magnitude constrained to 1 Newton. For this simulation the target orbit is nearly circular, so the gain $K_\omega$ is set to zero so that the argument of periapse error, $\delta \omega$ (which is undefined for circular orbits), does not contribute to the requested thrust. For the same reason, the argument of latitude error, $\delta \theta$, is used in the mean motion control.
For this simulation we use a mean motion control gain of $K_n = 1 \times 10^{-4}$ for the first 3.1 orbits, and $K_n = 0.1$ for the remainder of the simulation. Notice that the out-of-plane error is corrected after just over three orbits, but the radial and in-track errors remain until after the $K_n$ gain is increased. This result is not surprising. The starting choice of $K_n$ creates a small semi-major axis error (negative $\delta r_3$ represents error in the positive radial direction). Close inspection of the $\delta r_1$ data reveals that this small semi-major axis error is gradually correcting the in-track error. If the simulation is allowed to run long enough, without increasing $K_n$, the in-track error will eventually decrease to the error envelope, and the semi-major axis error, shown by radial error, will be corrected. This approach provides significant fuel savings, as the control law inherently allows drifting arcs in the maneuvering spacecraft’s trajectory.

Figure 4 shows the values of the semi-major axis, $a$, the target orbit semi-major axis, $a^*$, and the semi-major axis requested by the mean motion control, $a^{**}$, for the duration of the maneuver. Recall that the elemental control law feeds back $\delta a = a - a^{**}$. In the initial stage of the maneuver, the semi-major axis remains within a few kilometers of the target value. Once the phasing part of the maneuver begins, the mean motion control forces $\delta a$ up to almost 20 kilometers. As the in-track error is corrected, the value of $a^{**}$ gradually decreases to the original target value of $a^*$. Notice that after about 5.5 orbits, the value of $a$ starts to make steps towards the target value. These steps correspond to impulses correcting the semi-major axis. The level areas between the step changes correspond to coasting arcs, where no thrust is applied, as the in-track error is slowly corrected by a constant level of semi-major axis error.

The maneuver shown in Figs. 3 and 4 requires a $\Delta V$ of approximately 375 m/s. To establish the entire formation, using 1-N, scaled, classical element control, the individual spacecraft require between 281 and 925 m/s, each. This large variation in fuel usage is an important observation, as it means that during this maneuver the ballistic coefficients of the spacecraft change non-uniformly throughout the formation. This non-uniform ballistic
coefficient distribution results in differential atmospheric drag on the formation, and should be avoided.

While the formation-establishment maneuver is an interesting problem, low-thrust feedback control might not be the best choice for this stage of the mission. For a formation of this size, a more practical approach might be to use a launch vehicle upper stage to place each spacecraft in the stack, one at a time, on or near its desired orbit, as is commonly done with spacecraft constellations. Of course, the use of an upper stage loses its appeal as the difference between the orbital planes of the formation flyers increases. Ultimately, we want to choose a method that minimizes the total propellant mass at launch, but also maintains mass symmetry amongst the formation flyers, reducing the effect of differential drag.

**Leonardo formation-keeping.** The formation-keeping problem involves the maintenance of the formation geometry in the presence of differential orbital perturbations such as non-spherical central body effects, atmospheric drag, and solar radiation pressure. We assume that the Leonardo spacecraft are, for the most part, identical in size and mass, and therefore that the differential effects of atmospheric drag and solar radiation pressure are negligible. This assumption will be invalid once the effects of formation geometry on the propellant mass required by each spacecraft are considered; however, the assumption is useful in initial studies of formation-keeping requirements.

As shown in Eqs. (8–10), the magnitude of $J_2$ secular drift varies with inclination. Because the Leonardo spacecraft have different inclinations, the formation will experience differential secular drift due to Earth oblateness effects.

While it may be ideal, from a control engineer’s perspective, to correct orbital errors resultant from perturbation effects as they appear (indeed, a simple way to do this is to simply allow the spacecraft to thrust whenever the thruster on/off logic requests thrust), other spacecraft operations, such as performance of science missions, may require a much less aggressive approach. Discussions with NASA-Goddard engineers suggest that Leonardo formation-keeping operations might occur every two weeks. For this reason, we simulate
a maneuver from initial conditions defined by two weeks of secular $J_2$ drift. These initial conditions are calculated by applying drift to the appropriate spacecraft orbital elements, and recalculating the formation master orbit and desired orbital states from the perturbed formation states.

By calculating the master orbit for this maneuver from the perturbed orbit states, we can re-center the formation, so that the bulk drift is ignored, and only the differential secular drift is corrected.

Figure 5 illustrates the simulation of a two-week-$J_2$-error formation-keeping maneuver, using classical orbital element Lyapunov control scaled to 0.01 N. The figure shows the orbital-frame components of position error for all six spacecraft in the formation. As before, we use gain scheduling, with an emphasis on the out-of-plane error in the first stage of the maneuver, and the in-plane angular error in the second stage. We define the end of the first stage, and the beginning of the second stage as the point where the control law no longer requests thrust for 30 consecutive time steps. At this time, the elemental errors are within the error envelope, and mean motion control commences. Notice that gain scheduling is applied on different schedules for each of the spacecraft. In fact, individual gains should be calculated for each spacecraft using the methods described in this paper.

Table 3 shows the initial values of $\delta \Omega_j$ and $\delta \theta_j$, and the $\Delta V$ required to re-establish the formation after two weeks of $J_2$ drift. The results show a significant imbalance in fuel consumption, with spacecraft 3 and 4 performing the majority of the thrusting. This result is not surprising, as these spacecraft must overcome the largest initial elemental errors.

Comparing spacecraft 1 with spacecraft 2, we notice that while spacecraft 1 starts with larger initial elemental errors, it requires fewer thrust impulses. This anomaly is actually the result of the original equations of motion. The time rate of change of $\Omega$ as a result of orbit normal thrust is

$$\frac{d\Omega}{dt} = \frac{r \sin \theta}{h \sin i} u_\theta$$

(61)

Notice that decreasing inclination results in increasing $\dot{\Omega}$. Therefore, spacecraft with lower
inclinations require less thrust to correct the same $\Omega$ errors. This observation suggests that a shift of the master $\Omega$ towards the highly inclined orbits may reduce the total thrust use.

Table 4 shows the results of a formation-keeping maneuver with $\delta\Omega$ values shifted by 0.1° toward the spacecraft with higher inclinations. The results of this shift are significant: while there is little improvement in the thrust distribution (cf. Table 3), the total thrust count decreases by almost 10%. Furthermore, the thrust count decreases for every spacecraft in the formation, including those whose initial orbital element errors increase as a result of the $\Omega$ shift. The cause of this result is simple: by shifting the right ascension angle of the master point (and thus the target points, which are defined from the master orbit), the target orbit is moved closer to the specific spacecraft.

This effect is illustrated in Fig. 6. Notice that an increase of the right ascension error magnitude results in a decrease in the position error magnitude. The target point (‘$\circ$’) in the orbit rotated the most about the inertial z-axis (up), is much closer to the spacecraft (‘$\circ$’) than is the target point in the orbit rotated less about the z-axis. Because the element $\theta$ is measured from the node (‘$\bullet$’), and because the orbit is inclined so little, increasing the angle between the target node and the actual node, while holding $\delta\theta$ constant, actually brings the target point closer to the spacecraft. As a result, less thrust is required to perform the maneuver.

This effect is also illustrated by the results of previous simulations. If we had plotted the in-plane angle error during the formation-establishment maneuver, we would see that during the first phase of the maneuver, where the selected gains correct the out-of-plane error, the value of $\delta\theta$ decreases dramatically. This correction of the in-plane angular error is not a result of mean motion control, but a result of the movement of the node, the point from which $\theta$ is measured. As the $\delta\Omega$ error is corrected, the $\delta\theta$ error decreases, because the points from which $\theta$ and $\theta^*$ are measured get closer together. This observation is a strong argument in favor of using gain scheduling for this control. For this formation (and others with low inclinations), it is unwise to allow feedback of $\delta\theta$ initially, as the points from which
Conclusions

We develop nonlinear orbit control laws for use in spacecraft orbital maneuvers, and spacecraft formation flying. This nonlinear orbital element feedback control law, augmented by a new mean motion control strategy, successfully and efficiently maintains and modifies the full Keplerian motion of a large variety of spacecraft orbits. Unlike some previous element feedback controls, this law controls both the orbit size, shape, and orientation, and the spacecraft’s angular position within the orbit. The control also takes advantage of physical insight to minimize fuel consumption, by utilizing differential mean motion to correct in-plane angle errors. A new technique for choosing orbital element feedback gains appropriately scales the gains to provide efficient, chatter-free thrust performance. The master orbit algorithm serves as a starting point for a formation-establishment maneuver, and also a point from which we can calculate the desired spacecraft orbital elements. Simulation results show that for the Leonardo formation, the definition of the master orbit at the geometric center of the formation does not provide fuel optimal results. Future studies should seek other analytical methods for calculating a fuel optimal formation master orbit. Another interesting conclusion drawn from the simulation results is that the total thrust required to complete a maneuver is not necessarily proportional to the initial elemental errors. Because of the interaction between the elements, increasing some elemental errors might actually result in decreased thrust use. This observation suggests that the geometric center of the formation is not the best choice for a master reference orbit, and that the master formation orbit definition should be refined.

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References


Table Captions

Table 1: Leonardo formation elemental offsets

Table 2: Leonardo formation orbital elements

Table 3: Leonardo formation-keeping

Table 4: Leonardo formation-keeping with $\delta\Omega$ shifted by 0.1°
Table 1: Leonardo formation elemental offsets  
\((a = 6778 \text{ km}, e=0)\)

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Table 2: Leonardo formation orbital elements  
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### Table 3: Leonardo formation-keeping

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### Table 4: Leonardo formation-keeping with $\delta \Omega$ shifted by 0.1°

<table>
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<tr>
<th>s/c</th>
<th>$i$ [°]</th>
<th>$\delta i$ [°]</th>
<th>$\delta \Omega$ [°]</th>
<th>$\delta \theta$ [°]</th>
<th>$\Delta V$ [m/s]</th>
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<tr>
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<td></td>
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<td>23.7626</td>
</tr>
</tbody>
</table>
Figure Captions

Fig. 1: Formation simulation models: A) target-interceptor model; B) leader-follower model; C) cooperative model

Fig. 2: Combination orbit-raising and inclination shift maneuver. A) $K_a = 1$, $K_e = 1$, and $K_l = 50$ B) $K_a = 1$, $K_e = 1$, and $K_l = 5$

Fig. 3: Formation establishment using 1-N scaled elemental feedback control with gain scheduling

Fig. 4: Evolution of semi-major axis during mean motion controlled formation-establishment maneuver

Fig. 5: Two week $J_2$ perturbed formation-keeping using elemental control with gain scheduling

Fig. 6: Increasing right ascension error maps to decreasing position error
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