DYNAMICS AND CONTROL OF TETHERED SATELLITE SYSTEMS
FOR NASA’S SPECS MISSION

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The control problems of two different configurations of Tethered Satellite Systems (TSS) for NASA’s Submillimeter Probe of the Evolution of Cosmic Structure (SPECS) mission are studied and their control performances are compared. The configuration of main focus is the TetraStar model comprised of three controlled spacecraft and three uncontrolled counterweights. This system is compared to a triangular TSS consisting of three controlled spacecraft. The equations of motion are derived using Lagrange’s equations. Several mission scenarios for the SPECS mission considering the operation of an infrared telescope are introduced and asymptotic tracking laws based on Lyapunov control are developed.

INTRODUCTION

Over the past three decades, a variety of concepts have been proposed for space exploration using Tethered Satellite Systems (TSS). These include scientific experiments in the microgravity environment, upper atmospheric research, cargo transfer between orbiting bodies, generation of electricity, and deep space observation.1–3 Numerous missions have already been launched to verify the tethered system concept for space application. Important milestones include retrieval of a tether in space (TSS-1, 1992), successful deployment of a 20 km tether in space (SEDS-1, 1993), closed-loop control of a tether deployment (SEDS-2, 1994), and operation of an electrodynamic tether used in both power and thrust mode (PMG, 1993).4,5 The idea of interconnecting spacecraft by means of lightweight deployable tethers has been proven to be particularly attractive also for space observations for various reasons. Variable-baseline interferometric observations can be achieved by a carefully controlled deployment and retrieval procedure of the tethers. In addition, the observational plane can be densely covered by spinning the tethered system. Lastly, the high levels of propellant consumption demanded by separated spacecraft in formation can be significantly reduced by tension or length control of the interconnecting tethers.

The paper is organized as follows. We begin with a discussion of some relevant papers and articles on the dynamics of TSS and adaptive control of mechanical systems. In the following section we present a brief description of the science mission which motivated this study and introduce two particular satellite formations. Subsequently, the system model is introduced and the motion equations are derived. Relative equilibria are identified and mission scenarios are presented followed by a thorough discussion on control law development. Finally, the performance of the controllers are validated considering the different mission scenarios.

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LITERATURE REVIEW

The problem of dynamics and control of tethered satellite formations has attracted considerable attention over the past decade. Of particular interest in the context of this paper are publications by DeCou, Keshmiri and Misra, Misra et al. and Farquhar. For a thorough literature review we refer to Ref. 11.

Nonlinear adaptive control of dynamical systems has emerged as an increasingly important approach to nonlinear controller design in recent years. Major breakthroughs in adaptive output–feedback control were achieved amongst others by Marino and Tomei whose work triggered a remarkable development in the following years. In Ref. 12 the authors managed to remove the structural restriction that the nonlinearities in the output were not allowed to precede the input. A more general class of systems was considered in their companion paper where the system was not required to be linear with respect to the unknown parameter vector. The latter results were only obtained, however, for set–point regulation problems. In a more recent paper Marino and Tomei extended the results obtained in Ref. 12 for a class of systems with time–varying parameters. The paper by Kristić and Kokotović refined the control approaches developed by Marino and Tomei using such concepts as “tuning functions” and “swapping–based” schemes to allow for incorporation of any standard gradient update law and to deal with overparameterization. Khalil considered the problem of adaptive output–feedback control of single–input–single–output systems which can be nonlinearly dependent on the control input. However, only semiglobal stability could be achieved. Furthermore, a–priori knowledge of bounds on parameters and on initial conditions was required and, more importantly, persistence of system excitation was sufficient for both parameter and tracking error convergence. Loria presented one of the first papers on global output–feedback control for one–degree–of–freedom Euler–Lagrange systems. The control design exploited the properties of hyperbolic functions to define a “nonlinear approximate differentiation filter” to automatically enlarge the domain of attraction. Ref. 18 uses a similar approach to achieve global stability for general nth order uncertain systems and is the major reference for the control approach used here.

THE SPECS MISSION CONCEPT

NASA’s future Earth and Space science missions involve formation flying of multiple coordinated spacecraft. Several space science missions include distributed instruments, large phased arrays of lightweight reflectors, and long variable baseline space interferometers. An array of collectors and combiner spacecraft will form variable–baseline space interferometers for a variety of science missions such as the Submillimeter Probe of the Evolution of Cosmic Structure (SPECS). This particular mission concept was initiated by a NASA science team and proposes a 1 km baseline submillimeter interferometer (λ ≈ 40 – 500 µm). It comprises possibly as few as three 3 – 4 meter diameter mirrors and rotates about the primary optical axis collecting (3 – 0.25 eV) photons which are then preprocessed by a central beam collector. Since operating such systems in any kind of Earth orbit is not feasible due to extensive fuel consumption and unsatisfactory photon yield, the second Lagrangian point in the Sun–Earth system L2 was chosen as the operational environment. The SPECS spacecraft formation is intended to be placed in a Halo orbit about this libration point.

Possible satellite formations for SPECS

The two tethered satellite configuration we compare in this paper are a triangular system and a more complex formation called TetraStar. The latter system consists of a total of three controlled spacecraft and three uncontrolled counterweights. Figures 1(a–c) show a schematic layout of TetraStar during tether deployment. The three inner point masses denote the controlled spacecraft
and form what we refer to as the “inner” system, the outer point masses (“outer” system) mark the uncontrolled counterweights. Additionally, the three tethers interconnecting the controllable spacecraft are extensible whereas the remaining six tethers are assumed to be of fixed length. Note that the triangular configuration can be obtained from the TetraStar model by removing the outer point masses and the corresponding tethers.

![Diagram](image)

Figure 1: Deployment of the TetraStar system.

From a controls point of view TetraStar offers significant benefits over the triangular TSS. As we showed in an earlier work,\(^{11}\) the overall control effort for a triangular TSS can be divided into two parts. The first and major contribution to the total control impulse is required in angular direction and regulates the angular momentum of the TSS according to the desired trajectory. The second control part is a radial control component which significantly contributes to the overall control effort only during times when the TSS is in a transitional phase between steady-spin and tether deployment or retrieval and vice versa. The TetraStar model can improve the control performance on both counts. Firstly, the three counterweights in combination with the corresponding tethers provide additional control authority in the radial direction for \(\rho(t) < \pi/3\). In Figure 1(a) these additional force components are identified by full arrows, the broken arrow denotes a centripetal–Coriolis acceleration component. Secondly and more importantly, if dimensioned correctly these counterweights can act as a buffer to balance the increase and decrease in angular momentum during tether deployment and retrieval which is due to the controlled spacecraft.

**SYSTEM MODEL AND EQUATIONS OF MOTION**

In this section equations of motions (EOM) of the system are formulated using the system description developed in Ref. 22. The mechanical system considered is shown in Figure 2 with TetraStar. It is comprised of a system of \(n\) point masses interconnected arbitrarily by \(m\) idealized tethers. The tethers are assumed to be massless and extensible capable of exerting force only along the straight–line connecting the respective masses. Also, the tethers do not support compression or any components of shear forces or bending moments and are therefore assumed to be perfectly flexible. The constitutive character for the tethers is taken as visco–elastic, allowing intrinsic energy dissipation. Since the system is ultimately being operated at the second Lagrangian point \(L_2\) in the Sun–Earth system, gravitational and other environmental forces are assumed to be negligible, but are readily incorporated into the model if necessary.
The EOM are derived using Lagrange’s equations defined in a “prescribed motion” reference frame $\mathcal{F}_P$ rather than in an inertial reference frame $\mathcal{F}_I$. Cylindrical coordinates relative to $\mathcal{F}_P$ are chosen to describe the position of the point masses. As illustrated in Figure 2, $r_i$, $\theta_i$, and $z_i$ are the radial distance in the $e_x e_y$ plane, the angle between the $e_x$–axis and the projection of the position vector $q_i$ onto the $e_x e_y$ plane, and the axial distance in $e_z$ direction of the point mass $m_i$, respectively. Note that during an observation the TSS rotates in the so–called $uv$ plane which is defined as the plane perpendicular to the target direction. The motion equations for the controlled system are

$$M_i \ddot{q}_i = Q_i^{(e)} + N_i + S_i + U_i, \quad i = 1, 2, \ldots, n$$

where

$$M_i = m_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_i^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad q_i = \begin{pmatrix} r_i \\ \theta_i \\ z_i \end{pmatrix}, \quad Q_i^{(e)} = \begin{pmatrix} F_{ri} \\ r_i F_{ti} \\ F_{zi} \end{pmatrix}, \quad S_i = Q_i^{(i)}$$

In equations (1) $M_i$, $q_i$, $Q_i^{(e)}$, $N_i$, $S_i$, and $U_i$ are the mass matrix, the vector of generalized coordinates, and the vectors of external, coupling, spring (or internal), and control forces for the $i$th point mass, respectively. The spring forces in equations (1) are obtained assuming a Kelvin–Voigt constitutive law; that is, the tensile stress, $\tau_{ij}$, and the longitudinal strain, $\epsilon_{ij}$, are related by

$$\tau_{ij} = E_{ij} (\epsilon_{ij} + a_{ij} \dot{\epsilon}_{ij}) \quad i, j = 1, 2, \ldots, n \text{ and } i \neq j$$

where $E_{ij}$ is the effective Young’s modulus and $a_{ij}$ is a constant dissipation parameter for the tether connecting point masses $m_i$ and $m_j$. The longitudinal strain in the tether is defined as

$$\epsilon_{ij} = \frac{l_{ij} - d_{ij}}{d_{ij}}$$
where $l_{ij}$ and $d_{ij}$ are the actual and the unstrained tether length, respectively. Hence, if $A_{ij}$ is the effective cross sectional area of the tether then the tension is given by

$$f_{ij} = \begin{cases} A_{ij} \tau_{ij} & \text{if } \epsilon_{ij} \geq 0 \text{ and } \tau_{ij} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

which introduces discontinuities since the tethers do not support compression. Defining the vector of generalized coordinate vectors for the system as

$$q = (q_1^T, q_2^T, \ldots, q_{n-1}^T, q_n^T)^T$$

equations (1) can be combined to yield

$$M \ddot{q} = Q^{(e)} + N + S + U$$

where

$$M = \begin{bmatrix} M_1 & 0_3 & \cdots & 0_3 \\ 0_3 & M_2 & \cdots & 0_3 \\ \vdots & \vdots & \ddots & \vdots \\ 0_3 & 0_3 & \cdots & M_n \end{bmatrix}, \quad Q^{(e)} = \begin{bmatrix} Q_1^{(e)} \\ Q_2^{(e)} \\ \vdots \\ Q_n^{(e)} \end{bmatrix}, \quad N = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{bmatrix}, \quad S = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{bmatrix}, \quad U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}$$

where $0_3$ is the $3 \times 3$ null matrix. Notice that equations (7) are readily integrated after being transformed to a set of $2 \times 3n$ first order ODEs; that is, defining $x = (q, \dot{q})^T$, equations (7) can be written in compact form as

$$\dot{x} = f(x), \quad \text{where} \quad f(x) = \left( M^{-1}(Q^{(e)} + N + S + U) \right)$$

For the subsequent analysis we consider two different TSS configurations. The triangular system comprises $n = 3$ point masses and $m = 3$ tethers whereas the TetraStar system consists of $n = 6$ point masses and $m = 9$ tethers. In any case, we are only interested in symmetrical TSS. That is, for the triangular system all the point masses are equal ($m_i = m$). The same applies to all of the tether parameters. For TetraStar all point masses and tethers belonging to the “outer” system have similar characteristics and so do the point masses and tethers of the “inner” system. For the sake of convenience we introduce the following notation: subscripts $\circ$ and $\bullet$ are used to denote physical quantities such as states and parameters corresponding to the inner and outer system, respectively. For example for TetraStar:

$$m_i = \begin{cases} m_0 & \text{for } i = 1, 2, 3 \\ m_\bullet & \text{for } i = 4, 5, 6 \end{cases}$$

Figure 3 shows a schematic layout of the labelling of the tethers and point masses for the two satellite configurations. The “inner” subsystems are marked by a hatched border.

In the following section we identify relative equilibria for the triangular system and for TetraStar.

**RELATIVE EQUILIBRIA**

Reviewing Gates’ system description we note that the spring forces appearing in equations (1) can be written as

$$S_i = \sum_{j=1}^{n-2} \sum_{j \neq i}^{n-1} \left( \frac{\sigma_{ij}}{l_{ij}} \right) \left( \frac{\tau_{ij}}{l_{ij}} \right)$$

Figure 3 shows a schematic layout of the labelling of the tethers and point masses for the two satellite configurations. The “inner” subsystems are marked by a hatched border.

In the following section we identify relative equilibria for the triangular system and for TetraStar.
where we have introduced the quantities
\[
\begin{align*}
\sigma_{ij} &= r_j \cos(\theta_j - \theta_i) - r_i \\
\tau_{ij} &= r_i r_j \sin(\theta_j - \theta_i) \\
\zeta_{ij} &= z_j - z_i
\end{align*}
\] (12)

Additionally, in the rotating reference frame \(\mathcal{F}_p\) the relative equilibrium motion satisfies
\[
\dot{x}^e = f(x^e) = 0
\] (13)

where the superscript denotes the relative equilibrium state. Assuming perfectly symmetrical TSS configurations in the case of the triangular system it is sufficient to analyze either of the subsystems described by equations (1). For TetraStar two subsystems have to be solved simultaneously, one of which corresponds to an inner point mass. The second subsystem corresponds to one of the two corresponding outer point masses.

**Relative equilibria for a triangular system**

Solving equation (13) for \(x^e\) we find three conditions for each of the three point masses of the form
\[
\begin{align*}
r^e_i &= \text{const.} \\
\dot{\theta}^e_i(r^e_i) &= \sqrt{3 \mathcal{A}_0(\sqrt{3} \tau^e_i - d_i)} \\
z^e_i &= \text{const.}
\end{align*}
\] (14)

\[
\begin{align*}
\dot{\theta}^e_i(r^e_i) &= \sqrt{\frac{3 \mathcal{A}_0(\sqrt{3} \tau^e_i - d_i)}{m_0 r^e_i d_i}} \\
\text{note: } l^e_i &= \sqrt{3} r^e_i
\end{align*}
\] (15)

where we have introduced the variable \(\mathcal{A}_0 = A_0 E_0\). Therefore, the relative equilibrium angular velocity for the TSS is
\[
\dot{\theta}^e(\xi) = \sqrt{\frac{3 \mathcal{A}_0(\xi - d_i)}{m_0 \xi d_i}}
\] (17)

which shows that the rotating TSS has to be under tension (\(l^e_i > d_i\)).

**Relative equilibria for TetraStar**

In the case of TetraStar conditions for a relative equilibrium are obtained effortlessly using d’Alembert’s principle or the principle of virtual work.
These conditions are
\[ r_o^c = \text{const.} \]
\[ r_\star^c = \text{const.} \] (18)
\[ m_o (\dot{\theta}_o^c)^2 r_o^c = \sqrt{3} \frac{\mathcal{A}_o (l_o^c - d_o^c)}{d_o^c} - (r_o^c - 2r_\star^c) \frac{\mathcal{A}_o (l_\star^c - d_\star^c)}{l_\star^c} \] (19)
\[ m_\star (\dot{\theta}_\star^c)^2 r_\star^c = (2r_\star^c - r_o^c) \frac{\mathcal{A}_\star (l_\star^c - d_\star^c)}{l_\star^c} \] (20)
\[ z_\star^c = z_o^c = \text{const.} \] (21)
Note that for the symmetrical TetraStar system \( \dot{\theta}_o^c = \dot{\theta}_\star^c = \dot{\theta}_c \). Equation (21) can be rewritten to yield a quartic equation in \( r_\star^c \) of the following form
\[ (r_\star^c)^4 + (r_\star^c)^3 r_o^c \left( \frac{2\beta}{\alpha} - 1 \right) + (r_\star^c)^2 \left\{ \frac{r_o^c}{\alpha} \left( \frac{2\gamma}{\alpha} \right)^2 - \left( \frac{2\gamma}{\alpha} \right)^2 \right\} + \]
\[ r_o^c \left\{ \frac{r_o^c}{\alpha} \left( \frac{2\gamma}{\alpha} \right) + r_o^c \left( \frac{2\gamma}{\alpha} \right)^2 \right\} + (r_o^c)^2 \left\{ \frac{r_o^c}{\alpha} \left( \frac{2\gamma}{\alpha} \right) - \left( \frac{\gamma}{\alpha} \right)^2 \right\} = 0 \] (23)
where
\[ \alpha = m_\star (\dot{\theta}_c)^2 d_\star - 2\mathcal{A}_\star, \quad \beta = \mathcal{A}_\star, \quad \gamma = \mathcal{A}_o d_o \] (24)
For known tether parameters and a given fixed length of the outer tethers \( d_\star \), equation (23) can be solved for \( r_\star^c \) in terms of \( r_o^c \) using a Newton–Raphson method. Equations (20) and (21) can then be used to solve for the only remaining unknown – the unstretched tether length of the inner tethers \( d_o^c \)– which yields
\[ d_o^c = r_o^c \frac{3\mathcal{A}_o}{(\dot{\theta}_c)^2 \left( m_o r_o^c + m_\star r_\star^c \frac{r_\star^c - 2r_o^c}{2r_\star^c - r_o^c} \right) + \sqrt{3} \mathcal{A}_o} \] (25)
Note that in the limit as \( \mathcal{A}_o \to \infty \) (extremely stiff tethers, rigid TSS) \( d_o^c \to l_o^c = \sqrt{3} r_o^c \), as expected.

Figures 4 and 5 show uncontrolled initial conditions perturbation responses for the triangular system and for TetraStar. Unperturbed initial conditions result in relative equilibrium motion. Additionally, in Figure 5 we show the effect of tether damping for \( a_{ij} = 0.5 \) s. Comparing the two system responses we point out that coordinate errors measured relative to the unperturbed system motions are notably greater for TetraStar, especially in angular direction. The system response signals can be thought of as a superposition of three signal components which are best explained with the upper two plots of Figure 5. The high–frequency oscillations reflect tether vibrations which can be significantly reduced by increasing the dissipation parameters \( a_{ij} \). Notice, that the radial error signal for the counterweight carries an additional high–frequency signal component compared to the inner point mass which results from increased tether stiffness of the outer tethers (see Table 1). The signal component with a time period of approximately 95 s is due to the rotation of the TSS. The signal amplitudes are increasing since the choice of initial conditions results in a non–zero system linear momentum. The set of initial conditions perturbations chosen for TetraStar yields a relatively increased system angular momentum; hence the TSS rotates notably faster than the unperturbed satellite formation.

**MISSION SCENARIOS FOR SPECS**

This section briefly discusses two mission scenarios to validate the control law developed in the following section. For a detailed analysis we refer to Ref. 11. The first mission scenario considers the
stabilization of a particular relative equilibrium motion of the TSS. This case is of importance for those phases during the sequence of observations when the satellite formation needs to be reoriented to point at specific targets of interest. The second mission scenario represents a SPECS relevant trajectory including a tether deployment and retrieval procedure which allows the system to densely cover the observational plane in an optimal fashion. For this particular scenario an additional scientifically motivated constraint is formulated which requires that the instantaneous tangential velocity of the controlled point masses never exceeds a maximum value.

In Ref. 11 we commented on the importance of implementing “smooth” mission trajectories to minimize the required control input and we introduced smoothing functions for those critical periods of time when the TSS in a transitional phase between a relative equilibrium motion and a tether deployment or retrieval process.

Sizing of outer point masses for TetraStar

Up to this point the outer point masses have not entered the analysis. As mentioned before, their purpose is to provide additional control authority in radial direction and serve as a buffer for the system angular momentum during tether deployment and retrieval. To use the interferometer efficiently we require the instantaneous tangential velocity of the point masses to be equal to the maximum allowable velocity. However, we also require the outer tethers to be inextensible which leaves the outer point masses as the only free parameters to work with in order to obtain increased control performance compared to the triangular TSS. Note, that these constraints do not allow an optimal solution to the problem of deploying and retrieving the TSS such that the system angular momentum $h(t) = \text{const.}$ However, we can size the outer point masses such that $h(t_i) = h(t_f)$, where $t_i$ and $t_f$ are the initial and final times for tether deployment or retrieval. It is straightforward to show that,

$$h(t_i) = h(t_f) \iff m_\bullet (r_i^\bullet - r_f^\bullet) + m_\star \left( \frac{(r_i^\star)^2}{r_i^\circ} - \frac{(r_f^\star)^2}{r_f^\circ} \right) = 0$$  \hspace{1cm} (26)

Note that equation (26) contains three unknowns, namely $m_\bullet$, $r_i^\bullet$, and $r_f^\bullet$. Evaluating equation (23) at $t = t_i$ and $t = t_f$ adds two more equations and the resulting system of equations can be solved
using a Newton–Raphson method. For the system parameters listed in Table 1 and an unstretched and constant tether length of the outer tethers of $d_{o} = 65.0 \text{ m}$ we obtain an outer point mass of $m_{o} = 5.64 \text{ kg}$.

Applying the outlined methodology to a realistic 1 km baselength TSS with inner system point masses of $m_{i} = 500 \text{ kg}$ and an unstretched outer tether length of $d_{o} = 1100 \text{ m}$ results counterweights of $m_{c} = 3.56 \text{ kg}$ for a full aperture scan.

In the next section the tools necessary to develop a nonlinear adaptive output feedback controller are introduced.

**CONTROL LAW DEVELOPMENT**

A drawback of the observer–based control approach as used by the authors in an earlier paper is that an accurate physical model of the dynamical system is required to guarantee asymptotic stability. Although the mathematical structure of the system is usually well defined, the physical parameters are often not precisely known. Additionally, the cost of implementing a controller based on input–
state feedback typically includes the cost of motion sensors which in turn results in elevated sensor count. Adaptive output–feedback controllers utilizing a velocity–generated filter and an estimator for system parameters offer an elegant solution to the problem.

For this analysis the positions of the inner point masses \( m_i \) are chosen as outputs; that is, \( y_i \equiv q_i = (r_i, \theta_i, z_i)^T, \ i = 1, 2, 3 \). Also, due to the particular choice of control inputs in equations (1) one can show that the triangular system is feedback linearizable and therefore completely controllable.\(^{11}\) TetraStar, on the other hand is clearly uncontrollable due to a lack of control authority (outer system). However, we will show that for the mission scenarios described in the previous section the control of the inner system is sufficient. With these results we derive an adaptive output–feedback control law based on the work developed in Ref. 18.

**Problem statement**

Motivated by the subsequent analysis we rewrite the motion equations (7) in the following form

\[
\Omega(q, \dot{q}, \ddot{q})\Psi \equiv M(q)\ddot{q} + V_m(q, \dot{q})q + G(q) + F_d\dot{q} = U \tag{27}
\]

to collect and reorder corresponding terms in \( q, \dot{q}, \) and \( \ddot{q} \). We have used the fact that the dynamics equations are linearly parameterizable to define the product \( \Omega(q, \dot{q}, \ddot{q})\Psi \) where \( \Omega(q, \dot{q}, \ddot{q}) \) is the system regression matrix, which contains known functions of \( q, \dot{q}, \) and \( \ddot{q} \) and where \( \Psi \) is the constant system parameter vector. In equations (27) \( V_m(q, \dot{q}) \) represents the centripetal–Coriolis matrix, \( G(q) \) denotes gravity effects, and \( F_d \) is the viscous friction coefficient matrix.

The primary control objective is to design the control input \( U \) such that the tracking error vector \( e = y_d - y \to 0 \) as \( t \to \infty \), while the state remains bounded and with the constraint that only position measurements are available. Note that for both TSS under consideration only the inner point masses are controlled, thus we define

\[
e = (e_1, e_2, e_3)^T, \quad \text{where} \quad e_i = y_i^d - y_i, \quad i = 1, 2, 3 \tag{28}
\]

**Filter and controller design**

To aid the control design and analysis of this section, we define the vector function \( \tanh \xi \triangleq (\tanh \xi_1, \ldots, \tanh \xi_n)^T \) and the matrix function \( \cosh(\xi) \triangleq \text{diag}(\cosh \xi_1, \ldots, \cosh \xi_n) \) for \( \xi \in \mathbb{R}^n \). The filtered tracking error–like variable \( \eta \) is then defined as

\[
\eta = \dot{e} + \tanh e + \tanh e_f \tag{29}
\]

where we introduce the filter output \( e_f \) which is defined to have the following dynamics

\[
\dot{e}_f = -\tanh e_f + \tanh e - k \cosh^2(e_f)\eta, \quad \text{with} \quad e_f(0) = 0 \tag{30}
\]

and where \( k \) is a positive control gain. To obtain the open–loop dynamics in terms of the filtered tracking error–like variable we differentiate equations (29) with respect to time and premultiply the resulting expression by \( M(q) \) to obtain

\[
M(q)\dot{\eta} = M(q)\ddot{q}^d - M(q)\dot{q} + M(q) \left[ \cosh^2(e) \right]^{-1} (\eta - \tanh e - \tanh e_f) + M(q) \left[ \cosh^2(e_f) \right]^{-1} (\tanh e - \tanh e_f - k \cosh^2(e_f)\eta) \tag{31}
\]
where we have used equations (29) and (30). Next we use equation (27) to substitute for the term \(-M(q)\ddot{q}\) which contains time derivatives of the unmeasurable velocity vector. The centripetal–Coriolis term can be expanded into

\[
V_m(q, \dot{q}) = V_m(q, \dot{q})(\ddot{q} + \tanh e + \tanh e_f - \dot{e} - \tanh e - \tanh e_f) \tag{32}
\]

\[
= -V_m(q, \dot{q})(\dot{e} + \tanh e + \tanh e_f) \tag{33}
\]

\[
+ V_m(q, \dot{q} + \tanh e + \tanh e_f)(\ddot{q} - \dot{q}_d) \tag{34}
\]

\[
+ V_m(q, \dot{q}_d)(\dot{q}_d + \tanh e + \tanh e_f) \tag{35}
\]

\[
= -V_m(q, \dot{q})\eta + \Upsilon(q, \dot{q}_d, \eta, \tanh e, \tanh e_f) + V_m(q, \dot{q}_d)\dot{q}_d \tag{36}
\]

where we have used the symmetry property \(V_m(q, \dot{q})\zeta = V_m(q, \zeta)\xi\), repeatedly. The open-loop dynamics (31) then yields

\[
M(q)\dot{\eta} = -V_m(q, \dot{q})\eta - kM(q)\eta + \Omega_d\Psi + \chi + \ddot{Y} - U \tag{37}
\]

where \(\chi(q, \dot{q}_d, \eta, \tanh e, \tanh e_f)\) and \(\ddot{Y}(q, q_d, \dot{q}_d, \ddot{q}_d)\) are introduced as

\[
\chi = M(q)\left\{[\cosh^2(e)]^{-1}(\eta - \tanh e - \tanh e_f) + [\cosh^2(e_f)]^{-1}(\tanh e - \tanh e_f)\right\} \tag{38}
\]

\[
+ \Upsilon(q, \dot{q}_d, \eta, \tanh e, \tanh e_f)
\]

and

\[
\ddot{Y} = M(q)\ddot{q}_d + V_m(q, \dot{q}_d)\dot{q}_d + G(q) + F_d\dot{q} - \Omega_d\Psi \tag{39}
\]

In equations (37,39) we have introduced the desired regression matrix \(\Omega_d\) which is obtained from \(\Omega\) by exchanging \(q^{(i)} \leftrightarrow q_d^{(i)}\), \(i = 0, 1, 2\). Note that the desired regression matrix plays the role of the desired trajectory when using input–state feedback linearization, for instance. For the open-loop dynamics described by equation (37) it is proven in Ref. 18 that the following control input yields asymptotic stability:

\[
U = \Omega_d\ddot{\Psi} - k \cosh^2(e_f)\tanh e_f + \tanh e \tag{40}
\]

where the parameter estimate vector \(\dot{\Psi}\) is generated according to a gradient update law

\[
\dot{\Psi} = \Gamma\Omega_d^T\eta, \quad \text{with} \quad \Gamma = \Gamma^T > 0 \tag{41}
\]

Note that the control input in equations (40) requires the computation of \(e_f\) (or rather \(\eta\)) and therefore \(\dot{e}\). To obtain a control law with only position measurements define \(v = v_i \ddot{\Psi} / \tanh e_f\) and notice that

\[
\dot{v}_i = \frac{\dot{e}_f i}{\cosh^2(e_f i)} = (v_i^2 - 1)(v_i - \tanh e_i) - k(\dot{e}_i + \tanh e_i + v_i), \quad \text{with} \quad v_i(0) = 0 \tag{42}
\]

which can be rewritten as follows

\[
\dot{v}_i = \left\{(\phi_i - ke_i)^2 - 1\right\}(\phi_i - ke_i - \tanh e_i) - k(\tanh e_i + \phi_i - ke_i), \quad v_i = \phi_i - ke_i \tag{43}
\]

where \(\phi_i(0) = ke_i(0)\). With the help of equation (43) \(v\) and therefore \(\tanh e_f\) can be calculated with position measurements only. Using filter (43) the parameter update law can be integrated with respect to time to yield

\[
\dot{\Psi} = \Gamma\Omega_d^T e + \Gamma \int_0^t \left\{\Omega_d^T (\tanh e + v) - \dot{\Omega}_d^T e\right\} d\tau \tag{44}
\]
where we have used definition (29) and integrated by parts to replace the term containing time
derivatives of the error vector, namely $\dot{e}$. Finally, the control law (40) can be written componentwise

$$U_i = (\Omega_d \dot{\Psi}_i) - \frac{k_{\nu_i}}{1 - \frac{\nu_i}{\nu_i^2}} + \tanh e_i \quad (45)$$

Equations (43–45) represent the velocity filter, the parameter estimator, and the control law, respectively. Note that the controller (45) has the same basic structure as a conventional controller based on input–state feedback linearization. The first term is a feedforward term which contains the desired system dynamics and for $e = \dot{e} = 0$ exactly cancels the right–hand–side of the motion equations. The second and third term regulate the velocity–like and the position error vectors, respectively.

**EXAMPLES**

We developed and implemented a generic simulation code in C++ to validate the performance of the control laws presented in the previous sections. In particular, the mission scenarios described earlier were used to define desired trajectories. System and simulation parameters are listed in Table 1.

For the perturbed initial conditions response simulation the perturbation vectors were generated randomly with an upper bound of $\delta x \leq 10^{-3}$ for each element of the vectors. The initial and final radial distance of the point masses for mission scenarios 2 were chosen to be $R_0 = R_{0_i} = 15 \text{ m}$ and $R_f = R_{0_i} = 35 \text{ m}$. Note that we chose the outer tether length $d$ such that the angle between two corresponding outer tethers is $\rho(t) < \pi/3$ (Figure 1) at all times; hence the additional radial force component due to the outer tethers can assist in the deployment and control process even when the system is fully deployed.

Control parameters for all simulations were tuned to yield comparable control performance in terms of control effort and output error for the triangular system and TetraStar. In particular, the control parameters were chosen as $k = 40.0$ and $\Gamma = \text{diag}\{1,1,1,5,\ldots,5\}$, where $\Gamma$ is of proper dimension.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point masses $m_c$ in kg</td>
<td>50.0</td>
</tr>
<tr>
<td>Point masses $m_\circ$ in kg</td>
<td>5.644</td>
</tr>
<tr>
<td>Young’s moduli $E_c$ in N/mm$^2$</td>
<td>10,000</td>
</tr>
<tr>
<td>Young’s moduli $E_\circ$ in N/mm$^2$</td>
<td>15,000</td>
</tr>
<tr>
<td>Tether cross sections $A_c$ in mm$^2$</td>
<td>1.0</td>
</tr>
<tr>
<td>Tether cross sections $A_\circ$ in mm$^2$</td>
<td>1.0</td>
</tr>
<tr>
<td>Damping parameters $\alpha_c$ in s</td>
<td>0.0</td>
</tr>
<tr>
<td>Mirror diameters $D_i$ in m</td>
<td>3.0</td>
</tr>
<tr>
<td>Unstretched tether length $d$ in m</td>
<td>65.0</td>
</tr>
<tr>
<td>Initial radial distance $R_0$ in m</td>
<td>15.0</td>
</tr>
<tr>
<td>Final radial distance $R_f$ in m</td>
<td>35.0</td>
</tr>
<tr>
<td>Integration time step $dt$ in s</td>
<td>0.01</td>
</tr>
<tr>
<td>IC disturbances $\delta x_{i,j}$ in $[\delta x_{i,j}]$</td>
<td>$\leq 10^{-3}$</td>
</tr>
</tbody>
</table>

Figures 6–13 show simulation results for the mission scenarios introduced in a previous section. The initial conditions perturbation responses are illustrated in Figures 6 and 7 for the triangular system and in Figure 8 for TetraStar. Comparing Figures 6 and 8 the triangular system converges considerably faster towards “steady–state” conditions than TetraStar. The controllability of the former system on the one hand and the increased number of parameters to be estimated for the more complex TetraStar configuration on the other hand are the crucial factors resulting in an increase in overall required control input. Note that the parameter estimates for $m_i$ and $\mathcal{A}_\circ$, $i = 1,2,3$ in Figure 7 reach steady states which are, however, notably different from the actual parameter
values. According to Ref. 18 the difference in estimated and actual parameters is not uncommon for mechanical systems and lies in the fact that the primary goal of the controller is to track a desired output and not to accurately estimate the unknown parameter vector.

The results for the SPECS relevant mission scenarios are shown in Figures 9–13, Figures 9 and 11 illustrate the corresponding trajectories. Similar to the previous simulation results the triangular systems converges rapidly towards an “optimal” control state lacking high–frequency oscillations. Both control input and signal error are comparable in magnitude for both satellite formations, however, we observe a significant decrease in required control effort in radial and angular direction in the case of TetraStar. The bumps in the radial control signal shown in Figure 10 have disappeared in Figure 12 due to the additional radial force component of the outer tethers. More importantly, the control effort in angular direction is significantly reduced for TetraStar as shown in Figure 12. The corresponding (negative) control effort for the triangular system is shown here for comparison (dashed line). For this particular simulation we obtain a decrease in overall required control effort for TetraStar of about 75%, about 68% in angular direction. We point out, however, that the particular strategy of choosing the outer point masses following the approach developed in a previous section might not yield an optimal solution to the problem. Especially for large final base-lengtths with a ratio of $r_f/r_i \approx 40.0$ TetraStar shows significant benefits when the outer tethers are allowed to be extensible, as preliminary simulation results show. Figure 13 shows the parameter estimate history for the SPECS relevant mission scenario for TetraStar. Other than for mission scenario 1 the parameters are not approaching final constant values but show a cyclic symmetry owing to the symmetry of the TetraStar configuration. The time–dependence of the parameter vector comes to no surprise since for the SPECS relevant mission scenario we introduced explicit time dependence in the motion equations during tether deployment and retrieval.

**SUMMARY AND CONCLUSIONS**

Techniques to control the motion of TSS comprised of $n$ point masses and interconnected by $m$ idealized tethers are presented. Specifically, the control problems of a triangular TSS and a satellite configuration called TetraStar are discussed. Mission scenarios for NASA’s SPECS mission are introduced and asymptotic tracking laws based on output–feedback control are developed. The simulation results show that triangular system is superior in terms of parameter estimation compared to the more complex TetraStar formation which is mainly due to the fact that we require the outer spacecraft of TetraStar to be uncontrolled which renders the satellite formation uncontrollable and unstabilizable in general. From a controls point of view TetraStar shows significant benefits. The control effort in radial direction for TetraStar becomes negligible, in angular direction we obtain a significant decrease in overall control impulse by requiring that the system angular momentum at $t = t_i$ and $t = t_f$ are equal. Even though, this approach proofs to be a reasonable technique of choosing the mass of the outer spacecraft we come to the conclusion that the a better way of dramatically decreasing the overall control effort is to allow the outer tethers to be extensible. As a final notice, we point out that the lack of control authority for the TetraStar configuration with uncontrollable counterweights turns out to be immaterial for in–plane maneuvers. The uncontrollability of the system greatly effects, however, plane changes such as re–targeting maneuvers, which are certainly infeasible.

**ACKNOWLEDGEMENTS**

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Coordinate error $e_{r1} = r_1^d - r_1$

Coordinate error $e_{\theta 1} = \theta_1^d - \theta_1$

Control effort in $r$-direction for $m_1$

Control effort in $\theta$-direction for $m_1$

Control torque $u_{\theta 1}$ in Ns

Figure 6: Triangle: Output error and control vector history for an initial conditions perturbation response. The controlled system performs a relative equilibrium motion.

Parameter estimates for masses $m_1$, $m_2$, and $m_3$

Parameter estimates for $EA_1$, $EA_2$, and $EA_3$

Figure 7: Triangle: Parameter estimate history for a relative equilibrium motion.
Figure 8: TetraStar: Output error and control vector history for an initial conditions perturbation response. The controlled system performs a relative equilibrium motion.
Figure 9: Triangle: Trajectory of point mass $m_1$ for SPECS relevant mission scenario.

Figure 10: Triangle: Output error and control vector history for SPECS relevant mission scenario.
Figure 11: TetraStar: Trajectories of point masses $m_1$ and $m_6$ for SPECS relevant mission scenario.

Figure 12: TetraStar: Output error and control vector history for SPECS relevant mission scenario. Angular control effort for the triangular system (dashed line) and savings for TetraStar (dotted line) are shown for comparison.
Figure 13: TetraStar: Parameter estimate history for SPECS relevant mission scenario.
REFERENCES


