Minimum-Time Orbital Phasing Maneuvers

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MINIMUM-TIME ORBITAL PHASING MANEUVERS
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The minimum-time, constant-thrust, orbital phasing maneuver is studied numerically. Non-dimensionalization reduces the problem to one where thrust magnitude and phase angle are the only parameters. Extremal solutions are obtained for the entire range of practical values of thrust magnitude and phase angle. Plots of trajectories, thrust profiles, and loci of initial costates are used to identify a near-invariance principle that leads to a variety of conclusions about this class of problems. The types of thrust profiles are shown to fall into at least four types, two of which exist within the region of near-invariance. These two thrust profile types are distinguished by the time-of-flight, \( t_f \), and the transition between them takes place for a \( t_f \) of approximately one-half of an orbit. The relationship between \( t_f \) and the ratio of thrust to phase angle is also shown to be nearly invariant over a wide range of thrust magnitudes.

INTRODUCTION

We investigate a special class of coplanar time-optimal orbital maneuvers, in which the spacecraft is controlled to move ahead of or behind its orbit position using a constant thrust whose direction is the control variable. This minimum-time orbital phasing maneuver is conceptually simple, but its solution has some interesting characteristics, including an approximate invariance relating the dimensionless thrust, phase angle, dimensionless time-of-flight, and thrust profile when the maneuver occurs in less than about one orbit.

The minimum-time, constant-thrust, orbit transfer problem is well-established as one of the fundamental problems in control of spacecraft trajectories.\(^1\)\(^-\)\(^3\) Most researchers find extremal solutions that solve the first-order, or necessary, conditions. A recently published procedure is now available for applying second-order conditions,\(^4\) but is not applied in this paper. Many applications of this problem involve low-thrust propulsion systems where the orbit transfer takes place over a relatively long duration. For example, the minimum-time transfer of a 100 kg spacecraft from low-Earth orbit (LEO) to geostationary orbit (GEO) using a 1 N thruster takes about 5 days, as compared with about 5 hours for a Hohmann transfer, but the continuous-thrust case would typically use much less propellant. The thrust angle profile of a continuous thrust orbit transfer depends on both the thrust magnitude and the “size” of the orbit transfer. For example, Ref. 5 showed that there are problems involving small orbit changes using low thrust that are similar to large-thrust problems with

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larger orbit changes. In Ref. 6 the effects of thrust magnitude on trajectory behavior were described in some detail.

The development in Refs. 5 and 7 led to the identification of three different regimes of minimum-time, constant-thrust, coplanar orbit transfer problems, along with a means of approximating the initial conditions of the Lagrange multipliers required to determine the optimal control history. In an application of these approximations, Ref. 8 used piecewise time-optimal transfers to investigate the effectiveness of using energy storage during eclipse for low-thrust electric propulsion systems.

In the present paper, we consider the minimum-time orbital phasing maneuver using constant thrust with the thrust angle as the control variable. Specifically, we pose and obtain solutions to the problem of moving a point mass spacecraft from one point in a given circular orbit to a different point in the same orbit, differing only by a phase angle \( \phi \). This problem is of course the same as the same-orbit rendezvous problem. However, our motivation is not rendezvous, but rather the formation-establishment and formation-keeping maneuvers associated with formation flying missions.\(^9,10\) We want to compute minimum-time solutions for comparison with nonlinear feedback controllers\(^11-13\) designed to support such missions.

We begin by defining the idealized model and stating the equations of motion. The equations are non-dimensionalized so that the dimensionless thrust, \( T \), and the phase angle, \( \phi \), are the only parameters in the problem. We then establish the minimum-time transfer problem which leads to a two-point boundary value problem requiring the determination of the unknown initial conditions for the Lagrange multipliers or costates. Of course, these solutions only satisfy the necessary conditions, and thus are strictly only extremal solutions. We present some example solutions intended to illustrate a certain near-invariance principle that is found within the various families of solutions for varying thrust and phase angle. Specifically, we show that, for a broad range of \( T \) and \( \phi \), extremal solutions with the same time-of-flight, \( t_f \), have the same thrust profile, and furthermore that the relationship between \( t_f \) and \( T/\phi \) is nearly thrust-independent. This near-invariance is the primary point of this communication and should be useful in developing further results relevant to this problem. A series of locus plots for the initial conditions of the costates is used to illustrate the near-invariance, and we identify a transition between large-\( t_f \) and small-\( t_f \) trajectories.

**MODEL AND EQUATIONS OF MOTION**

We make the idealized assumptions of a point-mass spacecraft moving in a plane about a spherical primary, and being controlled with a constant thrust with variable direction. Figures 1 and 2 illustrate the geometry and variables of the problem. In Fig. 1, the initial conditions are represented. The filled circle represents the position of the spacecraft at the initial time, \( t_0 \), and the empty circle represents the target position separated from the initial position by phase angle \( \phi \). A possible initial thrust vector direction is shown, with angle \( \psi(0) \) between the inertial \( x \) axis and the thrust vector direction, measured counterclockwise as shown. Additionally, we define the thrust angle relative to the local horizontal as \( \bar{\psi} \). In Fig. 2, an arbitrary intermediate configuration is shown, after time \( t - t_0 \). The uncontrolled spacecraft position is shown as a light shaded circle on the circular orbit, whereas the actual spacecraft position is shown as a darker shaded circle removed from the circular orbit. The target position and the uncontrolled spacecraft position have both advanced through an
angle $n(t - t_0)$, where $n$ is the mean motion of the circular orbit. We also indicate the thrust angle referenced to the local horizontal, denoted by $\vec{\psi} = \psi - \theta - \pi/2$. If we use polar coordinates, the distinction between $\psi$ and $\vec{\psi}$ can be eliminated; however we choose to use cartesian coordinates in order to simplify the extension of this work to three-dimensional cases.

![Figure 1: Optimal phasing maneuver geometry initial condition](image)

The equations of motion are those for Keplerian motion in the plane, with additional acceleration terms due to the thruster. We assume the spacecraft has constant mass for the results presented in the paper, and the results are qualitatively unchanged for small $\dot{m}$. Using cartesian coordinates in the inertial reference frame shown in Figs. 1 and 2, the equations of motion are

\begin{align*}
\dot{x}^* &= v_x^* \\
\dot{y}^* &= v_y^* \\
v_x^* &= -\frac{\mu^*}{r^*^3} x^* + \frac{T^*}{m^*} \cos \psi \\
v_y^* &= -\frac{\mu^*}{r^*^3} y^* + \frac{T^*}{m^*} \sin \psi
\end{align*}

where $x^*$ and $y^*$ are the components of the position vector in the orbital plane, $r^* = \sqrt{x^*^2 + y^*^2}$, $\mu^*$ is the gravitational parameter, $\psi$ is the thrust angle measured from an inertial $x$-axis, $T^*$ is the applied thrust, and $m^*$ is the spacecraft mass. The superscript * denotes that these are dimensional variables. Using dimensional variables requires that we consider the effects of varying orbit size, thrust magnitude, and spacecraft mass. Non-dimensionalizing the equations reduces the problem to one with only dimensionless thrust as a parameter.

Non-dimensionalization is based on canonical units,\textsuperscript{14} so that the reference orbit semi-major axis is 1 DU (Distance Unit), the reference orbit period is $2\pi$ TU (Time Units), and
the gravitational parameter is $\mu = 1 \text{DU}^3/\text{TU}^2$. The spacecraft mass, $m^*$, is taken as the dimensionless mass unit, and it follows that a dimensionless thrust can be defined by

$$T = \frac{T^*}{m^*a^*n^*}$$

(5)

where $a^*$ is the reference orbit semi-major axis and $n^*$ is the reference mean motion in dimensional units. Thus, for example, a 1 N thruster acting on a 100 kg spacecraft in a 10,000 km radius circular orbit provides a dimensionless thrust of about $T = 0.0025$. In results illustrated below, we use the range $[0.000005, 0.5]$ as a practical limit on the range of thrusts of interest for this application. For a 100 kg spacecraft, this range corresponds to approximately $[4 \text{ mN}, 400 \text{ N}]$ in LEO and $[100 \mu\text{N}, 10 \text{ N}]$ in GEO. Specific values of dimensional thrust for three different altitudes are given in Table 1. Even smaller thrusts may be of interest, but the near-invariance described below can be used to obtain results for smaller thrusts in a straightforward way. The expression for $T$ can also be written as

$$T = \left(\frac{T^*}{m^*}\right) / \left(\frac{\mu^*}{a^*n^*}\right)$$

(6)

so that $T$ can be interpreted as the ratio of “thrust acceleration” to gravitational acceleration.

The form of the dimensionless equations is identical to that of the dimensional equations, with the asterisks omitted, and both $\mu$ and $m$ set to unity. Thus the dimensionless states are $(x, y, v_x, v_y)$, and the only parameter remaining in the equations of motion is the dimensionless thrust magnitude, $T$. The other non-state variable in the equations of motion is the thrust angle $\psi$, which is the control variable to be determined.
Table 1: Dimensionless Thrust for 100 kg Spacecraft

\begin{tabular}{cccc}
\hline
 & LEO (500 km) & MEO (10,000 km) & GEO (36,000 km) \\
\hline
$T$ & $T^* \text{(Newtons)}$ & & \\
0.000005 & 0.0042 & 0.00074 & 0.00011 \\
0.00005 & 0.042 & 0.0074 & 0.0011 \\
0.005 & 0.42 & 0.074 & 0.11 \\
0.05 & 4.2 & 7.4 & 1.1 \\
0.5 & 42 & 74 & 11 \\
\hline
\end{tabular}

OPTIMAL CONTROL FORMULATION AND BOUNDARY VALUE PROBLEM

The minimum-time formulation is well-known and may be found for example in Refs. 1, 2, 3, and 5. The formulation leads to four additional differential equations describing variables variously known as costates or Lagrange multipliers. These four differential equations are:

\begin{align}
\dot{\lambda}_x &= \frac{\lambda_{v_x}}{r^3} - \frac{3x}{r^5} (x\lambda_{v_x} + y\lambda_{v_y}) \\
\dot{\lambda}_y &= \frac{\lambda_{v_y}}{r^3} - \frac{3y}{r^5} (x\lambda_{v_x} + y\lambda_{v_y}) \\
\dot{\lambda}_{v_x} &= -\lambda_x \\
\dot{\lambda}_{v_y} &= -\lambda_y
\end{align}

and they are to be integrated simultaneously with the state equations (1–4). Note that we have made use of the fact that $\mu = 1$ in the dimensionless system.

The thrust angle is found from the optimal control formulation to be

$$\psi = \tan^{-1} \frac{-\lambda_{v_y}}{-\lambda_{v_x}}$$

and this angle is used as the control in Eqs. (3–4).

In a typical scenario, we know the initial conditions on the states, and at least something about the final conditions on the states. However, we do not know the initial or final conditions on the costates. Thus we establish a two-point boundary value problem wherein the initial conditions on the costates are the unknowns.

For the phasing maneuver considered here, the initial conditions for the states are $x = 1$, $y = 0$, $v_x = 0$, and $v_y = 1$, corresponding to a circular orbit with radius 1 DU beginning on the $x$ axis. The final conditions that the states must satisfy include that the spacecraft must return to the same circular orbit: i.e., $x^2 + y^2 = 1$, $v_x^2 + v_y^2 = 1$, and $xv_x + yv_y = 0$. In addition to these three conditions, we also have the condition corresponding to completion of the phasing maneuver through angle $\phi$. Another way to formulate the final conditions on the states is

\begin{align}
x(t_f) &= \cos(\phi + t_f) \\
y(t_f) &= \sin(\phi + t_f)
\end{align}
Table 2: Four Example Cases

<table>
<thead>
<tr>
<th>Case No.</th>
<th>$T$</th>
<th>$\phi$</th>
<th>$\lambda_{v_x}(0)$</th>
<th>$\lambda_{v_y}(0)$</th>
<th>$t_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
<td>0.2101</td>
<td>0.6906</td>
<td>0.5000</td>
<td>3.6599</td>
</tr>
<tr>
<td>2</td>
<td>0.00005</td>
<td>0.00002246</td>
<td>0.5548</td>
<td>0.5000</td>
<td>3.8195</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>3.0</td>
<td>0.4531</td>
<td>0.8745</td>
<td>8.7342</td>
</tr>
<tr>
<td>4</td>
<td>0.00005</td>
<td>0.3</td>
<td>0.0113</td>
<td>0.9960</td>
<td>89.3266</td>
</tr>
</tbody>
</table>

$$v_x(t_f) = -\sin(\phi + t_f)$$  \hspace{1cm} (14)  
$$v_y(t_f) = \cos(\phi + t_f)$$  \hspace{1cm} (15)

where $t_f$ is the time-of-flight, which is also unknown. An added benefit of the non-dimensionalization is that $t_f$ is the angle the spacecraft moves through during the maneuver.

Because the variational Hamiltonian is linear in the Lagrange multipliers, we can arbitrarily fix the initial condition of one of the multipliers, and we select $\lambda_x(0) = \pm 1$. The choice of $\lambda_x(0) = +1$ leads to the correct solution for the case where $0 < \phi < \pi$, whereas $\lambda_x(0) = -1$ provides the correct solution for the case where $-\pi < \phi < 0$. This skew-symmetry in the structure of solutions is described in more detail below.

Thus the unknowns in the boundary value problem are $\lambda_y(0), \lambda_{v_x}(0), \lambda_{v_y}(0)$, and $t_f$, so the problem may be stated as

$$F(\lambda_y(0), \lambda_{v_x}(0), \lambda_{v_y}(0), t_f) = \begin{bmatrix} x(t_f) - \cos(\phi + t_f) \\ y(t_f) - \sin(\phi + t_f) \\ v_x(t_f) + \sin(\phi + t_f) \\ v_y(t_f) - \cos(\phi + t_f) \end{bmatrix} = 0 \hspace{1cm} (16)$$

with $T$ a fixed constant appearing in the equations of motion, which are numerically integrated from $t = 0$ to $t = t_f$ to compute the final states, and $\phi$ a fixed constant appearing in the final-time boundary conditions. This problem can be solved using Newton’s method or any other appropriate numerical method. All solutions presented in this paper were computed using Matlab’s `fsolve` function with a tolerance setting of $10^{-11}$. The stringent accuracy was used to ensure stability in the continuation approach used to obtain solutions for many combinations of $T$ and $\phi$.

EXAMPLE SOLUTIONS

In this section we present four examples of minimum-time trajectories. The examples illustrate the nature of solutions to the optimal control problem with varying thrust $T$ and phase angle $\phi$. For each example we provide plots of the trajectory and the thrust angle relative to the local horizontal. As noted in the previous section and illustrated in Fig. 1, all trajectories begin with initial conditions $x = 1$, $y = 0$, $v_x = 0$, and $v_y = 1$. In Table 2, we provide the parameters $(T, \phi)$ for each case, as well as some data specific to the solution for each case.

The first example illustrates a maneuver through a small phase angle, $\phi = 0.21$, using a large thrust, $T = 0.05$ (Figs. 3a and 3b). The trajectory (solid line) and the circular orbit (dashed line) are both shown in Fig. 3a, where it is evident that the trajectory is
completed in just over one-half of an orbital period ($t_f \approx 4$). Figure 3a also includes a sequence of line segments indicating the thrust direction along the trajectory. Of the four costates, only the pair ($\lambda_{v_x}, \lambda_{v_y}$) is used in computing the thrust angle $\psi$, using Eq. (11). While the angle $\psi$ is useful in solving the boundary value problem and in plotting the thrust direction as in Fig. 3a, it is not particularly useful to plot the angle itself because of its interaction with the spacecraft’s position in the inertial frame. Therefore, we plot the thrust angle referenced to the local horizontal, denoted by $\overline{\psi}$, as indicated in Figs. 1 and 2. The thrust angle history is shown in Fig. 3b. Note that the initial thrust angle is in the second quadrant, which has the effect of lowering the orbit, thereby increasing the orbital speed so that the spacecraft “catches up” with the target. At about the halfway point, the thrust vector switches direction so that the orbit is raised to achieve the rendezvous. The thrust profile is easily seen in both of Figs. 3a and 3b.

Figure 3: Trajectory and thrust angle for Cases 1 ($T = 0.05, \phi = 0.21$) and 2 ($T = 0.000005, \phi = 0.000022$).

Case 2 illustrates a small phase angle maneuver using a small thrust (Figs. 3c and 3d). This case is interesting because of its similarity to Case 1, even though the $(T, \phi)$ parameters differ by four orders of magnitude: $(0.05, 0.21)$ vs. $(0.000005, 0.000022)$. The two trajectories shown in Figs. 3a and 3c are similar in that the thrust profiles and times of flight are nearly the same, although the lower thrust in Case 2 leads to a smaller change in radius during the maneuver. Note the similarities between the thrust angle histories, as seen in Figs. 3b and 3d. The costate histories (not shown here) exhibit the same equivalence. In both cases, the thrust begins in a rearward, inward direction, and swings through $\overline{\psi} = 0$ at about
the midway point, and thus includes a brief period of outward thrusting. The similarities between Cases 1 and 2 are evident in many other cases and we develop this idea further in the sequel.

The third case displays quite different behavior from Cases 1 and 2 (Figs. 4a and 4b). Case 3 uses the same thrust as in Case 1 ($T = 0.05$), but the maneuver is through a larger phase angle ($\phi = 3.0$). The maneuver takes almost two orbits. Figure 4b illustrates the thrust profile for this maneuver, which exhibits an interesting “porpoising” behavior.

![Thrust profile for Case 3.](image)

Figure 4: Trajectory and thrust angle for Cases 3 ($T = 0.05, \phi = 3.0$) and 4 ($T = 0.00005, \phi = 0.3$).

The final case illustrated in full is Case 4, where a relatively large phase angle maneuver, $\phi = 0.3$, is effected using a small thrust, $T = 0.00005$ (Figs. 4c and 4d). Here the thrust is nearly tangential throughout the maneuver, with the first half of the maneuver using orbit-lowering to “catch up” with the target, and the second half raising the orbit with a nearly tangential thrust in the velocity direction. The thrust vectors shown in Fig. 4c appear to be mostly inwards, but this misperception is because most of the thrusting is tangential, except for the period of roughly 10 TUs (about 1.5 orbits) at the halfway point where the thrust changes directions, as seen more readily in Fig. 4d. The tangential thrust vectors do not show up in Fig. 4c because they overlap the trajectory. This maneuver takes about 15 orbits to complete.

The four example cases presented in Figs. 3–4 and Table 2 are intended to illustrate the near-invariance that is discussed in more detail below, as well as to illustrate the different
types of trajectories that may occur for this problem. There are four basic types of trajectories that occur as extremal solutions to the minimum-time phasing maneuver, three of which are illustrated in these four cases: i) initial inward, rearward thrusting with a swing through $\psi = 0$ near the trajectory midpoint, as in Cases 1 and 2; ii) multiple-revolution trajectories with thrust angle $\psi$ passing through zero multiple times, as in Case 3; and iii) multiple-revolution trajectories with thrust angle $\psi$ rearward tangential for most of the maneuver, with a swing through $\psi = 90^\circ$ near the midpoint and forward tangential thrusting for the remainder of the trajectory, as in Case 4. The fourth type of trajectory is illustrated in the sequel.

**LOCUS OF COSTATE INITIAL CONDITIONS**

Once a solution for a given $(T, \phi)$ pair has been obtained, we obtain solutions for additional cases using numerical continuation. As an example, we illustrate a locus of the initial conditions of $(\lambda_{v_x}, \lambda_{v_y})$ in Fig. 5. The diagram shows how the initial costate locus can be interpreted as the initial thrust angle. That is, we juxtapose the costate locus and the cartesian coordinate frame so that the origin of the locus coincides with the initial spacecraft position. Then the vector from the initial costate to the spacecraft position defines the initial thrust vector direction. Initial costates with $\lambda_{v_y} > 0$ correspond to initially thrusting inward and rearward, whereas $\lambda_{v_y} < 0$ corresponds to initially thrusting inward and forward. In the diagram we show a segment of a typical locus plot, for a fixed thrust $T$, with $\phi$ varying on the interval $[\phi^-, \phi^+]$. For the two end points, we show how to construct the initial thrust vectors $\vec{T}^-$ and $\vec{T}^+$ and the associated initial thrust angles $\psi^-(0)$ and $\psi^+(0)$. Obviously, since $\phi^+ > \phi^-$, one expects the $t_f$ for $\phi^+$ to be greater than that for $\phi^-$, which is indeed the case. Thus, as one moves from $\phi^-$ to $\phi^+$ along the locus, $t_f$ increases, and, at least for the range shown, $\psi(0)$ increases. Each point on the locus corresponds to a specific initial thrust angle, as well as to a specific time-of-flight, $t_f$. Thus, we can identify each point on the locus with the mapping

$$(T, \phi) \mapsto (\lambda_{v_x}, \lambda_{v_y}, \psi, t_f)$$

We present a set of locus plots illustrating the entire range of reasonable values of $(T, \phi)$ and develop the near-invariance that arises in the mapping defined by Eq. (17).

We also note the skew-symmetry in the costate locus related to the sign of the phase angle. If $\phi \in (0, \pi)$, then the locus of initial costates is in the right-half of the $(\lambda_{v_x}, \lambda_{v_y})$ plane, and the choice of $\lambda_x(0) = 1$ (as described following Eqs. (12–15)) is correct. If $\phi \in (-\pi, 0)$, then the locus of initial costates is in the left-half of the plane, and the choice $\lambda_x(0) = -1$ is correct. Furthermore, $\phi \mapsto -\phi$ (within these ranges) corresponds to $(\lambda_{v_x}, \lambda_{v_y}) \mapsto (-\lambda_{v_x}, -\lambda_{v_y})$.

We use continuation to obtain solutions for $\phi \in (0, \pi)$ and $T \in [0.000005, 0.5]$, and use the results to construct locus plots of the initial conditions of $(\lambda_{v_x}, \lambda_{v_y})$. Note that for smaller values of $T$, we do not obtain solutions for larger values of $\phi$ because of the extreme times-of-flight required, and for larger values of $T$, we do not obtain solutions for smaller values of $\phi$ because of the extreme sensitivity. We also do not compute solutions for $\phi > \pi$ because one would normally choose the "short-way" solution, taking advantage of the skew-symmetry of the costate locus described above.
Figure 5: Example locus of costates illustrating thrust angle

Figure 6 presents the locus of initial costates for several values of $T \in [0.000005, 0.05]$. In each of these graphs, the initial values of $(\lambda_{v_x}, \lambda_{v_y})$ are plotted for a fixed value of $T$, while varying the phase angle $\phi$. Beginning in the lower half of one of these graphs (in the vicinity of $(0, -1)$) corresponds to a small phase angle, and the “end” of the graph (in the vicinity of the point (0,1)) corresponds to the largest value of $\phi$ for that graph. For the larger thrusts, the endpoint corresponds to $\phi = \pi$, whereas for the smaller thrust value of $T = 0.0005$, the endpoint corresponds to a value of $\phi = 2.25 < \pi$, and has a time-of-flight of 76.9597 TUs (about 12 orbits). The points labelled A, B, C, and D in the locus plots are described in detail below, and correspond to the data in Table 3.

As described above, one can view each point on each of these three graphs as corresponding to a specific thrust and phase angle, $(T, \phi)$, to a specific costate history, $\lambda(t)$, and to a specific time-of-flight, $t_f$. Several properties of the costate loci that are presented in Fig. 6 are of interest:

1. Each locus begins near the point $(0, -1)$ corresponding to tangential thrust in the direction of the velocity vector. This observation is consistent with the idea that using a large thrust to effect a small phase angle change would begin the maneuver by thrusting towards the destination, then switching thrust direction to decelerate to the desired state. This observation is true for small $T$ as well, but corresponds to a smaller phase angle than in the large $T$ cases. We refer to this region of the locus as the small-$t_f$ region; generally, increasing thrust or decreasing the phase angle corresponds to decreasing $t_f$ and moving along the locus towards $(0, -1)$.

2. Each locus passes close to the point $(0.73, 0)$, corresponding to thrust in the nadir
Figure 6: Locus of costates for $T \in [0.000005, 0.05]$, $\phi \in (0, \pi)$

direction. This point can be thought of as a transition point between the typical small-$t_f$ and large-$t_f$ scenarios. We refer to this region of the locus as the transition region; here the time-of-flight is about one-half of an orbit. A more precise transition point is identified below.

3. Each locus moves in a “looping” pattern towards the point $(0, 1)$ corresponding to tangential thrust in the direction opposite the velocity vector. This observation is consistent with the idea that using a small thrust to effect a large phase angle change would begin the maneuver by thrusting “backwards” to lower the orbit and “catch up” with the target. We refer to this region as the large-$t_f$ region; generally decreasing $T$ or increasing $\phi$ leads to an increase in $t_f$ and initial costates closer to the point $(0, 1)$.

4. The locus graphs are nearly identical in the lower phase angle portions of the graphs before the “looping” pattern begins. For smaller values of thrust, the locus plots are indistinguishable except in the fine structure of the “loops” near $(0, 1)$.

5. Not immediately evident in the graphs, but supported by the data in Table 3 and by subsequent plots, is the fact that the thrust profile and time-of-flight for a point in one of the locus plots is nearly identical to the thrust profiles and times-of-flight for the corresponding points on the other locus plots.

These properties of the locus graphs can be used to draw some more general conclusions about the minimum-time trajectories. In the next section, we develop the near-invariance principal using these locus graphs.

DISCUSSION OF THE NEAR-INVARIANC

The near-invariance is most valid for small thrust, so in the discussion that follows, we omit the $T = 0.5$ case. However, one can easily refer to Table 3 to compare the $T = 0.5$ case to the rest of the cases that are described in more detail.

On each of the locus graphs, we indicate four points labelled A, B, C, and D, that share the same initial values of $\lambda_{vy}$. That is, the points labelled A all have $\lambda_{vy} = 1.0$, and so forth,
as detailed in Table 3. The table also includes data for two smaller values of $T$, 0.00005 and 0.000005. The loci for these smaller thrusts are included in Fig. 6, where horizontal lines indicate the four points selected for comparison. These points are of interest because they help to illustrate the near-invariance that we wish to emphasize. Since the invariance is least accurate in the large-$t_f$ region, we begin by discussing the points A on the three locus graphs and the composite locus graph in Fig. 6.

In the $T = 0.05$ graph in Fig. 6, the point A, selected for $\lambda_{y}(0) = 1$, corresponds to $(T, \phi) = (0.05, 0.89)$. The initial costates are given in Table 3, along with $t_f$, which is approximately 5.6, corresponding to completing the maneuver in a little less than one orbit ($t_f = 2\pi$ corresponds to a one-orbit transfer). In the $T = 0.005$ graph, the thrust is one order of magnitude smaller (by design), as is the phase angle ($0.89 \approx 10 \times 0.10$, by observation), and the solution has similar values for the other two initial costates and for the time-of-flight (5.6 $\approx$ 6.2). In the $T = 0.0005$ graph, the phase angle corresponding to point A is one order of magnitude smaller than for $T = 0.005$, with two significant digits of agreement, and the time-of-flight is identical to two significant digits. This trend continues for the two smaller values of thrust, with the additional costates and time-of-flight agreeing exactly to within two significant digits, and the phase angle decreasing by the same order of magnitude as $T$. Note that we only include two significant digits in Table 3 specifically to highlight the near-invariance. We compute all solutions with a convergence tolerance
of $10^{-11}$; however, the data presented in the table are sufficient to achieve convergence in about four iterations in most cases. As thrust decreases, the near-invariance becomes more pronounced, and as thrust increases, the near-invariance becomes less significant. However, the time-of-flight for all five values of thrust is in agreement to one significant digit, and for the three smaller values of thrust, $t_f = 6.3$ to two significant digits. For points B, C, and D, the near-invariance is even more pronounced.

Since the initial costates are nearly the same for all the points labelled A, the initial thrust angle is also nearly the same for all of these points. Furthermore, the thrust profiles for the entire maneuver are nearly the same for all five points. The thrust profiles, $\bar{\psi}(t)$, are illustrated in Fig. 7 along with those for points B, C, and D. In the figure, the initial values, $\bar{\psi}(0)$, are all essentially identical, and although the thrust profiles are not all identical, they are quite similar. The high thrust case, $T = 0.05$, produces the profile that is most different from the others, with a significant difference near the mid-point, and with a shorter time-of-flight. However, the other four cases lead to almost identical thrust profiles for the entire maneuver, in spite of the orders-of-magnitude difference between the thrust values. These observations are increasingly true for the smaller times-of-flight for points B, C, and D.

The points labelled B in Fig. 6 are chosen for $\lambda_{\psi_B} = 0.5$. As shown in Table 3, $T \propto \phi$ for all five values of $T$. The thrust profiles are shown in Fig. 7, and are similar to those in Fig. 7. The thrust angle $\bar{\psi}$ begins as an inward, backward thrust, and the angle decreases, passing through zero and swinging around through $360^\circ$ to end with an inward, forward thrust. These thrust profiles are similar to those of points A, and include the two thrust profiles of Cases 1 and 2 illustrated in Figs. 3. These thrust profiles are characteristic of a subset of the large-$t_f$ set of trajectories. The near-invariance of the thrust profile and time-of-flight is more noticeable for points B than for points A. The maneuvers all take a
little more than half an orbit.

The points labelled C are chosen for $\lambda_{v_y} = 0$, which corresponds to initial thrust in the nadir direction. Again, $T \propto \phi$ evident from the data in Table 3. The thrust profiles, shown in Fig. 7, are substantially different from those of points A and B. The initial thrust is in the nadir direction, and the thrust angle initially decreases, but does not pass through zero as in the large-$t_f$ cases. Rather $\bar{v}$ reaches a minimum of about $25^\circ$, then rapidly increases to about $155^\circ$ before decreasing again to its final value of about $90^\circ$. These thrust profiles are characteristic of all of the small-$t_f$ trajectories. Furthermore, for some point between B and C, there is a transition between the large-$t_f$ and small-$t_f$ type trajectories, and we identify this transition behavior below.

The points labelled D are chosen for $\lambda_{v_y} = -0.5$, which corresponds to initial thrust in an inward, forward direction. The linear relationship between $T$ and $\phi$ evident from the data in Table 3. The thrust profiles, shown in Fig. 7, are similar to those for points C. These maneuvers all take about a quarter orbit to complete.

Comparing the thrust profiles associated with points A and B (large-$t_f$) with those of points C and D (small-$t_f$), it appears that a transition point exists between the two types of profiles. Using bisection, we find that the transition between small-$t_f$ and large-$t_f$ trajectories occurs for $t_f \approx 3.3$. Figure 8 illustrates the thrust profiles for points along the locus with $\lambda_{v_y}(0) \in \{0.34, 0.36, 0.37, 0.38\}$, showing that the transition occurs for slightly different values of $t_f$ for the different values of thrust. All four plots use the same limits to clarify the differences and similarities between the two types of profiles. In particular, note that except for the direction of the $180^\circ$ swing that occurs in each type of solution, the thrust profiles are essentially identical. In principle, we could add a line in Fig. 6 at $\lambda_{v_y} \approx 0.37$ to indicate this transition family of trajectories.

The final observation we make is to characterize the $t_f$ invariance that has been described above. In Fig. 9, we plot curves of constant $t_f$ in the $(T, \phi)$ plane, with $T \in (5 \times 10^{-6}, 4)$ and $\phi \in (0, 2\pi)$. The thin lines represent solutions for $t_f \in \{\pi/2, \pi, 3\pi/2, ..., 4\pi\}$. The thick lines use the four values of $t_f$ associated with $T = 0.000005$ for the points A, B, C, and D, and the specific A, B, C, and D points are identified on the plot. Even though the plot is a log-log plot, the linearity of $T$ vs. $\phi$ is remarkable, even across the small-$t_f$/large-$t_f$ transition. Only in the large-$T$/large-$\phi$ region does the linearity begin to break down. The A, B, C, D points clearly lie along the constant $t_f$ curves for most of the points shown. The curves are nearly linear until $\phi$ approaches $\pi$, and there are some interesting variations in the $\phi > \pi$ region, especially for larger thrust. We do not investigate these $\phi > \pi$ cases further, as one would normally use the $\phi - 2\pi$ trajectory and the skew-symmetry described earlier in this paper.

We further characterize the relationship between $T$, $\phi$, and $t_f$ by plotting $t_f$ vs. $T/\phi$ for the full range of these parameters presented in earlier plots and tables. The plot in Fig. 10 is a log-log plot of the time-of-flight vs. the ratio of thrust to phase angle for each of the different values of thrust (including $T = 0.5$). The thrust-to-phase angle ratio ranges between $10^{-4}$ and $10^4$, with resulting times-of-flight ranging between $10^{-2}$ and $10^2$. For most of this range, the graphs are linear, and are essentially identical for all the values of thrust. However, in the region where $T/\phi \in (0.01, 1)$, there is a nonlinear transition, which is simple for most of the values of $T$; for large $T$ (0.5), however, the graph exhibits a more complicated behavior. In Fig. 11, we plot $t_f$ vs. $T/\phi$ on a linear graph in the range
\( T/\phi \in [0, 2], \) which corresponds to the range where the nonlinear transition is evident in Fig. 10. The complicated relationship between \( t_f \) and \( T/\phi \) for \( T = 0.5 \) is evident here, whereas the relationship between \( t_f \) and \( T/\phi \) is quite simple for the smaller values of \( T \), which includes essentially all values of \( T \) of interest (cf. Table 1). Note that the transitional region in Figs. 10 and 11, where the plots of \( t_f \) vs. \( T/\phi \) are not nearly identical, begins near the transition time-of-flight, \( t_f \approx 3.3 \), identified in Fig. 8.

Finally, we note that the invariance illustrated here could be interpreted as simply the linearity associated with small perturbations from the desired position, and indeed we expect that a careful comparison with solutions of the linearized equations of motion would provide consistent results in the same regions where the near-invariance occurs. However, the fact that these results are obtained from the nonlinear equations of motion with a broad range of thrust and phase angles provides a deeper insight into this entire class of transfer problems.

Three questions that remain unanswered here are: i) are these extremal solutions actually optimal? ii) how do these solutions compare with time-optimal solutions derived from linearized equations of motion? and iii) what additional structure exists in the \( t_f > 2\pi \) region of the parameter space? We expect to address these questions in a subsequent communication.

**CONCLUSIONS**

The extremal solutions to the minimum-time, constant-thrust phasing maneuver problem have some interesting properties that have not previously been reported. Using the
equations of motion in a dimensionless form permits the investigation of essentially all problems of interest in one setting, with dimensionless thrust $T$ and phase angle $\phi$ as the only parameters. The initial costates have a uniform structure that can be exploited when seeking solutions for problems with a specific thrust and phase. There is a near-invariance property which relates the dimensionless time of flight, $t_f$ to the thrust and phase: for a wide range of parameter values, a nearly linear relationship exists between the thrust and phase for a given time of flight. This relationship persists even in the transition region that distinguishes maneuvers with only “inward” thrusting from those with some “outward” thrusting. The results presented here provide a thorough foundation for future studies of continuous thrust problems of this type.

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REFERENCES


Figure 10: Relationship between $t_f$ and $T/\phi$


Figure 11: Relationship between $t_f$ and $T/\phi$ in transition region


