Attitude Dynamics of Orbiting Gyrostats

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Abstract. Equilibrium attitudes of a rigid satellite with N rotors in a central gravitational field are investigated. The equations of motion are written as a noncanonical Hamiltonian system, where the Hamiltonian includes the potential, a volume integral over the body of the gyrostat. In practice, the Hamiltonian is approximated to partially decouple the position and attitude equations. The equilibria of this system of equations represent the steady motions of the body as seen in the body frame, and correspond to stationary points of the Hamiltonian constrained by the Casimir functions. This defines an algorithm for computing equilibria. In contrast to other approaches, this algorithm provides stability information directly, since the calculations required to solve the constrained minimization problem are also involved in computing the positive definiteness of the Hamiltonian as a Lyapunov function.

Keywords: attitude dynamics, gyrostat, Hamiltonian

1. Introduction

In this paper we study a subset of the equilibrium attitudes of a rigid satellite with N rotors in a central gravitational field. The work presented herein is an extension of similar results for a rigid body (Beck and Hall, 1998), with the added complexity of the flywheels or rotors representing reaction or momentum wheels. The model of a rigid body with axisymmetric wheels is termed a gyrostat. As a result of numerous studies [*e.g.*, (Volterra, 1899; Krishnaprasad and Berenstein, 1984)], the global torque-free motion of a gyrostat is understood in cases with freely spinning rotors or with rotors constrained to spin at a constant speed relative to the platform. The basic results are well-known (Hughes, 1986).

There are also many reports relevant to orbiting gyrostats, where the gravity gradient torque is included (Kane and Mingori, 1965; Roberson and Sarychev, 1985; Anchev, 1973). These papers characterize the relative equilibrium motions of gyrostats in circular orbits; as a result, the steady motions of orbiting gyrostats, the subject of this paper, are fairly well understood. The main results are typically given as a set of special cases (Roberson and Sarychev, 1985; Hughes, 1986). Note that the gravitational moment used in all these studies is obtained by truncating the gravitational potential in a way that has been shown to be inconsistent (Wang et al., 1991). The significance of the inconsistency has been shown to be negligible for "ordinary" asymmetric, rigid, gravity-gradient spacecraft (Beck and Hall, 1998). The gyrostat case has been investigated and some stability criteria have been obtained for a variety of cases (Wang et al., 1995). However, their analysis was based on the constant-speed rotor case, and is not directly applicable to the problem of performing rotational maneuvers.

Most papers parameterize a rotor's motion by its constant angular velocity relative to the body. However, the rotor's absolute angular velocity about its spin axis is more important in developing control torques to perform rotational maneuvers. The problem of performing rotational maneuvers using flywheels has been investigated by numerous authors, and a brief literature review has been presented by the present author (Hall, 1995). Only one article (Anchev, 1973) has used information about equilibrium motions to develop reorientation control laws.

Here we develop the general equations of motion for an N-rotor gyrostat in a central gravitational field. We specialize the equations of motion to the problem of a Keplerian circular orbit, and develop a new noncanonical Hamiltonian formulation similar to that developed for a rigid body (Beck and Hall, 1998). This formulation is equivalent to the equations that have been used by others to study equilibria of orbiting gyrostats, but has the advantage that standard methods can be used to obtain equilibria and to characterize their stability. We then develop the stability criteria for the simplest case of the cylindrical equilibria.

2. System and Equations of Motion

The system model is a gyrostat ($\mathcal{G}$): a rigid body ($\mathcal{B}$) with N axisymmetric flywheels ($\mathcal{R}_j$), with spin axes fixed in the body frame, $\mathcal{F}_b$ (see Fig. 1). The system inertia tensor is $\mathbf{I}$, expressed in $\mathcal{F}_b$. The wheels have axial moments of inertia, I_{sj}, $j = 1, ..., N$, which are collected into a diagonal matrix $\mathbf{I}_s = \mathrm{diag}[I_{s1}...I_{sN}]$, and their spin axes are defined by the vectors $\mathbf{a}_j$, $j = 1, ..., N$, expressed in $\mathcal{F}_b$, and collected into a $3 \times N$ matrix as $\mathbf{A} = [\mathbf{a}_1...\mathbf{a}_N]$. The inertia-like matrix $\mathbf{J} = \mathbf{I} - \mathbf{A}\mathbf{I}_s\mathbf{A}^\mathsf{T}$ is symmetric and positive definite.

The gyrostat rotational equations of motion may be written as

$$\dot{\mathbf{h}} = \mathbf{h}^\times \mathbf{J}^{-1}(\mathbf{h} - \mathbf{A}\mathbf{h}_a) + \mathbf{g}_e \quad (1)$$

$$\dot{\mathbf{h}}_a = \mathbf{g}_a \quad (2)$$

where $\mathbf{h}$ is the 3×1 angular momentum vector, $\mathbf{h}_a$ is the $N \times 1$ vector of the absolute axial momenta of the wheels, "$\times$" denotes the skew-symmetric matrix form of a vector, $\mathbf{g}_e$ is the 3×1 vector of external

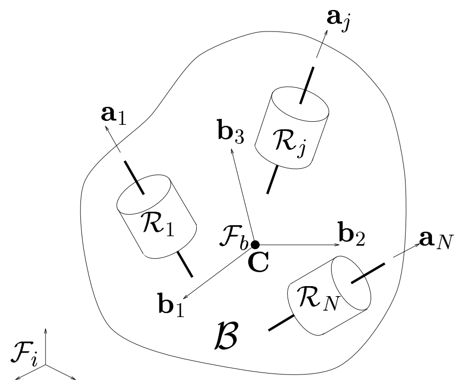

Figure 1. Gyrostat $\mathcal{G}$ with body $\mathcal{B}$ and N momentum wheels $\mathcal{R}_j$

torques, and $\mathbf{g}_a$ is the $N \times 1$ matrix containing the axial torques applied by the platform on the rotors. The term $\mathbf{J}^{-1}(\mathbf{h} - \mathbf{A}\mathbf{h}_a)$ is recognized as the angular velocity $\boldsymbol{\omega}$ of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$. To study equilibrium motions, we set $\mathbf{g}_a = \mathbf{0}$, and to study rotational maneuvers, we choose a suitable control law for the rotor torques. In this paper, we deal exclusively with the $\mathbf{g}_a = \mathbf{0}$ case.

The dynamics of Eqs. (1–2) must be augmented if the external or internal torques depend on the position or orientation of the gyrostat. For example, the gravity gradient torque is (Hughes, 1986):

$$\mathbf{g}_e = \mathbf{r}^{\times} \nabla_{\mathbf{r}} V(\mathbf{r}), \quad V(\mathbf{r}) = -\int_{\mathcal{G}} \frac{\mu}{\|\mathbf{r} + \boldsymbol{\rho}\|} \, dm \tag{3}$$

and the force acting on the body is $-\nabla_{\mathbf{r}} V(\mathbf{r})$. Here μ is the gravitational constant for the central body, $\mathbf{r}$ is the position vector from the center of attraction to the gyrostat mass center, and $\boldsymbol{\rho}$ is the position vector from the mass center to a mass element dm. This integral depends on the orientation of the gyrostat. A standard approximation, assuming $\|\boldsymbol{\rho}\| \ll \|\mathbf{r}\|$ is

$$\mathbf{g}_e = 3 \frac{\mu}{\|\mathbf{r}\|^3} \mathbf{o}_3^{\times} \mathbf{I} \mathbf{o}_3 \tag{4}$$

where $\mathbf{o}_3$ is the nadir vector; *i.e.* $\mathbf{o}_3 = -\mathbf{r}/\|\mathbf{r}\|$ (see Fig. 2). Here $\mathbf{o}_3$ denotes the third column of the rotation matrix $\mathbf{R}^{bo}$ that takes vectors from the orbital frame, $\mathcal{F}_o$, to $\mathcal{F}_b$. The orbital frame's remaining unit vectors are arranged so that $\mathbf{o}_2$ is in the negative orbit normal direction, and $\mathbf{o}_1 = \mathbf{o}_2^{\times} \mathbf{o}_3$. For circular orbits, $\mathbf{o}_1$ is in the velocity direction. Furthermore, for circular orbits, the term $\mu/\|\mathbf{r}\|^3$ is constant, and is denoted by ω_c^2. In the circular case, we append $\dot{\mathbf{o}}_3 = \mathbf{o}_3^{\times} \mathbf{J}^{-1}(\mathbf{h} - \mathbf{A}\mathbf{h}_a)$ so that the current state of $\mathbf{o}_3$ is available for computing the gravity gradient torque; otherwise, the translational equations of motion are required to describe the variable radius vector $\mathbf{r}$.

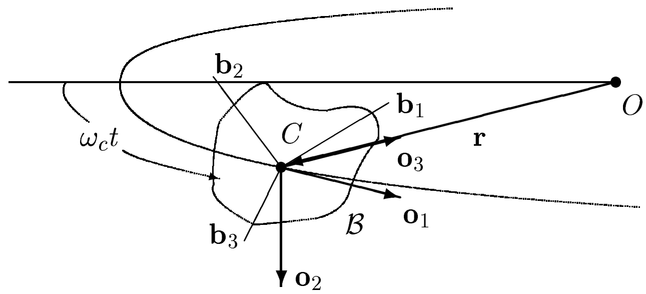

Figure 2. Configuration of Keplerian Orbit and Orbital Frame

The full equations can be recognized as a noncanonical Hamiltonian system (Maddocks, 1991; Beck and Hall, 1998):

$$\dot{\mathbf{z}} = \mathcal{J}(\mathbf{z})\nabla H(\mathbf{z}) \tag{5}$$

where $\mathbf{z}$ is the vector of states, $\mathcal{J}(\mathbf{z})$ is the skew-symmetric Poisson tensor or structure matrix, $H(\mathbf{z})$ is the Hamiltonian, and ∇ represents the gradient of H with respect to $\mathbf{z}$.

In systems of the form of Eq. (5), there are special first integrals, known as Casimir functions, whose gradients span the nullspace of the structure matrix, $\mathcal{J}(\mathbf{z})$. For the general equations of motion for an orbiting gyrostat, the nullspace of $\mathcal{J}(\mathbf{z})$ is one-dimensional and is spanned by the vector ∇C, where C is the total angular momentum about the center of the attracting body. This is the only Casimir-type first integral for this system. A second integral of the motion is the Hamiltonian, which because of the skew symmetry of the structure matrix, is easily seen to be constant:

$$\dot{H} = \nabla H^{\mathsf{T}}\dot{\mathbf{z}} = \nabla H^{\mathsf{T}}\mathcal{J}(\mathbf{z})\nabla H = 0$$

Thus the system of equations admits two first integrals. If the internal wheel torques, $\mathbf{g}_a$, are not all zero, then Eq. (2) must be used with the general equations. In this case, the Casimir function is still a first integral, since it is independent of the particular form of the Hamiltonian; however, the Hamiltonian is not conserved, but satisfies the following differential equation:

$$\dot{H} = [\partial H/\partial \mathbf{h}_a]\,\dot{\mathbf{h}}_a = -\mathbf{h}^{\mathsf{T}}\mathbf{J}^{-1}\mathbf{A}\mathbf{g}_a \tag{6}$$

Elsewhere, we have used this relationship with the method of averaging to reduce the "spinup" problem of torque-free gyrostats from five dimensions to two (Hall, 1995).

The general form of the equations of motion for a gyrostat in a central gravitational field is usually approximated in some way. Three types of approximations are usually necessary to obtain useful results (Beck and Hall, 1998): a $3 \times 3 \times \infty$ "matrix" of approximations is possible, where the three dimensions are approximation of the potential, restriction of the mass center motion, and material symmetry of the body. Here we study the second-order potential approximation for an arbitrary body moving in a circular Keplerian orbit.

3. Second-Order Keplerian Approximation

We assume the gyrostat is in a circular Keplerian orbit, and approximate the potential with a second-order expansion, obtaining results that are equivalent to work previously reported. However, our approach is distinct from the classical approach in that we develop a new noncanonical formulation similar to that developed for the rigid body problem (Beck and Hall, 1998). We define the relative angular velocity and angular momentum of the gyrostat with respect to the rotating orbital reference frame, $\mathcal{F}_o$:

$$\boldsymbol{\omega}_r = \boldsymbol{\omega} + \omega_c \mathbf{o}_2 = \mathbf{J}^{-1}(\mathbf{h}_r - \mathbf{A}\mathbf{h}_a) \tag{7}$$

$$\mathbf{h}_r = \mathbf{h} + \omega_c \mathbf{J}\mathbf{o}_2 = \mathbf{J}\boldsymbol{\omega}_r + \mathbf{A}\mathbf{h}_a \tag{8}$$

We also use a second-order approximation of the potential:

$$V_2(\mathbf{o}_3) = \frac{3}{2}\omega_c^2 \mathbf{o}_3{}^{\mathsf{T}} \mathbf{I} \mathbf{o}_3 \tag{9}$$

Using these definitions, and the mass, length, and time scales

$$m = \overline{m} = \int_{\mathcal{G}} d\overline{m} \qquad l = \left(\frac{\operatorname{tr}\overline{\mathbf{J}}}{\overline{m}}\right)^{\frac{1}{2}} \qquad t = \overline{\omega_c}^{-1}. \tag{10}$$

the dimensionless equations of motion become

$$\begin{bmatrix} \dot{\mathbf{h}}_r \\ \dot{\mathbf{o}}_2 \\ \dot{\mathbf{o}}_3 \end{bmatrix} = \begin{bmatrix} [\mathbf{h}_r + \{(\mathbf{1} - 2\mathbf{J})\mathbf{o}_2\}]^\times & \mathbf{o}_2^\times & \mathbf{o}_3^\times \\ \mathbf{o}_2^\times & \mathbf{0} & \mathbf{0} \\ \mathbf{o}_3^\times & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{J}^{-1}(\mathbf{h}_r - \mathbf{A}\mathbf{h}_a) \\ -\mathbf{J}\mathbf{o}_2 + \mathbf{A}\mathbf{h}_a \\ 3\mathbf{I}\mathbf{o}_3 \end{bmatrix} \tag{11}$$

This system is in the form of Eq. (5), with $\mathbf{z} = (\mathbf{h}_r, \mathbf{o}_2, \mathbf{o}_3)^{\mathsf{T}}$, and

$$H = \frac{1}{2}\mathbf{h}_r{}^{\mathsf{T}}\mathbf{J}^{-1}\mathbf{h}_r - \mathbf{h}_r{}^{\mathsf{T}}\mathbf{J}^{-1}\mathbf{A}\mathbf{h}_a - \frac{1}{2}\mathbf{o}_2{}^{\mathsf{T}}\mathbf{J}\mathbf{o}_2 + \mathbf{o}_2{}^{\mathsf{T}}\mathbf{A}\mathbf{h}_a + V(\mathbf{o}_3) \tag{12}$$

This system admits three Casimir functions, as the nullspace of the structure matrix is spanned by the three vectors:

$$\mathcal{N}[\mathcal{J}(\mathbf{z})] = \operatorname{span}\left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{o}_2 \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{o}_3 \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{o}_3 \\ \mathbf{o}_2 \end{pmatrix} \right\} \tag{13}$$

From the spanning vectors, we identify three independent Casimir functions:

$$C_1(\mathbf{z}) = \mathbf{o}_2{}^\mathsf{T}\mathbf{o}_2 \qquad C_2(\mathbf{z}) = \mathbf{o}_3{}^\mathsf{T}\mathbf{o}_3 \qquad C_3(\mathbf{z}) = \mathbf{o}_2{}^\mathsf{T}\mathbf{o}_3 \tag{14}$$

In this problem, all three Casimir functions are trivial since $\mathbf{o}_2$ and $\mathbf{o}_3$ are columns of the rotation matrix $\mathbf{R}^{bo}$ so that

$$\mathbf{o}_2{}^\mathsf{T}\mathbf{o}_2 \equiv 1 \qquad \mathbf{o}_3{}^\mathsf{T}\mathbf{o}_3 \equiv 1 \qquad \mathbf{o}_2{}^\mathsf{T}\mathbf{o}_3 \equiv 0 \tag{15}$$

The Hamiltonian is also constant (if $\mathbf{g}_a = \mathbf{0}$). We thus have a ninth-order system with four known first integrals.

4. Equilibria

In canonical Hamiltonian systems, equilibria are found as the critical points of the Hamiltonian; *i.e.*, by setting $\nabla H = \mathbf{0}$, and computing $\mathbf{q}_e$ and $\mathbf{p}_e$. In the noncanonical case, the structure matrix can be singular, so that equilibria can also satisfy $\nabla H \in \mathcal{N}[\mathcal{J}(\mathbf{z})]$. Since the gradients of the Casimir functions lie in $\mathcal{N}[\mathcal{J}(\mathbf{z})]$, equilibria may be expressed as the critical points of a "variational Lagrangian:"

$$F(\mathbf{z},\boldsymbol{\mu}) = H(\mathbf{z}) - \mu_1 (C_1(\mathbf{z}) - 1) - \mu_2 (C_2(\mathbf{z}) - 1) - \mu_3 C_3(\mathbf{z}) \tag{16}$$

subject to the constraints that the Casimir functions are constant. For the system represented by Eqs. (11), setting $\nabla F = \mathbf{0}$ leads to a nonlinear algebraic system with 12 unknowns: $(\mathbf{z},\boldsymbol{\mu}) = (\mathbf{h}_r,\ \mathbf{o}_2,\ \mathbf{o}_3,\ \mu_1,\ \mu_2,\ \mu_3$.

A typical problem involves fixing the wheel momenta $\mathbf{h}_a$, and computing the associated equilibria. Using Newton's method (Seydel, 1988) requires the Hessian $\nabla^2 F(\mathbf{z}_e,\boldsymbol{\mu}_e)$. The full 12×12 Hessian is required for numerical computation of equilibria, but as we will see below, the Hessian also plays a role in computing the stability of equilibria, and only the upper left 9×9 block associated with the states is required for the stability calculations.

5. Linearization, Linear Stability, and Nonlinear Stability

There are two approaches to computing stability in Hamiltonian systems (Beck and Hall, 1998): linear or spectral stability, and nonlinear stability. The former is based on linearizing the equations of motion about equilibrium, and the latter is based on establishing a suitable Lyapunov function.

The linearization of Eq. (5) about an equilibrium, $\mathbf{z}_e$ leads to

$$\mathbf{A}_e(\mathbf{z}_e) = \mathcal{J}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e) \tag{17}$$

Thus, one checks the linear stability of an equilibrium by computing the eigenvalues of $\mathbf{A}_e(\mathbf{z}_e)$. Because the eigenvalues of a Hamiltonian system occur in pairs that are symmetric about both the real and imaginary axes, this approach only provides conditions for instability.

Since the Hamiltonian and Casimir functions are constants, the variational Lagrangian, $F(\mathbf{z})$, is a candidate Lyapunov function. Thus, the eigenvalues of $\nabla^2 F(\mathbf{z}_e)$ can be used to determine nonlinear stability.

Considering only the upper left 9×9 block of $\nabla^2 F$, one finds that the matrix is block diagonal. The upper left 3×3 block, $\mathbf{J}^{-1}$, is positive definite. If the eigenvalues of the remaining 6×6 block are positive, then the equilibrium is stable. The Lagrange multipliers appear in these terms, and are important in determining stability.

Possibly $\nabla^2 F$ is indefinite and hence F is not useful as a Lyapunov function. However, the equilibrium may be viewed as a constrained extremum of the Hamiltonian subject to the constant values of the C_i (Beck and Hall, 1998). We introduce the orthogonal projection matrix, $\mathbf{P}(\mathbf{z})$ onto the range of $\mathbf{A}(\mathbf{z})$ as follows. Define

$$\mathbf{K}(\mathbf{z}) = [\nabla C_1(\mathbf{z}) \;\; \nabla C_2(\mathbf{z}) \;\; \nabla C_3(\mathbf{z})] \tag{18}$$

and let $\mathbf{Q}(\mathbf{z})$ be the projection onto $\mathcal{N}\left[\mathbf{A}^{\mathsf{T}}(\mathbf{z})\right]$:

$$\mathbf{Q}(\mathbf{z}) = \mathbf{K}(\mathbf{z}) \left(\mathbf{K}^{\mathsf{T}}(\mathbf{z})\mathbf{K}(\mathbf{z})\right)^{-1} \mathbf{K}^{\mathsf{T}}(\mathbf{z}) \tag{19}$$

Then the desired projection operator is

$$\mathbf{P}(\mathbf{z}) = \mathbf{1} - \mathbf{Q}(\mathbf{z}) \tag{20}$$

The projected Hessian is then given by $\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)$. This matrix has three zero eigenvalues associated with the nullspace of $\mathbf{A}(\mathbf{z}_e)$, and with the three Casimir functions. If its remaining eigenvalues are all positive, then the equilibrium is a constrained minimum and the relative equilibrium is nonlinearly stable.

6. An Example: Cylindrical Equilibria

Most presentations consider the equilibria of orbiting gyrostats as a variety of cases, including cylindrical, conical, hyperbolic, and offset hyperbolic. We examine the existence of and stability of the cylindrical case, with a single wheel aligned with the $\mathbf{b}_2$ axis. Thus $\mathbf{I}$ and $\mathbf{J}$ are diagonal, and the diagonal elements of $\mathbf{J}$ are $\{I_1, I_2 - I_s, I_3\}$. Using the principal reference frame, the standard gravity gradient equilibria have $\mathcal{F}_b$ aligned with $\mathcal{F}_o$, whence $\boldsymbol{\omega}_r = \mathbf{0}$. We assume without loss of generality that $\mathbf{o}_2 = [0,1,0]^\mathsf{T}$ and $\mathbf{o}_3 = [0,0,1]^\mathsf{T}$. The equilibrium conditions lead to restrictions on the first and third elements of the wheel momenta in the body frame:

$$[\mathbf{A}\mathbf{h}_a]_1 = 0 \qquad [\mathbf{A}\mathbf{h}_a]_3 = 0 \tag{21}$$

The second element, $[\mathbf{A}\mathbf{h}_a]_2 = h_{r2}$, is arbitrary, but affects the stability of the steady attitude. The Lagrange multipliers are

$$\mu_1 = -J_2 + h_{r2} \qquad \mu_2 = 3I_3 \qquad \mu_3 = 0 \tag{22}$$

Thus the 9×9 block of $\nabla^2 F$ simplifies to

$$\nabla^2 F = \begin{bmatrix} \mathbf{J}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{J} - \mu_1 \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 3\mathbf{I} - \mu_2 \mathbf{1} \end{bmatrix} \tag{23}$$

which is diagonal, but is not positive definite, since the last diagonal element is zero. Using the values of the Lagrange multipliers from Eq. (22), the Hessian has a zero eigenvalue, 3 positive eigenvalues $(1/I_1, 1/(I_2 - I_s), 1/I_3)$, two eigenvalues that place constraints on the moments of inertia $(3(I_1 - I_3), 3(I_2 - I_3))$, and 3 eigenvalues that involve the rotor momentum $(J_2 - I_1 - h_{r2}, -h_{r2}, J_2 - I_3 - h_{r2})$. Introducing the Smelt parameters $k_1 = (I_2 - I_3)/I_1$ and $k_3 = (I_1 - I_2)/I_3$, these conditions can be written as $k_1 > 0$, $k_1 > k_3$, and

$$\tfrac{(1-k_1)k_3}{3-k_3-k_1(1+k_3)} - h \;>\; 0 \quad \text{and} \quad \tfrac{(1-k_3)k_1}{3-k_3-k_1(1+k_3)} - h \;>\; 0 \tag{24}$$

where $h = I_s + h_{r2}$. When $h = 0$, the third of these four conditions is equivalent to $k_3 > 0$, and the fourth is equivalent to $k_1 > k_3$. When $h \neq 0$, the third condition becomes $k_3 > h(3 - k_1)/(1 - k_1 + h(1 + k_1))$, and the fourth becomes $k_3 > (h(3 - k_1) - k_1)/(h(1 - k_1) - k_1)$. Notice that the well-known DeBra-Delp region is not predicted by these stability conditions. The criteria provided here are based on using F as a Lyapunov function, and are therefore sufficient conditions only. (However, since $\nabla^2 F$ is only positive semidefinite, strictly speaking, we cannot even make these weak conclusions.)

The projected Hessian, $\mathbf{P}\nabla^2 F\mathbf{P}$ yields sharper stability criteria. For the single-rotor, cylindrical equilibria case, the important eigenvalues of the projected Hessian are:

$$\{I_2 - I_1 - h, 2(I_2 - I_3) - h/2, 3(I_1 - I_3)\} \tag{25}$$

There are three zero eigenvalues associated with the Casimir functions, and 3 eigenvalues that are always positive. If three stated eigenvalues are positive, then the equilibrium is stable (this is a sufficient condition). This leads to the following sufficient conditions: $k_1 > k_3$, $(k_3(1-k_1))/(3-k_3-k_1(1+k_3)) - h > 0$, and $(4k_1(1-k_3))/(3-k_3-k_1(1+k_3)) - h > 0$. As stated above, the eigenvalues of the projected Hessian give sharper stability conditions than those of the Hessian. The corresponding stability regions are shown in Fig. 3. The principal benefit is that the $k_1 > 0$ condition is replaced with a criterion that provides an additional region of nonlinear stability. Furthermore, the conditions derived from the projected Hessian permit stability even in the case of $h_{r2} > 0$, which immediately leads to a negative eigenvalue of the unprojected Hessian. The stability region is shown in Fig. 3, with the unshaded region indicating the region for which the projected Hessian guarantees nonlinear stability. The stable region in the lower left quadrant is the additional region of stability provided by the sharper conditions obtained with the projected Hessian.

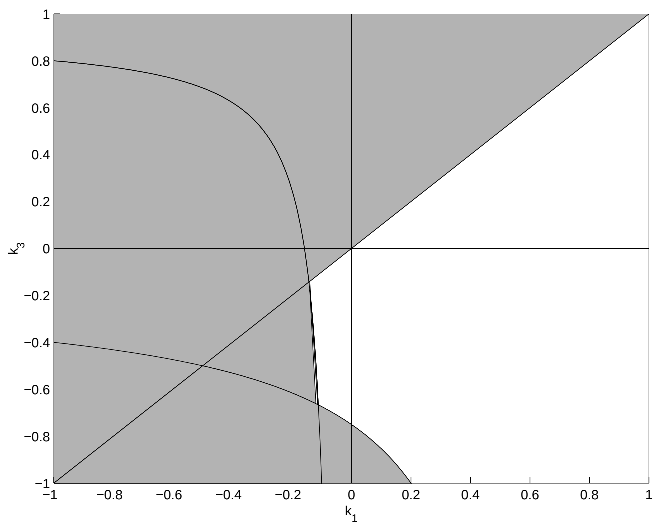

Figure 3. Smelt Parameter Plane Using Projected Hessian, with $h = -0.2$.

7. Conclusions

A noncanonical Hamiltonian formulation of the equations of motion for orbiting gyrostats leads to straightforward algorithms for computing relative equilibria and determining their stability. Results that have been obtained previously using a variety of manipulations of the equations of motion and their conserved quantities are obtained in a more straightforward fashion when the equations are put into a noncanonical form. Although we have treated only the simplest case of gyrostat equilibria in this paper, this approach should provide the means to unify the various cases that are usually treated separately. Our goal is to apply this approach to unify the treatment of both rigid body and gyrostat relative equilibria.

Acknowledgements

This work was supported by Arje Nachman of the Air Force Office of Scientific Research and Alison Flatau of the National Science Foundation.

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