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Singular Direction Avoidance Steering for Control-Moment Gyros

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A new form of the equations of motion for a spacecraft with single gimbal control-moment gyros (CMG) is developed using a momentum approach. This set of four vector equations describing the rotational motion of the system is of order $2N + 7$, where N is the number of CMGs. The control input is an $N \times 1$ column vector of torques applied to the gimbal axes. A modification to a singularity robust Lyapunov control law is examined and compared to that control law. Specifically, the attempt to avoid singular gimbal configurations is abandoned in favor of simply avoiding movement in the singular direction. The singular value decomposition is used to compute a pseudoinverse that prevents large gimbal rate commands near or at actual singularities.

Introduction

MOMENTUM exchange devices provide an effective means of reorienting spacecraft. Major benefits of such systems over external thrusters include cleanliness (no expulsion of gases) and replenishability of the power source (electricity instead of fuel). The complexity associated with the nonlinearity of the attitude control problem, along with the gyric terms introduced with the spinning flywheels, poses the largest obstacle to widespread use in current spacecraft designs.

The flywheel of a control moment gyro (CMG) spins at a constant speed and a gimbal arrangement allows variation of the spin axis in the body reference frame. A double-gimbal arrangement can permit the spin axis of the CMG to assume any direction in the body. The single-gimbal CMG (SGCMG) allows reorientation of the spin axis only in a plane that is perpendicular to the gimbal axis. The advantage of the SGCMG is the well-known torque amplification property. Essentially, a rate about the gimbal axis can produce an output torque orthogonal to both the gimbal and spin axes, which is much greater than the gimbal axis torque.

In this paper we examine control laws for the operation of a cluster of SGCMGs to reorient a rigid spacecraft. We first develop the equations of motion for a cluster of SGCMGs including all of the gimbal inertia terms—terms often neglected in the analysis of SGCMG problems. A physically realistic problem requires this, as the control input is the set of gimbal axis torques. The steering law examined in this paper therefore falls into the category of an acceleration steering law as opposed to the more common (but incomplete) velocity or rate steering law. Whereas the complete equations of motion have been presented previously by Oh and Vadali,[1] the derivation here is in a different and more concise form.

The Lyapunov control law proposed by Oh and Vadali[1] is also examined with the goal of improving the performance when approaching a singularity in the gimbal cluster. A singularity is encountered when there exists some direction in the body in which the cluster is not capable of producing torque. Alternatively, we could say that the range of the transformation from gimbal rates to output torque of the cluster is two-dimensional (planar). The singularity robust steering law investigated by Oh and Vadali seeks to avoid these singular configurations. Operating near these singularities can result in unreasonably high gimbal rate commands and undesirable system response if one wishes to generate a torque in the singular direction. If the required torque is orthogonal to the singular direction, there is no reason to avoid the singularity during a reorientation profile. Based on the elegant results of the singular value decomposition (SVD), we are able to steer through some singularities, resulting in a smoother reorientation. An excellent presentation of the principles

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of the SVD and its relationship to the typical pseudoinverse techniques was given by Junkins and Kim.[2]

The development of the equations of motion in matrix form was motivated by a work of Hughes[3] in which he developed the equations of motion for the gyrostat using the absolute spin axis momentum as a state of the system. This approach was further developed by Hall[4] for the case of a multirotor gyrostat.

A paper by Oh and Vadali,[1] in which they provided what is apparently the only complete derivation of the equations of motion for a body with multiple CMGs, was the foundation for this paper. Whereas most developments assume that gimbal rates can be controlled as desired, Oh and Vadali recognized the need to consider gimbal accelerations and developed a feedback control law that handles it. They also presented the utility of the singularity robust control law for avoiding singularities in the case of a set of CMGs.

The work by Margulies and Aubrun[5] established the fundamentals for the study of clusters of CMGs, including consideration of singular configurations, both internal and external. A method for finding sets of singular gimbal angles was provided. They also introduced the possibilities for null motion when the number of CMGs exceeds three.

Several authors have studied methods for avoiding singularity problems in CMG clusters. Hoelscher and Vadali[6] considered open-loop and feedback control laws that minimized control effort, maneuver time, and proximity to singularities. Vadali and Krishnan[7] narrowed the focus exclusively to avoiding these singularities by parameterizing gimbal rates as polynomial functions of time and optimizing the parameters with respect to a singularity avoidance objective function. Going one step further, Vadali et al.[8] developed a method for determining a family of initial gimbal angles that would avoid singularities during a maneuver. Paradiso[9] developed a computer algorithm that was capable of globally avoiding singular states in a feedforward steering law.

Bedrossian et al.[10,11] developed a way of adding the null motion into the control algorithm to avoid singularities. They also showed how the singular values of the matrix transforming gimbal rates into output torques are affected by the singularity robust steering law.

We begin by presenting a new development of equations of motion for a rigid body with a SGCMG cluster. We then recall the Lyapunov control law of Oh and Vadali[1] and present it in terms of the variables of our development. Next the SVD is discussed as an aid to understanding the relationship between gimbal angle rates and output torques. The problem of computing a pseudoinverse to compute suitable gimbal rate commands is discussed, which leads to the singular direction avoidance control law. A modification to the singularity avoidance parameter is suggested. Finally, we present some examples that illustrate how qualitative improvements in the reorientation trajectories can be achieved.

System Equations of Motion

Model and Definitions

Consider a rigid body in which N CMGs are embedded (Fig. 1). The CMGs are designated $W_1, W_2, \ldots, W_N$, and the rigid platform is identified as B. The platform is in general not symmetric. A reference frame $\mathcal{F}_b$ is established in the body with basis $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$. The body is free to translate and rotate with respect to the inertially fixed reference frame $\mathcal{F}_i$ with basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.

The wheels spin about their individual axes of symmetry, which are defined by the unit vectors $\vec{a}_{s1}, \vec{a}_{s2}, \ldots, \vec{a}_{sN}$ (Fig. 2). The directions of the spin axis unit vectors vary with the gimbal angles, as developed next. The gimbal axes are always orthogonal to the spin axes and are denoted by the unit vectors $\vec{a}_{g1}, \vec{a}_{g2}, \ldots, \vec{a}_{gN}$. A third set of unit vectors given by $\vec{a}_{t1}, \vec{a}_{t2}, \ldots, \vec{a}_{tN}$ (subscript representing transverse), where $\vec{a}_{tj} = \vec{a}_{sj} \times \vec{a}_{gj}$, will prove useful in the derivation.

We define a matrix $\boldsymbol{A}_s$ such that the columns of $\boldsymbol{A}_s$ are the column matrices $\boldsymbol{a}_{sj}$ $(j = 1, \ldots, N)$, which specify the orientations of the spin axes of the wheels W_j $(j = 1, \ldots, N)$, in the vehicle body frame $\mathcal{F}_b$. That is,

$$\boldsymbol{A}_s = [\boldsymbol{a}_{s1} \quad \boldsymbol{a}_{s2} \quad \cdots \quad \boldsymbol{a}_{sN}] \tag{1}$$

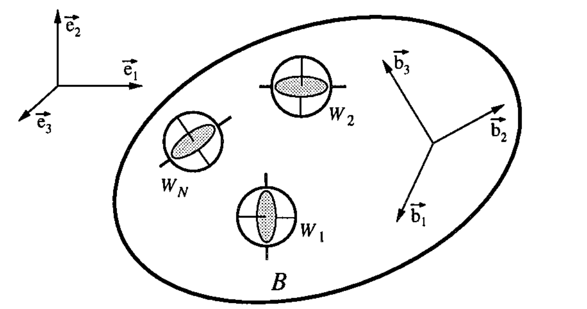

Fig. 1 Body with multiple CMGs.

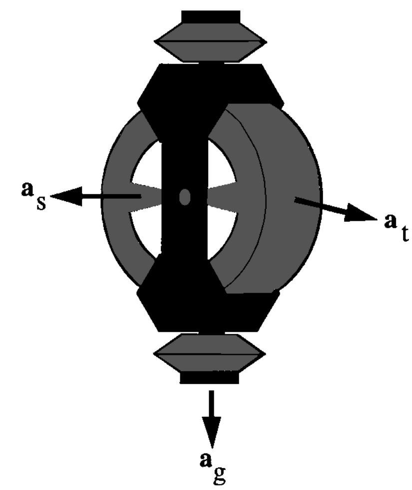

Fig. 2 Control-moment gyroscope.

The matrices $\boldsymbol{A}_g$ and $\boldsymbol{A}_t$ are defined similarly. Whereas $\boldsymbol{A}_g$ is a constant matrix, the matrices $\boldsymbol{A}_s$ and $\boldsymbol{A}_t$ depend on the gimbal angles.

The moment of inertia for the spacecraft is assumed constant except for the change caused by variations in the gimbal angles. The assumption is also made that the center of mass of the spacecraft is fixed in the body and does not vary with gimbal angles. An inertia dyadic $\vec{\boldsymbol{I}}$ is formed from the body inertia dyadic plus the parallel axis contributions of the wheels. It is given by

$$\vec{\boldsymbol{I}} = \vec{\boldsymbol{I}}_B + \sum_{j=1}^{N} m_j (\vec{\boldsymbol{r}}_j \cdot \vec{\boldsymbol{r}}_j \vec{\boldsymbol{1}} - \vec{\boldsymbol{r}}_j \vec{\boldsymbol{r}}_j) \tag{2}$$

where m_j is the mass and $\vec{\boldsymbol{r}}_j$ the fixed location of the center of mass of the jth CMG.

We define the terms I_{sj}, I_{gj}, and I_{tj} to be the total spin axis inertia, the total gimbal axis inertia, and the total transverse axis inertia of the jth CMG (including the gimbal frame). The total spin axis inertia of the CMG is the sum of the gimbal frame inertia I_{sgj} and wheel inertia I_{swj}:

$$I_{\mathrm{sj}} = I_{\mathrm{swj}} + I_{\mathrm{sgj}} \tag{3}$$

We form $\boldsymbol{I}_{\mathrm{sw}}$ as a diagonal matrix composed of the spin axis moments of inertia of the CMG wheels:

$$\boldsymbol{I}_{\mathrm{sw}} = \mathrm{diag}(I_{\mathrm{sw}1}, I_{\mathrm{sw}2}, \ldots, I_{\mathrm{sw}N}) \tag{4}$$

Four other $N \times N$ inertia matrices $\boldsymbol{I}_s, \boldsymbol{I}_g, \boldsymbol{I}_t$, and $\boldsymbol{I}_{\mathrm{sg}}$ are defined in a similar manner.

Angular Momentum

Assuming that the linear momentum of the system is constant (and without loss of generality, zero), then when the body is subjected to the external torque vector $\vec{\boldsymbol{g}}_e$ Euler's Equation for the angular momentum is

$$\dot{\vec{\boldsymbol{h}}} = \vec{\boldsymbol{g}}_e \tag{5}$$

where $\vec{h}$ denotes the angular momentum vector (which is independent of reference frame) and $\boldsymbol{h}$ is the vector's components in the body-fixed reference frame. When expressed in the body coordinate frame, Euler's Equation becomes

$$\dot{\boldsymbol{h}} + \boldsymbol{\omega}^{\times}\boldsymbol{h} = \boldsymbol{g}_e \tag{6}$$

where we use the superscript $\times$ to denote the skew-symmetric form of the column matrix $\boldsymbol{\omega}$. That is,

$$\boldsymbol{\omega}^{\times} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{7}$$

The angular momentum of the body plus its collection of CMGs can be expressed as

$$\vec{h} = \vec{I} \cdot \vec{\omega} + \sum_{j=1}^{N} \vec{h}_{\mathrm{aj}} \tag{8}$$

where $\vec{h}_{\mathrm{aj}}$ is the absolute angular momentum of the jth CMG about its own center of mass. Instead of grouping the CMG contributions to the angular momentum by CMG as did Oh and Vadali,[1] we decompose the CMG contributions to angular momentum into components in the spin, gimbal, and transverse directions. This allows for a concise matrix form for the angular momentum and subsequently a simple matrix form for the equations of motion. Using body-frame components with the origin at the center of mass,

$$\boldsymbol{h} = \boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{A}_s\boldsymbol{h}_{\mathrm{sa}} + \boldsymbol{A}_g\boldsymbol{h}_{\mathrm{ga}} + \boldsymbol{A}_t\boldsymbol{h}_{\mathrm{ta}} \tag{9}$$

The terms $\boldsymbol{h}_{\mathrm{sa}}$, $\boldsymbol{h}_{\mathrm{ga}}$, and $\boldsymbol{h}_{\mathrm{ta}}$ are $N \times 1$ column vectors that represent the components of absolute angular momentum of the CMGs about their spin axes, gimbal axes, and transverse axes, respectively. Here the $\boldsymbol{A}$ matrices are seen to be transformations of these absolute CMG momentum components into the body frame.

We split $\boldsymbol{h}_{\mathrm{sa}}$ into two terms as

$$\boldsymbol{h}_{\mathrm{sa}} = \boldsymbol{h}_{\mathrm{swa}} + \boldsymbol{h}_{\mathrm{sga}} \tag{10}$$

where $\boldsymbol{h}_{\mathrm{swa}}$ is the matrix of absolute angular momenta of the wheels about their spin axes and $\boldsymbol{h}_{\mathrm{sga}}$ is the matrix of absolute angular momenta of the gimbal frames about their spin axes.

The column vectors of the absolute angular momenta are given by

$$\boldsymbol{h}_{\mathrm{ga}} = \boldsymbol{I}_g\boldsymbol{A}_g^T\boldsymbol{\omega} + \boldsymbol{h}_{\mathrm{gr}} \tag{11}$$

$$\boldsymbol{h}_{\mathrm{swa}} = \boldsymbol{I}_{\mathrm{sw}}\boldsymbol{A}_s^T\boldsymbol{\omega} + \boldsymbol{h}_{\mathrm{swr}} \tag{12}$$

$$\boldsymbol{h}_{\mathrm{ta}} = \boldsymbol{I}_t\boldsymbol{A}_t^T\boldsymbol{\omega} + \boldsymbol{h}_{\mathrm{tr}} \tag{13}$$

$$\boldsymbol{h}_{\mathrm{sga}} = \boldsymbol{I}_{\mathrm{sg}}\boldsymbol{A}_s^T\boldsymbol{\omega} + \boldsymbol{h}_{\mathrm{sgr}} \tag{14}$$

which relate the absolute angular momenta components to the relative momenta components for the CMGs.

The CMG is constrained to rotation about the gimbal axis so that

$$\boldsymbol{h}_{\mathrm{tr}} = \boldsymbol{h}_{\mathrm{sgr}} = 0 \tag{15}$$

Because the rotors are assumed to spin at a constant speed relative to the body, $\boldsymbol{h}_{\mathrm{swr}}$ is a constant with the result that

$$\boldsymbol{h} = \left(\boldsymbol{I} + \boldsymbol{A}_t\boldsymbol{I}_t\boldsymbol{A}_t^T + \boldsymbol{A}_s\boldsymbol{I}_s\boldsymbol{A}_s^T\right)\boldsymbol{\omega} + \boldsymbol{A}_s\boldsymbol{h}_{\mathrm{swr}} + \boldsymbol{A}_g\boldsymbol{h}_{\mathrm{ga}} \tag{16}$$

The coefficient of $\boldsymbol{\omega}$ is not a constant in Eq. (16), but a function of the gimbal angles.

Gimbal Angles

We note from Eq. (11) that the relative angular momentum of the CMG about the gimbal axis is

$$\boldsymbol{h}_{\mathrm{gr}} = \boldsymbol{h}_{\mathrm{ga}} - \boldsymbol{I}_g\boldsymbol{A}_g^T\boldsymbol{\omega} = \boldsymbol{I}_g\boldsymbol{\omega}_{\mathrm{gr}}$$

so that we find the gimbal angle rates from the expression

$$\dot{\boldsymbol{\delta}} = \boldsymbol{\omega}_{\mathrm{gr}} = \boldsymbol{I}_g^{-1}\boldsymbol{h}_{\mathrm{ga}} - \boldsymbol{A}_g^T\boldsymbol{\omega} \tag{17}$$

which is itself a differential equation that must be integrated to produce the gimbal angles. These gimbal angles are required to compute the matrices $\boldsymbol{A}_s$ and $\boldsymbol{A}_t$. We assign the gimbal angles to be zero when the spin axes point in a set of arbitrary directions given by $\vec{a}_{s1_0}, \vec{a}_{s2_0}, \ldots, \vec{a}_{sN_0}$.

We define some new terms to aid in the numerical computation of the $3 \times N$ matrices $\boldsymbol{A}_s$ and $\boldsymbol{A}_t$. From the $N \times 1$ vector $\boldsymbol{\delta}$ of gimbal angles, we compute the $N \times N$ matrices Δ^c and Δ^s where

$$\Delta^c = \mathrm{diag}(\cos\boldsymbol{\delta}) \tag{18}$$

$$\Delta^s = \mathrm{diag}(\sin\boldsymbol{\delta}) \tag{19}$$

and $\cos\boldsymbol{\delta}$ and $\sin\boldsymbol{\delta}$ are column vectors of the cosines and sines taken term by term of the column vector $\boldsymbol{\delta}$.

By defining the matrices $\boldsymbol{A}_{s0}$ and $\boldsymbol{A}_{t0}$ as the values of $\boldsymbol{A}_s$ and $\boldsymbol{A}_t$ when the gimbal angles are all zero, then

$$\boldsymbol{A}_s = \boldsymbol{A}_{s0}\Delta^c - \boldsymbol{A}_{t0}\Delta^s \tag{20}$$

$$\boldsymbol{A}_t = \boldsymbol{A}_{t0}\Delta^c + \boldsymbol{A}_{s0}\Delta^s \tag{21}$$

For single-gimbal CMGs $\boldsymbol{A}_g = \boldsymbol{A}_{g0}$ is fixed so that its time rate of change is zero. The rates of change of $\boldsymbol{A}_s$ and $\boldsymbol{A}_t$, however, can be shown to be

$$\dot{\boldsymbol{A}}_s = -\boldsymbol{A}_t\mathrm{diag}(\dot{\boldsymbol{\delta}}) \tag{22}$$

$$\dot{\boldsymbol{A}}_t = \boldsymbol{A}_s\mathrm{diag}(\dot{\boldsymbol{\delta}}) \tag{23}$$

Gimbal Axis Momenta

To find the effect of gimbal torques on the equations of motion, we define the gimbal axis torque for a single CMG as

$$g_g = \vec{a}_g \cdot \vec{g}_{b\,\mathrm{cmg}} \tag{24}$$

where $\vec{g}_{b\,\mathrm{cmg}}$ is the torque applied by the body to the CMG. We include the gimbal frame because we are interested in the change in angular momentum of the spinning wheel plus its frame. Because

$$h_{\mathrm{ga}} = \vec{a}_g \cdot \vec{h}_{\mathrm{cmg}} \tag{25}$$

it follows that

$$\dot{h}_{\mathrm{ga}} = \dot{\vec{a}}_g \cdot \vec{h}_{\mathrm{cmg}} + g_g \tag{26}$$

where $\dot{\vec{a}}_g = \vec{\omega} \times \vec{a}_g$.

We make the reasonable assumption that the spin, gimbal, and transverse axes for the CMG are principal axes for the CMG. These assumptions, after considerable algebra,[12] result in the relationship

$$\dot{h}_{\mathrm{ga}} = \left[(I_t - I_{\mathrm{sg}})\boldsymbol{a}_s^T\boldsymbol{\omega} - h_{\mathrm{swa}}\right]\left(\boldsymbol{a}_t^T\boldsymbol{\omega}\right) + g_g \tag{27}$$

Developing this expression for each CMG leads to an $N \times 1$ vector differential equation for $\boldsymbol{h}_{\mathrm{ga}}$ in the form

$$\dot{\boldsymbol{h}}_{\mathrm{ga}} = \left[(\boldsymbol{I}_t - \boldsymbol{I}_{\mathrm{sg}})\boldsymbol{A}_s^T\boldsymbol{\omega} - \boldsymbol{h}_{\mathrm{swa}}\right] * \left(\boldsymbol{A}_t^T\boldsymbol{\omega}\right) + \boldsymbol{g}_g \tag{28}$$

where the operator $*$ represents term-by-term mutiplication of the two adjacent $N \times 1$ column vectors. Finally, substitution of Eq. (12) into Eq. (28) results in

$$\dot{\boldsymbol{h}}_{\mathrm{ga}} = \left[(\boldsymbol{I}_t - \boldsymbol{I}_s)\boldsymbol{A}_s^T\boldsymbol{\omega} - \boldsymbol{h}_{\mathrm{swr}}\right] * \left(\boldsymbol{A}_t^T\boldsymbol{\omega}\right) + \boldsymbol{g}_g \tag{29}$$

This vector equation is equivalent to N-scalar ordinary differential equations relating the gimbal axis momentum to the spin axis relative momentum, the platform angular velocity, and the gimbal axis torques. No dynamical terms have been excluded from this equation of motion.

Equation Summary

The rotational dynamics of the system therefore can be summarized by the following $2N + 3$ nonlinear equations of motion:

$$\dot{\boldsymbol{h}} = \boldsymbol{h}^{\times}\boldsymbol{\omega} + \boldsymbol{g}_e \tag{30}$$

$$\dot{\boldsymbol{h}}_{\text{ga}} = \left[(\boldsymbol{I}_t - \boldsymbol{I}_s)\boldsymbol{A}_s^T\boldsymbol{\omega} - \boldsymbol{h}_{\text{swr}}\right] * \left(\boldsymbol{A}_t^T\boldsymbol{\omega}\right) + \boldsymbol{g}_g \tag{31}$$

$$\dot{\boldsymbol{\delta}} = \boldsymbol{I}_g^{-1}\boldsymbol{h}_{\text{ga}} - \boldsymbol{A}_g^T\boldsymbol{\omega} \tag{32}$$

where

$$\boldsymbol{\omega} = \left(\boldsymbol{I} + \boldsymbol{A}_t\boldsymbol{I}_t\boldsymbol{A}_t^T + \boldsymbol{A}_s\boldsymbol{I}_s\boldsymbol{A}_s^T\right)^{-1}(\boldsymbol{h} - \boldsymbol{A}_s\boldsymbol{h}_{\text{swr}} - \boldsymbol{A}_g\boldsymbol{h}_{\text{ga}}) \tag{33}$$

$$\boldsymbol{A}_s = \boldsymbol{A}_{s0}\boldsymbol{\Delta}^c - \boldsymbol{A}_{t0}\boldsymbol{\Delta}^s \tag{34}$$

$$\boldsymbol{A}_t = \boldsymbol{A}_{t0}\boldsymbol{\Delta}^c + \boldsymbol{A}_{s0}\boldsymbol{\Delta}^s \tag{35}$$

We use quaternions to represent the attitude of the spacecraft defined by the eigenangle ϕ and eigenaxis $\hat{\boldsymbol{a}}$ as

$$q_0 = \cos(\phi/2) \tag{36}$$

and

$$\tilde{\boldsymbol{q}} = \hat{\boldsymbol{a}}\sin(\phi/2) \tag{37}$$

and define the four-vector $\boldsymbol{q}$ as

$$\boldsymbol{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{pmatrix} q_0 \\ \tilde{\boldsymbol{q}} \end{pmatrix} \in \Re^4 \tag{38}$$

The kinematical equations relating the rates of change of quaternions to the angular velocity of the spacecraft are

$$\dot{\boldsymbol{q}} = \tfrac{1}{2}\boldsymbol{G}(\boldsymbol{q})\boldsymbol{\omega} \tag{39}$$

where $\boldsymbol{q} = [q_0 \;\; q_1 \;\; q_2 \;\; q_3]^T$ and

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix} \tag{40}$$

These four additional equations must be integrated for attitude information. The dynamical equations (30–32) and kinematical equation (39) completely describe the rotational motion of the spacecraft/CMG system.

Lyapunov Feedback Control Law

In this section we develop the stabilizing Lyapunov control law from Oh and Vadali,[1] but using the variables of the current development. We define a positive definite Lyapunov function

$$V = k\boldsymbol{e}_1^T\boldsymbol{e}_1 + \tfrac{1}{2}\boldsymbol{e}_2^T\boldsymbol{I}_{\text{tot}}\boldsymbol{e}_2 \tag{41}$$

where the attitude error is $\boldsymbol{e}_1 = \boldsymbol{q} - \boldsymbol{q}_f$, the angular velocity error is $\boldsymbol{e}_2 = \boldsymbol{\omega} - \boldsymbol{\omega}_f$, and k is a positive scalar constant. The constant terms $\boldsymbol{q}_f$ and $\boldsymbol{\omega}_f$ are the desired final values for the reorientation maneuver.

The total inertia is positive definite and given by

$$\boldsymbol{I}_{\text{tot}} = \boldsymbol{I} + \boldsymbol{A}_t^T\boldsymbol{I}_t\boldsymbol{A}_t + \boldsymbol{A}_s^T\boldsymbol{I}_s\boldsymbol{A}_s + \boldsymbol{A}_g^T\boldsymbol{I}_g\boldsymbol{A}_g \tag{42}$$

The derivative of the Lyapunov function can be expressed in the form

$$\dot{V} = -(\boldsymbol{\omega} - \boldsymbol{\omega}_f)^T\left[-k\boldsymbol{G}^T(\boldsymbol{q}_f)\boldsymbol{q} + \boldsymbol{I}_{\text{tot}}\dot{\boldsymbol{\omega}}_f - \boldsymbol{I}_{\text{tot}}\dot{\boldsymbol{\omega}} + \tfrac{1}{2}\dot{\boldsymbol{I}}_{\text{tot}}\boldsymbol{\omega}_f - \tfrac{1}{2}\dot{\boldsymbol{I}}_{\text{tot}}\boldsymbol{\omega}\right] \tag{43}$$

We see that $\dot{V}$ can be guaranteed negative semidefinite when we ensure that

$$-k\boldsymbol{G}^T(\boldsymbol{q}_f)\boldsymbol{q} + \boldsymbol{I}_{\text{tot}}\dot{\boldsymbol{\omega}}_f - \boldsymbol{I}_{\text{tot}}\dot{\boldsymbol{\omega}} - \tfrac{1}{2}\dot{\boldsymbol{I}}_{\text{tot}}\boldsymbol{\omega} + \tfrac{1}{2}\dot{\boldsymbol{I}}_{\text{tot}}\boldsymbol{\omega}_f = \boldsymbol{K}(\boldsymbol{\omega} - \boldsymbol{\omega}_f) \tag{44}$$

where $\boldsymbol{K}$ is a positive definite gain matrix.

It is possible to show, using Eq. (33), that

$$\dot{\boldsymbol{\omega}} = \boldsymbol{I}_{\text{tot}}^{-1}(\dot{\boldsymbol{h}} - \dot{\boldsymbol{A}}_s\boldsymbol{h}_{\text{swr}} - \boldsymbol{A}_g\dot{\boldsymbol{h}}_{\text{gr}}) - \boldsymbol{I}_{\text{tot}}^{-1}\dot{\boldsymbol{I}}_{\text{tot}}\boldsymbol{\omega} \tag{45}$$

which leads us to the relationship

$$\tfrac{1}{2}\dot{\boldsymbol{I}}_{\text{tot}}(\boldsymbol{\omega} + \boldsymbol{\omega}_f) + \dot{\boldsymbol{A}}_s\boldsymbol{h}_{\text{swr}} + \boldsymbol{A}_g\dot{\boldsymbol{h}}_{\text{gr}} = \boldsymbol{K}(\boldsymbol{\omega} - \boldsymbol{\omega}_f) + k\boldsymbol{G}^T(\boldsymbol{q}_f)\boldsymbol{q} - \boldsymbol{I}\dot{\boldsymbol{\omega}}_f + \boldsymbol{h}^{\times}\boldsymbol{\omega} + \boldsymbol{g}_e \tag{46}$$

Equation (46) is in a useful form because the terms on the left-hand side all depend on either the gimbal angle rates or accelerations. In fact, if we define

$$\boldsymbol{L}_r \equiv \boldsymbol{K}(\boldsymbol{\omega} - \boldsymbol{\omega}_f) + k\boldsymbol{G}^T(\boldsymbol{q}_f)\boldsymbol{q} - \boldsymbol{I}\dot{\boldsymbol{\omega}}_f + \boldsymbol{h}^{\times}\boldsymbol{\omega} + \boldsymbol{g}_e \tag{47}$$

then Eq. (46) can be written as

$$\boldsymbol{B}\ddot{\boldsymbol{\delta}} + \boldsymbol{D}\dot{\boldsymbol{\delta}} = \boldsymbol{L}_r \tag{48}$$

Because $\dot{\boldsymbol{h}}_{\text{gr}} = \boldsymbol{I}_g\ddot{\boldsymbol{\delta}}$, the matrix coefficient $\boldsymbol{B}$ is simply

$$\boldsymbol{B} = \boldsymbol{A}_g\boldsymbol{I}_g \tag{49}$$

whereas the $\boldsymbol{D}$ matrix coefficient of $\dot{\boldsymbol{\delta}}$ in Eq. (48) is defined by

$$\boldsymbol{D}\dot{\boldsymbol{\delta}} = \tfrac{1}{2}\dot{\boldsymbol{I}}_{\text{tot}}(\boldsymbol{\omega} + \boldsymbol{\omega}_f) + \dot{\boldsymbol{A}}_s\boldsymbol{h}_{\text{swr}} \tag{50}$$

Both $\boldsymbol{B}$ and $\boldsymbol{D}$ are $3 \times N$ matrices. As developed next, $\boldsymbol{B}$ is constant, and $\boldsymbol{D}$ depends on both $\boldsymbol{\omega}$ and $\boldsymbol{\delta}$.

To compute $\boldsymbol{D}$, note that $\boldsymbol{I}_{\text{tot}}$ is a function of the gimbal rates:

$$\begin{aligned}\dot{\boldsymbol{I}}_{\text{tot}} &= \boldsymbol{A}_s\text{diag}\,\dot{\boldsymbol{\delta}}\boldsymbol{I}_t\boldsymbol{A}_t^T + \boldsymbol{A}_t\text{diag}\,\dot{\boldsymbol{\delta}}\boldsymbol{I}_t\dot{\boldsymbol{\delta}}\boldsymbol{A}_s^T \\ &\quad - \boldsymbol{A}_t\text{diag}\,\dot{\boldsymbol{\delta}}\boldsymbol{I}_s\boldsymbol{A}_s^T - \boldsymbol{A}_s\text{diag}\,\dot{\boldsymbol{\delta}}\boldsymbol{I}_s\boldsymbol{A}_t^T \\ &= \boldsymbol{A}_s\text{diag}\,\dot{\boldsymbol{\delta}}(\boldsymbol{I}_t - \boldsymbol{I}_s)\boldsymbol{A}_t^T + \boldsymbol{A}_t\text{diag}\,\dot{\boldsymbol{\delta}}(\boldsymbol{I}_t - \boldsymbol{I}_s)\boldsymbol{A}_s^T\end{aligned} \tag{51}$$

It is possible, for example, to write the product of the first term on the right-hand side of Eq. (51) with $\boldsymbol{\omega} + \boldsymbol{\omega}_f$ as

$$\begin{aligned}\boldsymbol{A}_s\text{diag}\,\dot{\boldsymbol{\delta}}(\boldsymbol{I}_t - \boldsymbol{I}_s)\boldsymbol{A}_t^T(\boldsymbol{\omega} + \boldsymbol{\omega}_f) &= \sum_{j=1}^{N}\boldsymbol{a}_{\text{sj}}\boldsymbol{a}_{\text{tj}}^T(\boldsymbol{\omega} + \boldsymbol{\omega}_f)(I_{\text{tj}} - I_{\text{sj}})\dot{\delta}_j \\ &= \left[\boldsymbol{a}_{s1}\boldsymbol{a}_{t1}^T(\boldsymbol{\omega} + \boldsymbol{\omega}_f) \quad \boldsymbol{a}_{s2}\boldsymbol{a}_{t2}^T(\boldsymbol{\omega} + \boldsymbol{\omega}_f) \quad \cdots \quad \boldsymbol{a}_{sN}\boldsymbol{a}_{tN}^T(\boldsymbol{\omega} + \boldsymbol{\omega}_f)\right] \\ &\quad \times(\boldsymbol{I}_t - \boldsymbol{I}_s)\dot{\boldsymbol{\delta}}\end{aligned} \tag{52}$$

Note also that

$$\dot{\boldsymbol{A}}_s\boldsymbol{h}_{\text{swr}} = -\boldsymbol{A}_t\text{diag}(\dot{\boldsymbol{\delta}})\boldsymbol{h}_{\text{swr}} = -\boldsymbol{A}_t\text{diag}(\boldsymbol{h}_{\text{swr}})\dot{\boldsymbol{\delta}} \tag{53}$$

so that

$$\boldsymbol{D} = -\boldsymbol{A}_t\text{diag}(\boldsymbol{h}_{\text{swr}}) + \tfrac{1}{2}\left[\left(\boldsymbol{a}_{s1}\boldsymbol{a}_{t1}^T + \boldsymbol{a}_{t1}\boldsymbol{a}_{s1}^T\right)(\boldsymbol{\omega} + \boldsymbol{\omega}_f) \quad \cdots \quad \left(\boldsymbol{a}_{sN}\boldsymbol{a}_{tN}^T + \boldsymbol{a}_{tN}\boldsymbol{a}_{sN}^T\right)(\boldsymbol{\omega} + \boldsymbol{\omega}_f)\right](\boldsymbol{I}_t - \boldsymbol{I}_s) \tag{54}$$

In general, because $\boldsymbol{h}_{\text{swr}}$ is large, the contributions to $\boldsymbol{D}$ from the first term of Eq. (54) are of much greater magnitude than those of the second term (especially for small $\boldsymbol{\omega} + \boldsymbol{\omega}_f$ or when $\boldsymbol{I}_t \approx \boldsymbol{I}_s$).

Choosing the gimbal rates and accelerations to satisfy Eq. (48) will ensure that the system is stable, that is, $\boldsymbol{\omega} \rightarrow \boldsymbol{\omega}_f$ and $\boldsymbol{q} \rightarrow \boldsymbol{q}_f$

with increasing time. Of course, $\boldsymbol{\delta}$ is actually determined by the integration of $\dot{\boldsymbol{\delta}}$ and is therefore a state of the system. The advantage of SGCMGs, however, lies in the torque amplification resulting from the gimbal rates. We therefore seek to drive the gimbals so that

$$\boldsymbol{D}\dot{\boldsymbol{\delta}} = \boldsymbol{L}_r \tag{55}$$

as closely as possible.

Oh and Vadali[1] showed that by choosing

$$\ddot{\boldsymbol{\delta}} = \boldsymbol{K}_\delta(\boldsymbol{D}^\dagger \boldsymbol{L}_r - \dot{\boldsymbol{\delta}}) \tag{56}$$

if desired gimbal accelerations are assumed small, we can keep $\dot{\boldsymbol{\delta}}$ close to that required by $\boldsymbol{D}^\dagger \boldsymbol{L}_r$, ensuring that the control takes advantage of the torques generated from the gimbal rates. Knowing the desired gimbal accelerations allows computation of the required gimbal axis torques, because from Eqs. (31) and (32)

$$\boldsymbol{g}_g = \boldsymbol{I}_g\ddot{\boldsymbol{\delta}} + \boldsymbol{I}_g\boldsymbol{A}_g^T\dot{\boldsymbol{\omega}} - \left[(\boldsymbol{I}_t - \boldsymbol{I}_s)\boldsymbol{A}_s^T\boldsymbol{\omega} - \boldsymbol{h}_{\mathrm{swr}} * \left(\boldsymbol{A}_t^T\boldsymbol{\omega}\right)\right] \tag{57}$$

The superscript † is used in Eq. (56) to indicate that when $\boldsymbol{D}$ is not invertible, either because it is not square or the gimbals are in a singular configuration, then a suitable pseudoinverse should be used. We turn our attention to developing one.

Singularity Problem

A singularity is encountered when there exists some direction in the body in which the cluster is not capable of producing torque. This occurs for a particular direction in the body frame when the spin axes of all CMGs in the cluster are either maximally or minimally projected in that direction. In this condition all of the transverse axes of the CMGs are perpendicular to this direction. Thus, the columns of $\boldsymbol{A}_t$ are coplanar, and $\boldsymbol{D}$ is rank 2. We could also say that the range of the transformation from gimbal rates to output torque of the cluster is two-dimensional. Operating near these singularities can result in unreasonably high gimbal rate commands and undesirable system response.

Of course, nothing physically significant actually happens at a singularity. There is no gimbal lock phenomenon or other adverse effect associated with these configurations. There is, however, a problem if one wishes to generate a torque in the singular direction. If the required torque is orthogonal to the singular direction, there is no reason to avoid the singularity during the reorientation profile. Using the SVD, we are able to steer through some singularities, resulting in a much smoother reorientation. We begin with an analysis of the singularity-robust (SR) steering law before showing how it is modified into a singular direction avoidance law.

SR Steering Law

For a system of three CMGs (with noncoplanar gimbal axes), and a nonsingular matrix $\boldsymbol{D}$, Eq. (55) can be solved for the desired gimbal rates as

$$\dot{\boldsymbol{\delta}} = \boldsymbol{D}^{-1}\boldsymbol{L}_r \tag{58}$$

When there are more than three CMGs in the cluster, then the solution for $\dot{\boldsymbol{\delta}}$ is underdetermined, and a minimum two-norm solution can be calculated from

$$\dot{\boldsymbol{\delta}} = \boldsymbol{D}^T(\boldsymbol{D}\boldsymbol{D}^T)^{-1}\boldsymbol{L}_r \tag{59}$$

Equation (59) fails when $\boldsymbol{D}$ drops below rank three because $\boldsymbol{D}\boldsymbol{D}^T$ becomes singular.

Oh and Vadali[1] showed that a modification to Eq. (59) when the determinant of $\boldsymbol{D}\boldsymbol{D}^T$ goes to zero can provide a usable solution for the gimbal rates when near a singularity. They proposed using the SR steering law (originally developed by Nakamura and Hanafusa[13]) given by

$$\dot{\boldsymbol{\delta}} = \boldsymbol{D}^T(\boldsymbol{D}\boldsymbol{D}^T + \alpha\mathbf{1})^{-1}\boldsymbol{L}_r \tag{60}$$

where $\mathbf{1}$ is the 3×3 identity matrix and α is a scalar parameter (called the singularity avoidance parameter), which is negligible when $\boldsymbol{D}\boldsymbol{D}^T$ is nonsingular but increases as a singularity is approached. They proposed, for example, letting

$$\alpha = \alpha_o e^{-\det(\boldsymbol{D}\boldsymbol{D}^T)} \tag{61}$$

where α_o is some small constant. This ensures that a solution for the gimbal rates always exists. Equation (60) is commonly referred to as the SR steering law.

Near or at a singularity, the computed gimbal rates do not produce the required torque. We are forced to accept some deviation, as it may be impossible to produce the desired torque with finite gimbal rates. In some cases, however, the required torque might be possible if it lies in the range of $\boldsymbol{D}$, even though $\boldsymbol{D}$ is singular. The SR steering law will produce an errant torque, even when the desired torque is achievable. The SVD allows for an examination of the structure of the SR steering law and provides some insight, which allows for an improvement in the steering law near or at a singularity.

SVD

In general, any $m\times n$ matrix can be decomposed into the product of three special matrices:

$$\boldsymbol{D} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T \tag{62}$$

where $\boldsymbol{U}$ is an $m\times m$ unitary matrix, $\boldsymbol{V}$ is an $n\times n$ unitary matrix, and $\boldsymbol{S}$ is an $m\times n$ matrix that is diagonal. If $\boldsymbol{S}$ has more columns than rows (i.e., $m < n$), then the last $n-m$ columns are zeroes.

Expressing $\boldsymbol{D}$ in this form makes it easy to compute $\boldsymbol{D}^{-1}$ because in the case where $\boldsymbol{D}$ is square

$$\boldsymbol{D}^{-1} = \boldsymbol{V}\boldsymbol{S}^{-1}\boldsymbol{U}^T \tag{63}$$

and $\boldsymbol{S}$ is an easily inverted diagonal matrix. When $m < n$, then the matrix $\boldsymbol{D}^T(\boldsymbol{D}\boldsymbol{D}^T)^{-1}$ can be calculated from the elements of the SVD by discarding the last $n-m$ zero columns of $\boldsymbol{S}$ and $\boldsymbol{V}$, inverting the new $m\times m$ square matrix $\boldsymbol{S}$, and forming

$$\boldsymbol{D}^\dagger = \boldsymbol{V}_t\boldsymbol{S}_t^{-1}\boldsymbol{U}^T \tag{64}$$

where $\boldsymbol{V}_t$ and $\boldsymbol{S}_t$ are the truncated matrices.

Equation (64) gives the minimum norm solution to the undetermined case and works well for the CMG problem when $\mathrm{rank}(\boldsymbol{D}) = 3$. When the gimbals become singular, however, $\boldsymbol{S}_t^{-1}$ does not exist because a singular value goes to zero.

We now ask how adding the singularity avoidance parameter to the underdetermined steering equation given by Eq. (60) affects the singular value parameters (as did Bedrossian[11]). We proceed as follows:

$$\begin{aligned}\boldsymbol{D}^T(\boldsymbol{D}\boldsymbol{D}^T + \alpha\mathbf{1})^{-1} &= \boldsymbol{V}\boldsymbol{S}^T\boldsymbol{U}^T(\boldsymbol{U}\boldsymbol{S}\boldsymbol{S}^T\boldsymbol{U}^T + \alpha\boldsymbol{U}\boldsymbol{U}^T)^{-1}\\ &= \boldsymbol{V}\boldsymbol{S}^T(\boldsymbol{S}\boldsymbol{S}^T + \alpha\mathbf{1})^{-1}\boldsymbol{U}^T = \boldsymbol{V}_t\boldsymbol{S}_t\left(\boldsymbol{S}_t^2 + \alpha\mathbf{1}\right)^{-1}\boldsymbol{U}^T = \boldsymbol{V}_t\boldsymbol{S}_{\mathrm{SR}}^\dagger\boldsymbol{U}^T\end{aligned} \tag{65}$$

Both $\boldsymbol{S}_t$ and $\boldsymbol{S}_t^2 + \alpha\mathbf{1}$ are diagonal matrices so that

$$\boldsymbol{S}_{\mathrm{SR}}^\dagger = \mathrm{diag}\left(\frac{S_{11}}{S_{11}^2+\alpha}, \frac{S_{22}}{S_{22}^2+\alpha}, \ldots, \frac{S_{mm}}{S_{mm}^2+\alpha}\right) \tag{66}$$

We now see how the singularity avoidance parameter affects the pseudoinverse. If α were zero, there would be no impact on the pseudoinverse; when it is nonzero, all of the effective singular values are altered, a result which we will argue is neither required nor desired.

Looking back at Eq. (61), it is interesting to note that because

$$\boldsymbol{D}\boldsymbol{D}^T = \boldsymbol{U}\boldsymbol{S}^T\boldsymbol{S}\boldsymbol{U}^T$$

then

$$\det(\boldsymbol{D}\boldsymbol{D}^T) = \det(\boldsymbol{U})\cdot\det(\boldsymbol{S}^T\boldsymbol{S})\cdot\det(\boldsymbol{U}^T) = \det(\boldsymbol{S}^T\boldsymbol{S}) = \prod_{i=1}^{m} S_{ii}^2 \tag{67}$$

so that the exponent term in the singularity avoidance parameter α is a function of all of the singular values. Next, we propose a singularity avoidance parameter that depends only on S_{33}, the critical singular value for this problem.

Singular Direction Avoidance

By convention, all singular values are positive and ordered such that

$$S_{11} \geq S_{22} \geq \cdots \geq S_{mm} \geq 0 \tag{68}$$

For a transformation matrix, such as $\boldsymbol{D}$, which is a continuous function of several variables, the singular values are continuous functions as well. Because the $3 \times N$ matrix $\boldsymbol{D}$ maps the $N \times 1$ vector of gimbal rates into the 3×1 vector of output torques, $\boldsymbol{D}$ is rank 3 for nonsingular gimbal angles. If a singularity is encountered, then $\boldsymbol{D}$ will fall to rank 2, with $S_{33} = 0$. The SVD provides some useful information through the unitary matrices $\boldsymbol{U}$ and $\boldsymbol{V}$. With $\boldsymbol{D}$ a $3 \times N$ matrix the last $N - 3$ columns of $\boldsymbol{V}$ span the null space of $\boldsymbol{D}$. These columns form a basis for the gimbal null motions. The third column of $\boldsymbol{V}$ always represents the direction mapped through the smallest singular value. When $S_{33} = 0$, the third column of $\boldsymbol{V}$ is also in the null space of $\boldsymbol{D}$.

The columns of $\boldsymbol{U}$ form the basis for the range of $\boldsymbol{D}$ when $\boldsymbol{D}$ is rank 3. However, when $\boldsymbol{D}$ falls to rank 2, only the first two columns of $\boldsymbol{U}$ span the range of $\boldsymbol{D}$, whereas the last column is orthogonal to the range of $\boldsymbol{D}$. The direction of the third column of $\boldsymbol{U}$ is the direction in which no output torque is possible when rank($\boldsymbol{D}$) = 2.

It is always the third singular value that goes to zero as a singular gimbal configuration is approached. We therefore suggest that preventing S_{33} from going to zero will ensure that the commanded gimbal rates are finite. Furthermore, we argue that it is not only unnecessary, but undesirable, to change all of the gimbal rate commands when a singularity is approached, as is obviously accomplished through the SR control law [see Eqs. (65) and (66)].

The output torque requested by the Lyapunov control law, Eq. (47), is not affected by the singularity of the gimbal cluster. It is the torque we would like to generate if possible. When we cannot because of a singular condition, there is no apparent reason to alter all of the gimbal rates to avoid the singularity.

When $\boldsymbol{D}$ is decomposed by SVD, Eq. (55) becomes

$$\boldsymbol{L}_r = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T\dot{\boldsymbol{\delta}} \tag{69}$$

Some useful insight is provided when rewritten in the form

$$\boldsymbol{U}^T\boldsymbol{L}_r = \boldsymbol{S}\boldsymbol{V}^T\dot{\boldsymbol{\delta}} \tag{70}$$

When $S_{33} = 0$, we observe that the range of $\boldsymbol{S}\boldsymbol{V}^T$ is two-dimensional and is the plane spanned by the first two columns of $\boldsymbol{U}$. The first two elements of the vector $\boldsymbol{U}^T\boldsymbol{L}_r$ remain the desired torque components in the range of $\boldsymbol{S}\boldsymbol{V}^T$, and it is not apparent why we would want to alter the torque components in the range of $\boldsymbol{D}$. We argue that it is best not to alter S_{11} or S_{22} before inversion.

The question becomes one of determining what to do with the possible gimbal motions in the null space of $\boldsymbol{D}$. The singularity robust inverse technique would suggest letting

$$\left(\frac{1}{S_{33}}\right) \rightarrow \left(\frac{S_{33}}{S_{33}^2 + \alpha}\right)$$

This function approximates $(1/S_{33})$ for large S_{33}, but goes back to zero at a singularity (for $\alpha \neq 0$).

It seems reasonable to expect maintaining the gimbal rates present prior to encountering the singularity would be preferable to attempting to stop the gimbal movement in the null direction as the singularity approaches. Whereas this logic is sound in theory, because we are actually forced to deal with an acceleration steering law the gimbal rates are never exactly what we desire, and it is impossible to move through a singularity in a direction exactly orthogonal to the singular direction.

We therefore examine the rate steering equation of the form

$$\dot{\boldsymbol{\delta}} = \boldsymbol{V}_t\boldsymbol{S}_{\mathrm{SDA}}^{\dagger}\boldsymbol{U}^T\boldsymbol{L}_r \tag{71}$$

where

$$\boldsymbol{S}_{\mathrm{SDA}}^{\dagger} = \mathrm{diag}\left(\frac{1}{S_{11}}, \frac{1}{S_{22}}, \frac{S_{33}}{S_{33}^2 + \alpha}\right) \tag{72}$$

We shall refer to the gimbal rates computed by Eq. (71) as the singular direction avoidance (SDA) steering law.

Of course, we should justify why choosing the gimbal rates in accordance with Eq. (71) is an improvement over those computed by Eq. (60), the SR steering law.

Define as $\boldsymbol{L}_p$ the torque produced by a cluster of CMGs. That is,

$$\boldsymbol{L}_p = \boldsymbol{D}\dot{\boldsymbol{\delta}} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T\dot{\boldsymbol{\delta}} = \boldsymbol{U}\boldsymbol{S}_t\boldsymbol{V}_t^T\dot{\boldsymbol{\delta}} \tag{73}$$

Choosing the gimbal rates in accordance with the SR steering law yields the gimbal rates given by Eq. (65) so that the torque produced is

$$\boldsymbol{L}_{p\mathrm{SR}} = \boldsymbol{U}\boldsymbol{S}_t\boldsymbol{S}_{\mathrm{SR}}^{\dagger}\boldsymbol{U}^T\boldsymbol{L}_r \tag{74}$$

Near a singularity, this steering law does not produce the torque required. The error is

$$\boldsymbol{L}_r - \boldsymbol{L}_{p\mathrm{SR}} = \boldsymbol{U}\left(\boldsymbol{1} - \boldsymbol{S}_t\boldsymbol{S}_{\mathrm{SR}}^{\dagger}\right)\boldsymbol{U}^T\boldsymbol{L}_r \tag{75}$$

The norm of the error in the torque produced is therefore

$$\|\boldsymbol{L}_r - \boldsymbol{L}_{p\mathrm{SR}}\| = \left\| \begin{pmatrix} \alpha/\left(S_{11}^2 + \alpha\right) & 0 & 0 \\ 0 & \alpha/\left(S_{22}^2 + \alpha\right) & 0 \\ 0 & 0 & \alpha/\left(S_{33}^2 + \alpha\right) \end{pmatrix} \boldsymbol{U}^T\boldsymbol{L}_r \right\| \tag{76}$$

We use the same reduction to show that for the SDA approach

$$\|\boldsymbol{L}_r - \boldsymbol{L}_{p\mathrm{SDA}}\| = \left\| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha/\left(S_{33}^2 + \alpha\right) \end{pmatrix} \boldsymbol{U}^T\boldsymbol{L}_r \right\| \tag{77}$$

The error in the torque produced is therefore directly proportional to the magnitude of the torque required, but varies with direction. If $\boldsymbol{L}_r$ is a unit vector, then the maximum error occurs in either case when $\boldsymbol{L}_r$ lies solely in the direction of the smallest singular value (and is the same for either control law). The minimum error, however, occurs for the SR law when $\boldsymbol{L}_r$ lies in the direction of the maximum singular value, but is always nonzero (assuming $\alpha \neq 0$). For the SDA law the minimum error occurs if $\boldsymbol{L}_r$ lies in the plane orthogonal to the direction of the minimum singular value (that is, in the range of the first two columns of $\boldsymbol{U}$).

We conclude that for the SR law

$$\frac{\alpha}{S_{11}^2 + \alpha} \leq \frac{\|\boldsymbol{L}_r - \boldsymbol{L}_{p\mathrm{SR}}\|}{\|\boldsymbol{L}_r\|} \leq \frac{\alpha}{S_{33}^2 + \alpha} \tag{78}$$

whereas for the SDA law

$$0 \leq \frac{\|\boldsymbol{L}_r - \boldsymbol{L}_{p\mathrm{SDA}}\|}{\|\boldsymbol{L}_r\|} \leq \frac{\alpha}{S_{33}^2 + \alpha} \tag{79}$$

and that for any $\boldsymbol{L}_r$

$$\|\boldsymbol{L}_r - \boldsymbol{L}_{p\mathrm{SDA}}\| \leq \|\boldsymbol{L}_r - \boldsymbol{L}_{p\mathrm{SR}}\| \tag{80}$$

The error in the torque output resulting from the SDA control law is always less than or equal to that produced by the SR law. In some cases the error might even be zero, whereas the SR law always produces some error.

Singularity Avoidance Parameter

The choice of the singularity avoidance parameter α can of course have a major impact on a typical reorientation profile. Because the intent of the singularity avoidance is to effect a smooth reorientation and prevent large gimbal accelerations (and torque commands), a method of judging the smoothness of a maneuver is to observe gimbal angle histories or possibly the output torque history. Numerous simulations using a singularity avoidance parameter based on Eq. (61) near singular configurations indicated a tendency toward abrupt gimbal angle commands and angular velocity histories because the exponential term tends to rise abruptly as the singularity is approached. Furthermore, the singularity parameter given by Eq. (61) depends on the physical size of the system because $\det(\boldsymbol{D}\boldsymbol{D}^T)$ has units of angular momentum squared. We therefore choose to nondimensionalize the exponent.

For a typical CMG system the contributions to $\boldsymbol{D}$ from Eq. (53) are much greater than those from Eq. (52). For the physical model and reorientation profile explored in the next section, the difference is roughly seven orders of magnitude. Therefore

$$\boldsymbol{D} \approx -\boldsymbol{A}_t \operatorname{diag}(\boldsymbol{h}_{\text{swr}}) \tag{81}$$

It is possible to show that when $\boldsymbol{D}$ is computed as in Eq. (81)

$$S_{11}^2 + S_{22}^2 + S_{33}^2 = \boldsymbol{h}_{\text{swr}}^T \boldsymbol{h}_{\text{swr}} = \text{const} \tag{82}$$

and when all CMGs have the same spin momenta h_{swr}, then for a cluster of N CMGs

$$\sum_{i=1}^{3} S_{ii}^2 = N h_{\text{swr}}^2 \tag{83}$$

This allows us to put a bound on the upper limit of the smallest singular value S_{33} because when S_{33} is at its maximum $S_{11} = S_{22} = S_{33}$. Therefore

$$S_{33} \leq \sqrt{(N/3)} h_{\text{swr}} \tag{84}$$

We define a nondimensional variable

$$\sigma_{33} \equiv \sqrt{(3/N)}(S_{33}/h_{\text{swr}}) \tag{85}$$

so that

$$0 \leq \sigma_{33} \leq 1 \tag{86}$$

and choose as a singularity avoidance parameter

$$\alpha = \alpha_0 e^{-k_\sigma \sigma_{33}^2} \tag{87}$$

so that the response of the system is independent of system size. The constant k_σ may be selected as desired. The coefficient term α_0 also has units of angular momentum squared and should be adjusted to match the dimension of the system. In the following section we compare the SR and SDA maneuvers, and we also compare maneuvers using the two singularity avoidance parameters.

Maneuver Examples

We now present some examples that are chosen to illustrate the advantages of the proposed steering law. The physical model chosen is a rigid spacecraft with a cluster of four CMGs arranged in a pyramid configuration (see Oh and Vadali[1]). The parameters are taken directly from the example used by Oh and Vadali to compare selected results. The model data are listed in Table 1.

Table 1 Physical data

Item	Value	Units
$\boldsymbol{h}_{\text{swr}}$	$[1.8, 1.8, 1.8, 1.8]^T$	kg-m^2/s
$\boldsymbol{I}$	diag{86.2, 85.1, 113.6}	kg-m^2
$\boldsymbol{I}_{\text{sw}}$	diag{0.05, 0.05, 0.05, 0.05}	kg-m^2
$\boldsymbol{I}_{\text{tw}}$	diag{0.03, 0.03, 0.03, 0.03}	kg-m^2
$\boldsymbol{I}_{\text{tg}}$	diag{0.01, 0.01, 0.01, 0.01}	kg-m^2
$\boldsymbol{I}_{\text{sg}}, \boldsymbol{I}_{\text{gg}}$	diag{0, 0, 0, 0}	kg-m^2
Pyramid angle	54.74	deg

Table 2 Example 1 simulation data

Item	Value	Units
$\boldsymbol{K}_\delta$	diag{100, 100, 100, 100}	s^{-1}
α_0	0.5	kg^2-m^4/s^2
$\boldsymbol{\delta}_0$	$[13.5, -13.5, -54.3, 54.3]^T$	deg
$\dot{\boldsymbol{\delta}}_0$	$[5.54, 9.20, 10.82, -1.98]^T$	deg/s
$\boldsymbol{\omega}_0$	$[0, 0, 0]^T$	deg/s
$\boldsymbol{L}_r$	$[0, -0.5, 0.5]^T$	N-m

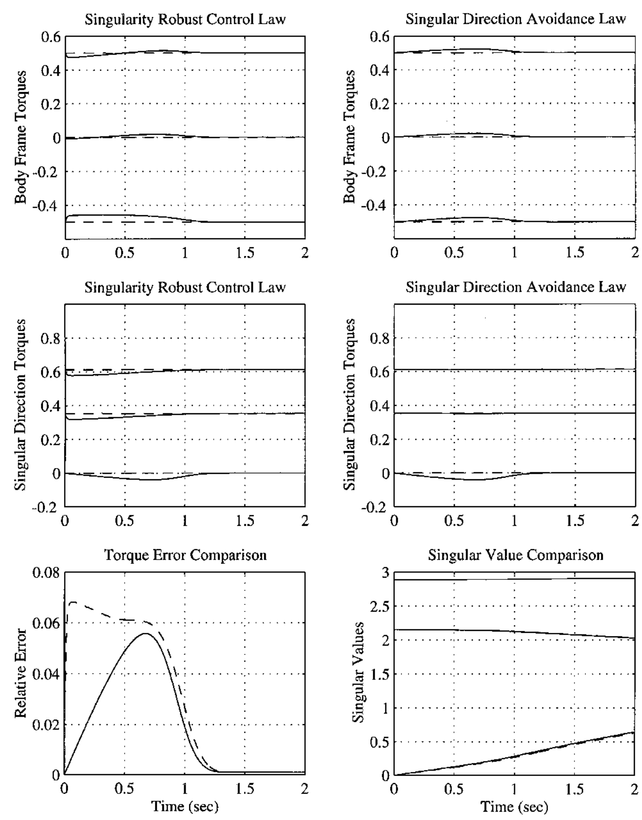

Fig. 3 Comparison of torque output near a singularity.

Singular Direction Avoidance Example

We begin by examining the advantage of singular direction avoidance only. For this example we choose a singular configuration of the CMGs that permits no torque output in the $[1, 1, 1]^T$ direction. Requesting a torque solely in this direction will result in no gimbal rate commands (and no motion). To show one advantage of the SDA law, we request the constant torque $\boldsymbol{L}_r = [0, -0.5, 0.5]^T$, which is perpendicular to the singular direction. We also begin at a set of gimbal rates that are providing the desired torque at the beginning of the maneuver. Initial conditions and control law gain parameters are shown in Table 2. Although this example is illustrative of the difference in the laws, it is somewhat artificial in that a singular condition is forced to exist at the start of the maneuver, a condition which is normally avoided during a reorientation.

Simulation results are shown in Fig. 3. The top row of plots shows the desired (dashed line) and produced (solid line) torques in the body reference frame for the two control laws. The second row of plots shows the same torques presented in a reference frame using the columns of $\boldsymbol{U}$ as basis vectors. Resolving components into this frame depicts how the control law affects the torque produced in the directions corresponding to the singular values. Whereas the SR law produces torque deviations in all axes, the SDA law produces zero error in the plane orthogonal to the singular direction.

The deviation in the direction of the singularity deserves explanation. As the simulation progresses, the gimbals do not remain in a singular configuration. The direction of the smallest singular value changes, and it is the torque in the direction of the smallest singular value, which is distorted because of the SDA control law. As the smallest singular value direction changes, $\boldsymbol{L}_r$ is no longer exactly in the plane orthogonal to this most singular direction, whereupon the pseudoinverse calculated using the SDA law begins to produce a deviation.

Figure 3 also presents a comparison of the magnitude of the torque error for both laws. It is evident that the norm of the error is reduced for the SDA law (solid line) when compared with the SR law (dashed line). The residual error for both cases is caused by the actual gimbal rates lagging the desired rates. It can be reduced by increasing $\boldsymbol{K}_\delta$. The singular value histories for both laws are also presented to illustrate the rate at which both control laws drive the gimbals away from the singularity.

Effect of the Singularity Avoidance Parameter

Figure 4 compares the torque produced (solid) and desired (dashed) for a reorientation maneuver started from a singular gimbal state. The simulation parameters are presented in Table 3. Case 1 uses the avoidance parameter based on Eq. (61), whereas case 2 uses Eq. (87) with $k_\sigma = 10$. Cases 1 and 2 use $\alpha_0 = 0.1$. Note the improvement in smoothness of the produced torque. The third column of plots presents the same reorientation using Eq. (87) and $\alpha_0 = 0.5$.

Table 3 Example 2 simulation data

Item	Value	Units
k	1.0	N-m
$\boldsymbol{K}$	diag{1, 1, 1, 1}	kg-m^2/s
$\boldsymbol{K}_\delta$	diag{1, 1, 1, 1}	s^{-1}
α_0, cases 1 and 2	0.1	kg^2-m^4/s^2
α_0, case 3	0.5	kg^2-m^4/s^2
$\boldsymbol{\delta}_0$	$[-90, 0, 90, 0]^T$	deg
$\boldsymbol{\delta}_0$	$[0, 0, 0, 0]^T$	deg/s
$\boldsymbol{\omega}_0$, $\boldsymbol{\omega}_f$	$[0, 0, 0]^T$	deg/s
$\boldsymbol{q}_0$	$[0.5, 0.5, 0.5, 0.5]^T$	——
$\boldsymbol{q}_f$	$[0, 0, 0, 1]^T$	——

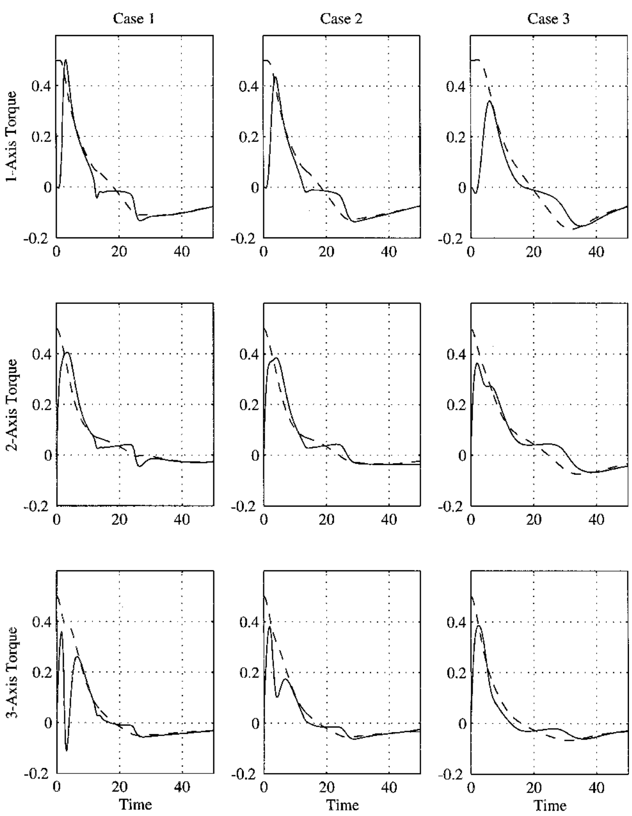

Fig. 4 Comparison of singularity avoidance parameters.

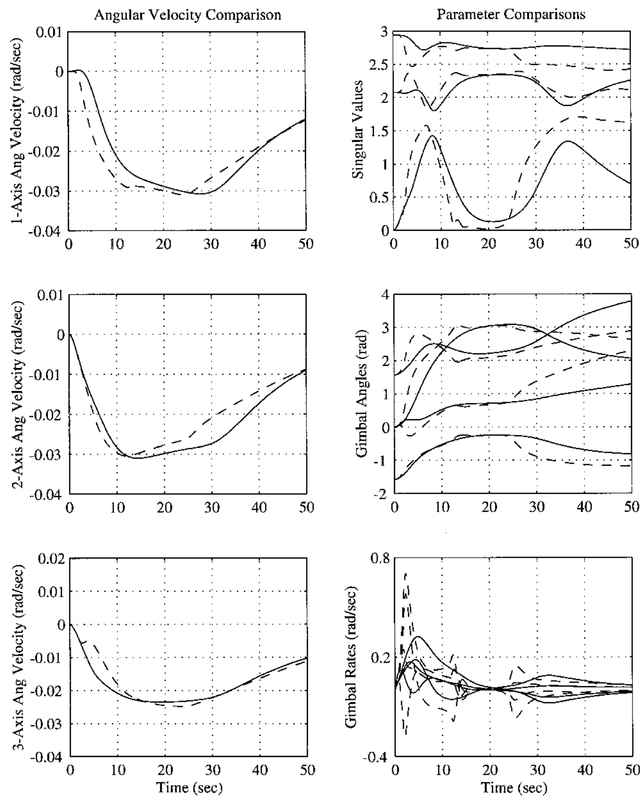

Fig. 5 Comparison of SR (- - -) and SDA case 3 (——) reorientation parameters.

The improvements evident in cases 2 and 3 can be explained by the smoother and shallower rise of the singularity avoidance parameter function as the singularity approaches.

Another way to gauge an improvement in the steering law is the smoothness of the angular velocity history. The frequency content in the angular velocity components, for example, gives some insight into the tendency for a reorientation to excite vibrations in flexible appendages.[14] An improvement here is evident from the angular velocity components presented in Fig. 5. Singular value and gimbal angle histories are also presented. The gimbal angle history is considerably smoother for the parameters of case 3. Although the quaternion history is not presented, they are similar and nearly identical at the 50-s point.

Conclusions

The development of a new form of the exact equations of motion for a rigid spacecraft with multiple CMGs leads to new insights, particularly in the relationship between the gimbal axis torques and the spacecraft attitude dynamics. The SVD was used to propose an SDA control law, which was shown to more closely produce the desired torque near singular configurations of the gimbal cluster. The use of a singular avoidance parameter based on the smallest singular value was used to show that a reorientation maneuver can be made smoother while in general not sacrificing the speed of the reorientation.

It seems likely that these qualitative improvements in the rigid body reorientation trajectories would be even more useful for the reorientation of a flexible spacecraft. A comparison of frequency content in the torque produced, or possibly even angular velocity profiles, might indicate the possibility of reduced oscillations from flexible appendages during reorientations.

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