An Analysis in the Efficiency of Various Orbit Transfer Methods

by

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Introduction.

Upon the launch of a satellite into a temporary parking orbit, there are many means of transfer methods to get the satellite to its final destination. The transfer depends upon many factors, including the satellite's mission, design and the time needed to reach the mission orbit. This paper focuses on the optimization of transfer by comparing different transfer methods and efficiency in energy and time of each.

A specific model problem is used here to demonstrate the various transfer methods. This model is a satellite whose parking orbit is in a low earth orbit with a given inclination, and whose final orbit is a geosynchronous, equatorial one. Three different transfer methods are examined. The Hohmann transfer is one of the most energy efficient transfer, yet is costly in time. Hyperbolic orbits are timely but costly in energy. The three burn transfer (bi-elliptic) saves in energy but also costs in time. Although these facts are generally known, a comparison can demonstrate how much is lost or gained through each transfer method. This comparison will include the effect of the inclination change. Analysis of this can aid in determining early mission design plans for satellites.

Basic Transfer Equations.

The inclination or plane change velocity differences (delta-v's) discussed here are large enough to enable great orbit change. The equations used to do this are those derived from the motion of the two-body problem which are used to describe a satellite's motion in orbit. The equations used here are,

$$E=\frac{v^2}{2}-\frac{\mu}{r}=-\frac{\mu}{2a} \quad \textbf{(1)}$$

$$r=\frac{a(1-e^2)}{1+e\cos\nu} \quad \textbf{(2)}$$

where, E=specific mechanical energy,
v = velocity in an orbit at distance r from earth center,
r = position of satellite in its orbit,
μ = gravitational parameter,
a = semi-major axis,
e = eccentricity,
ν = position of satellite from periapsis.

Equation 1 is used to calculate the velocity at any point in any orbit as long as the position and semi-major axis of the orbit are known. The total change in velocities needed for each transfer will be the measurement of the energy efficiency used to compare. It is assumed the reader is familiar with these equations, therefore detail is limited.

Other equations involve the transfer itself. For two velocity vectors at a burn point, the law of cosines can be applied, to determine the delta-v needed. The equation is then,

$$\Delta v^2 = v_1^2 + v_2^2 - 2 v_1 v_2 \cos\phi \tag{3}$$

where, Δv = thrust needed,
v_1= velocity of initial orbit,
v_2= transfer velocity needed
ϕ = angle between two velocity vectors.

The most simplified case is when the two orbits are tangential (ie. phi=0) and after factoring, the equation now becomes,

$$\Delta v = v_2 - v_1 \tag{4}$$

When an inclination change is involved, the location of the orbital burns is more crucial. If the only angle difference between the two velocity vectors is the inclination change, then this can be easily substituted into Equation 3. If the velocity magnitudes remain the same and an inclination change (Δ i) is necessary, a simple plane change is all that is needed. This equation is,

$$\Delta v = 2\ v \sin\frac{\Delta i}{2} \tag{5}$$

In some cases both an inclination change and a phi are involved. Here, the vectors of each velocity need to be known in order to use the dot product to determine the angle in between. This angle is then,

$$\alpha = \arccos\frac{\vec{v}_1 \cdot \vec{v}_2}{v_1 v_2} \tag{6}$$

which is then used in Equation 3 to determine the thrust needed to get into a new orbit at v_2.

These equations are presented first, since all transfer methods analyzed in this paper will use them. In addition, time of flight equations are used and rely more on the type of transfer orbit. These will be introduced in the individual transfer section. Also, fundamental geometry relationships for a circle, hyperbola, and ellipse are manipulated.

Assumptions.

In order to analyze specific transfer orbits, several simplifying assumptions are made:

(1) Two-body problem relationships (as the above equations) and all of its assumptions hold (ie. spherical earth and no other external or internal forces acting on the two body system besides gravitation).

(2) No special perturbations enter into the problem, therefore, all orbits are "perfect" circles, hyperbolas, and ellipses.

(3) Satellite is treated as a point mass, therefore ignoring spacecraft rigid body dynamics.

(4) Rotation of the earth is ignored.

Transfer method cases will be examined using these constraints:

(1) The satellite begins in a low earth circular orbit of radius of 1.02898 DU, inclination of 28.5 degrees, and ascending node at 180 degrees (standard shuttle parking orbit).

(2) Final orbit is geosynchronous, equatorial of radius 6.609915 DU.

(3) All calculations in computer programs and paper are in canonical units (where 1 DU = earth equatorial radius and 1 TU = 806.8118744 seconds).

It is this problem which will be analyzed using the different transfer methods.

Hohmann Transfer.

A Hohmann transfer between two circular orbits can be pictorially represented as in Figure 1. The transfer is an elliptical transfer method in which two tangential burns are done to put the satellite into its final orbit. Because the transfers are tangential, a total Hohmann delta-v can be acquired using the following equations,

$$\Delta v_1 = v_{T1} - v_1 \quad (7)$$

$$\Delta v_2 = v_2 - v_{T2} \quad (8)$$

$$\Delta v_{HOH} = \Delta v_1 + \Delta v_2 \quad (9)$$

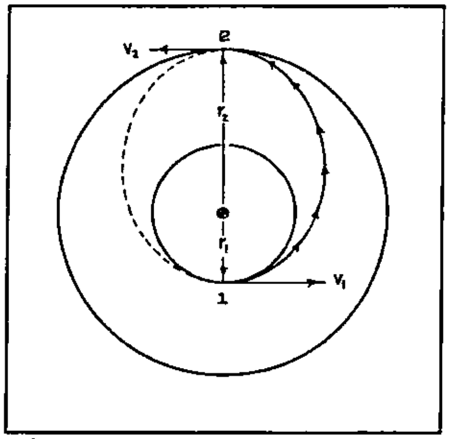

Figure 1: Hohmann Transfer

where, Δv_1 = delta-v needed at first tangential point,
Δv_2 = delta-v needed at second tangential point,
v_1 = velocity at initial orbit,
v_2 = velocity at final orbit,
v_{T1} = velocity at perigee of transfer ellipse,
v_{T2} = velocity at apogee of transfer ellipse.

Given an initial and final radius orbit, the velocities above can be calculated by manipulating Equation 1 and solving for velocity. The time for transfer (TOF) can be easily determined, since it is half of the ellipse transfer,

$$TOF=\Pi\sqrt{\frac{a_t}{\mu}} \quad \textbf{(10)}$$

where, a_t = semi-major axis of transfer.

Complications can occur when including a plane change on the transfer. The plane change must occur on the nodal line (either ascending or descending node). To minimize both the time of flight and the delta-v needed, it is best to consider a combination of plane changes and Hohmann transfers at both burn points. Since it is not known how much inclination change should be done at each burn, consider a fraction, f, on the first burn and 1-f on the second burn so that the equations become,

$$\Delta v_1^2=v_1^2+v_{T1}^2-2v_1v_{T1}\cos(f\Delta i) \quad \textbf{(11)}$$

$$\Delta v_2^2=v_2^2+v_{T2}^2-2v_2v_{T2}\cos((1-f)\Delta i) \quad \textbf{(12)}$$

Therefore, for each initial and final orbit, and inclination given, if f can be iterated, it can be determined at which f value will give the minimum delta-v for that inclination change.

To determine this, an iterative program was written (Appendix A). This program calculates the total delta-v needed at each iteration of f between 0 and 1, given an input of initial and final orbit radius and inclination change. By observation of a graph plotting f and delta-v, a minimum value for delta-v and that associated f can be determined.

Results: Figure 2 shows the graph of the transfers needed for the model problem. As can be observed, it is more economical to do most of the inclination change at the second burn. This is probably obvious from the equations, since an inclination change associated with larger velocities will produce a higher delta-v (first burn) then an inclination change associated with smaller delta-v's (second burn point). What is not so obvious, is that there is a payoff (however small) to do some of the inclination change during the first burn.

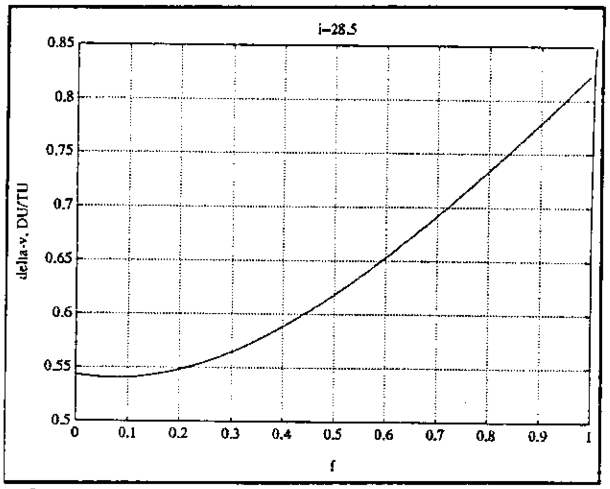

Figure 2: Hohmann Transfer with Plane Change Effect

An iteration run using the computer algorithm, shows the minimum delta-v of .54045 DU/TU when f=.076, indicating a 2.16 degree plane change on the first burn and 26.34 degree plane change on the second burn. When running this program for different inclination changes, similar graphs result, but with differing f values: for i=10 degrees, f=.09 and for i=90 degrees, f=.029. Therefore, in addition to it being more economical to do some inclination change on the first burn, it is beneficial to do more of the change at smaller inclinations.

The Hohmann transfer minimum velocity value of .54045 at f=.076 will be the value of which all other transfer orbits will be compared. At any f value, the time of transfer will be the same (using Equation 10, TOF=23.4224 DU). This indicates that perhaps the Hohmann may be the most efficient in minimizing total energy, it is not in minimizing transfer time.

Hyperbolic Transfers.

The interest in hyperbolic transfer orbits stem from the desire to minimize the time of transfer of a satellite. In wartime, when it may be crucial to replace or add a satellite in minimal time, the benefits of a shorter transfer time may outweigh a minimum delta-v (that is, if there is enough delta-v to do a hyperbolic transfer in the first place).

Hyperbolic transfers are treated similarly to elliptic transfers. Hyperbola velocity equations were derived through the manipulation of Equation 1. Using the relationship of the perigee position in orbit (ie. r = a(1-e)), the velocity desired to put the satellite in a hyperbolic trajectory becomes,

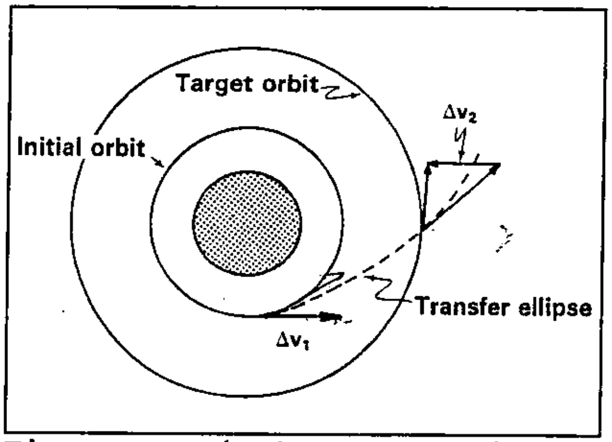

Figure 3: Simple Hyperbolic Transfer

$$v_{H1}=v_1\sqrt{1+e} \qquad \textbf{(13)}$$

where e represents the transfer eccentricity (greater then 1). Figure 3 shows a simple hyperbolic transfer. Three cases were examined, varying the point at which the inclination change was done and the initial and final burn points. Energy efficiency and time to transfer were examined.

Case I: Three burns were considered for this case. The first burn includes a fraction of the inclination change in combination with the hyperbolic transfer. Burn is done at the ascending node. The second burn circularizes the orbit at a new inclination, indicated by the first burn fraction, and at the final radius. Since a hyperbolic transfer will not intercept the final orbit at a node line, a third burn will be necessary at the ascending node of this orbit in order to correctly place this in the final equatorial orbit. These three burns can be expressed in equations as follows:

$$\Delta v_1^2=v_1^2+v_{H1}^2-2v_1v_{H1}\cos(f\Delta i) \qquad \textbf{(14)}$$

$$\Delta v_2^2=v_2^2+v_{H2}^2-2v_2v_{H2}\cos\phi \qquad \textbf{(15)}$$

$$\Delta v_3=2v_2\sin\frac{(1-f)\Delta i}{2} \qquad \textbf{(16)}$$

Phi is determined when knowing the vector quantities of v_2 and v_{H2}. Since the parameters of each orbit is known, vectors expressed in the geocentric-equatorial system are easily found, therefore, the angle in between is easily determined.

For this case, an algorithm similar to the Hohmann transfer was used, however, this time the eccentricity of the hyperbolic transfer was varied, keeping f constant. The f used here was the same as the one discovered for the Hohmann transfer case (f=.076), making the assumption that this too would be the best division of the inclination change (on the first and third burns). For each eccentricity (varying from 1.1 to 5.0) the values for the individual and total delta-v's were calculated using inputs from the model problem. Time of transfer was also determined. Equations used are;

$$\cosh F = \frac{e+\cos\nu_2}{1+e\cos\nu_2} \tag{17}$$

$$TOF_{Hyper} = \sqrt{\frac{-a^3}{\mu}}\,[e\sinh F - F] \tag{18}$$

$$TOF_{part} = (\Pi - \nu_2)\,\frac{r_2^{\frac{3}{2}}}{\sqrt{\mu}} \tag{19}$$

where, F = hyperbolic eccentric anomaly,
ν_2 = position satellite is from periapsis of transfer,
TOF_{hyper} = time it satellite is in hyperbolic transfer,
TOF_{part} = time satellite is in inclined circular orbit prior to doing a plane change to equatorial.

Therefore, the sum of these times gives the total transfer time. Transfer time will vary with eccentricity since varying e's will give trajectories that intersect the circular inclined radius at different true anomalies.

Results: It was expected to see the total delta-v increase as eccentricity grew large; however, time of transfer increased as well (see Figures in Appendix B). Figure 4 shows the break down of the two times required after each burn for changing e. The large time of transfer is due to the fact that as the eccentricity of the transfer orbit got larger, the true anomaly gets smaller, therefore intersecting the outer radius at an earlier position. Therefore, the time spent in this outer radius increased, which eventually became the domineering term on the total time transfer.

When observing the runs at the smallest eccentric value of 1.1, total delta-v measured 1.2371 DU/TU, over twice the velocity of the Hohmann transfer. Time of flight in this "best" case totalled 24.1980 TU, a little over the Hohmann transfer value. Therefore, this case of hyperbolic transfer is not worth pursuing if looking for a transfer method efficient in either energy or time.

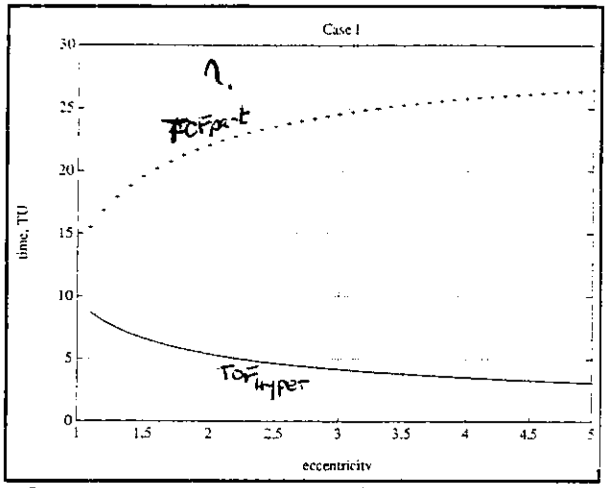

Figure 4: Hyperbolic, Case I; Comparison of Transfer Times for Thrusts

Case 2: Two burns were considered for this case. The first burn included a total inclination change (no fraction) done at the ascending node in combination with a hyperbolic transfer. The second burn circularized the orbit at the final radius position (since the transfer orbit is without an inclination). The equations used here are;

$$\Delta v_1^2 = v_1^2 + v_{H1}^2 - 2 v_1 v_{H1} \cos \Delta i \quad (20)$$

$$\Delta v_2^2 = v_2^2 + v_{H2}^2 - 2 v_2 v_{H2} \cos \phi \quad (21)$$

Again, phi is known by solving the geocentric-equatorial velocity vectors involved by knowing the orbital parameters for each orbit. Since transfer time will be the time between the two burns, it will only involve the hyperbolic transfer time as expressed in Equation 18.

As in case I, a similar modified algorithm was written and the problem model was tested (Appendix C). Eccentricity of the hyperbolic transfer was varied from 1.1 to 5.0, and the delta-v's and time for transfer were examined.

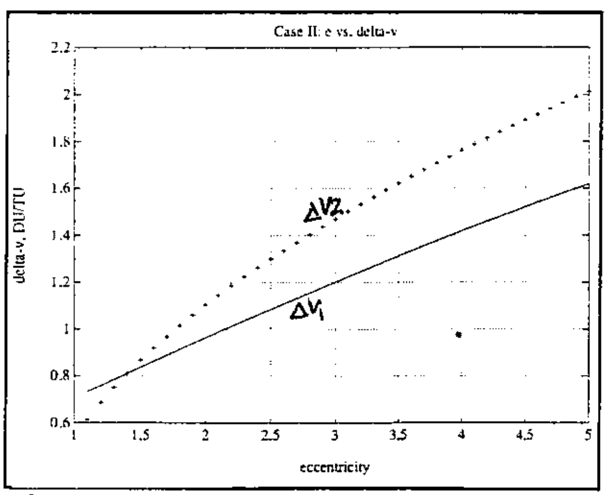

Figure 5: Hyperbolic, Case II; Comparison of Delta-V's for Each Thrust.

Results: As expected this time, total delta-v increased and total time of flight decreased while e gets larger (see Figures in Appendix C). Each delta-v component was observed closer. Figure 5 shows the burn components versus e. This demonstrates that at lower hyperbolic eccentricities, the first burn dominates the total delta-v needed while at higher hyperbolic eccentricities, greater energy is needed to overcome the phi angle. At the "best" value for minimum efficiency, where e= 1.1, total delta-v is 1.3479 DU/TU, more then 2.5 times the Hohmann transfer "best" velocity and slightly more then the first case. The real advantage of this case is in the time to transfer. At this e value ("worst" for time efficiency), time to transfer is 8.6938 TU, almost one third of the Hohmann transfer time. Therefore, if the mass of the satellite allowed the amount of the fuel needed to make this burn (or a high specific impulse fuel is used), this transfer would be the most advantageous if time was important.

Case III: Two burns are used in this case. The first burn, not done at the ascending node, is a hyperbolic transfer tangent to the first orbit, with the burn 270 degrees from the ascending node. The second burn includes the total inclination change done at the point where the hyperbolic transfer intersects the final equatorial orbit at r_2. The equations for the two burns are as follows:

$$\Delta v_v = v_{H1} - v_1 \tag{22}$$

$$\Delta v_2^2 = v_2^2 + v_{H2}^2 - 2 v_2 v_{H2} \cos\alpha \tag{23}$$

Again, equation 18 is used to determine the hyperbolic transfer time, since the only path taken to transfer is the hyperbola.

An algorithm similar to the other cases was developed (Appendix D). Again, for values of e from 1.1 to 5.0, total delta-v's and transfer time were examined.

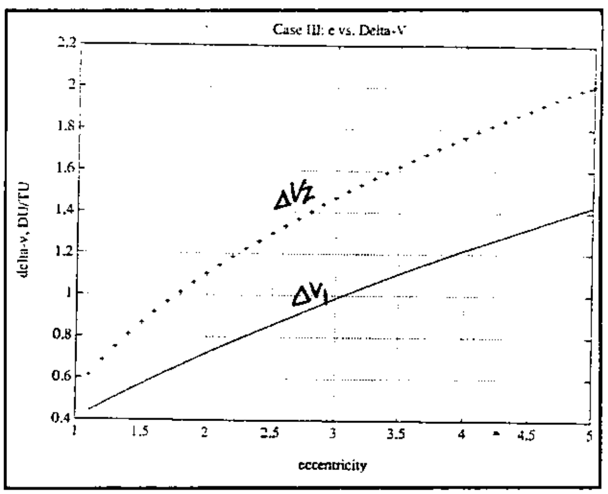

Figure 6: Hyperbolic, Case III; Comparison of Delta-V's for Each Thrust

Results: Since the hyperbolic paths were the same as for ones in case II, it was expected that the time of flights remained the same, decreasing with increasing e. Also, total delta-v increased with increasing e. However, the total delta-v's for each e in this case proved lower then in Case II (see figure in Appendix D). Therefore, it seems more economical to do the inclination change at the second burn when velocities of both the final and transfer orbit are lower. This could be predicted, as discovered in the Hohmann case. Again looking at the delta-v components, Figure 6 demonstrates that at lower hyperbolic transfers, the first burn thrust never dominates the second burn, as in Case II. It was also observed that the second burn thrust magnitudes for each e value matched that of case II. It is unknown why this has occurred, since the angle between these velocity vectors at the second burn point is different between the two cases. Perhaps the inclination angle is dominated by phi, giving an alpha approximately equal to phi. It would be interesting to analyze this problem for higher inclination orbits.

Did you compare α and ϕ?

In this case, the "best" energy efficient value for e=1.1 was 1.0576 DU/TU which is slightly less then twice the Hohmann, better then the second case, and somewhat acceptable. Its time to transfer was 8.6938 TU, same as Case II. This case proved to be the best choice for hyperbolic transfers and is comparable to the Hohmann, if time is important, providing one-third the time to transfer at almost twice the energy cost.

Bi-elliptic Transfer.

A bi-elliptic transfer is a three-burn transfer, intended to lower the energy efficiency needed for orbital transfer. Figure 7 illustrates a simple coplanar three burn transfer. The burns can be described as follows: the first burn ($\Delta v_A'$) is a coplanar maneuver which places the satellite into a transfer orbit with an apogee much higher then the final orbit. At this apogee, another orbit burn ($\Delta v_C'$) is accomplished, placing the satellite into a second transfer orbit, coplanar to the final orbit. Finally, when the satellite reaches perigee of the transfer, a third burn ($\Delta v_B'$), another coplanar maneuver, is done, placing the satellite into its final orbit. This transfer is most efficient when a plane change is involved with most of it best completed at the second burn (as seen from the other transfers, it becomes more economical to accomplish this where the velocity is the least). The best fraction of the inclination change will be examined here as was in the Hohmann.

The idea behind this transfer was initiated by looking at the extreme case of a transfer orbit, the case where the two transfer apogees are infinity. Using equation 1, and determining the velocity needed to escape from a circular radius of r_1 to a transfer orbit of a=infinity (parabola), and determining the velocity needed to return from this "apogee" and transfer into the final orbit, the equation becomes:

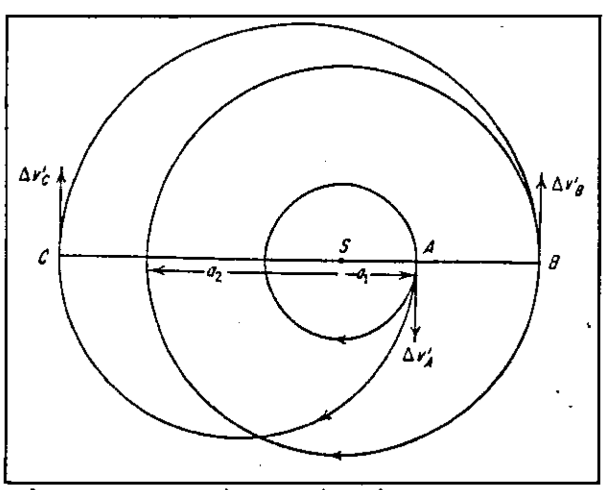

Figure 7: Bi-elliptic Transfer

$$\Delta v_{total} = \Delta v_{escape} + \Delta v_{return} = (\sqrt{2}-1)\ (v_1 + v_2) \qquad \textbf{(24)}$$

where, v_1 and v_2 represent the velocities of the circular orbits. The total delta-v is the total for the three burns, since conceptually, at the infinity second burn transfer position, any velocity change will be zero. By substituting $R=r_2/r_1$, the final velocity can be expressed as,

$$v_2 = v_1 \sqrt{\frac{1}{R}} \qquad \textbf{(25)}$$

then the total three burn transfer can be expressed as,

$$\frac{\Delta v_{tot,B}}{v_1} = (\sqrt{2}-1)\ (1+\sqrt{\frac{1}{R}}) \qquad \textbf{(26)}$$

Similarly, the total velocity needed for the Hohmann transfer can be manipulated and expressed as,

$$\Delta \frac{v_{tot,H}}{v_1} = \sqrt{\frac{2R}{1+R}}\,(1-\frac{1}{R}) + \sqrt{\frac{1}{R}} - 1 \qquad \textbf{(27)}$$

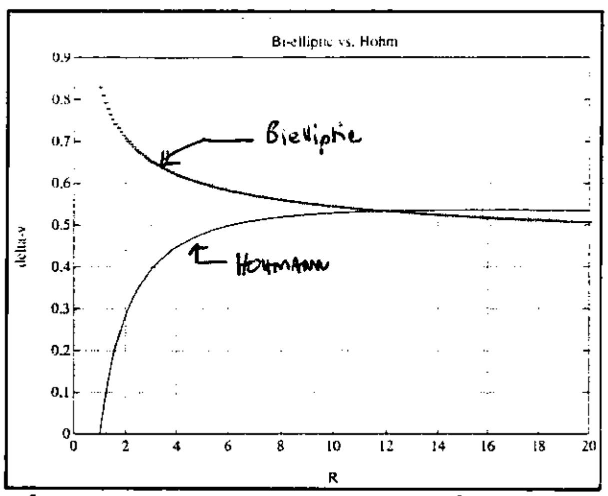

Figure 8: Hohmann vs Bi-elliptic Orbits: R vs. Delta-V Ratio

These equations can be used to determine the total velocities at any given initial and final satellite positions, and the velocities can be compared to determine the most efficient transfer method for that position. Knowing that these functions intersect, the equations can be set equal to find this intersection point. Figure 8 demonstrates this intersection. This occurs at R = 11.939 (This approximation was completed using MATHEMATICA solve function). This R value then corresponds to the point at which a three burn transfer becomes more efficient then the Hohmann transfer. At the model's current initial orbit radius (1.2 DU), the final orbit would have to be approximately 14.326 DU in order for the three burn transfer to be more energy efficient. Therefore, the

Hohmann transfer will probably be the most effective transfer to minimize energy for the given model problem. However, in the above analysis, plane changes were not taken into account. As in the Hohmann, analysis for this transfer will focus on this effect.

The effect of plane change will be examined by observing a fraction of the change on the second burn and the rest on the third. This division will most likely be the optimal (since a plane change on the first burn would probably result in an increased total velocity). Already knowing the circular velocities for the model problem, the transfer apogee and perigee velocities can be determined manipulating equation 1;

$$v_{pt1}=\sqrt{2\mu\left(\frac{1}{r_1}-\frac{1}{r_1+r_t}\right)} \quad (28)$$

$$v_{at1}=\sqrt{2\mu\left(\frac{1}{r_t}-\frac{1}{r_1+r_t}\right)} \quad (29)$$

$$v_{at2}=\sqrt{2\mu\left(\frac{1}{r_t}-\frac{1}{r_2+r_t}\right)} \quad (30)$$

$$v_{pt2}=\sqrt{2\mu\left(\frac{1}{r_2}-\frac{1}{r_2+r_t}\right)} \quad (31)$$

where, r_t = transfer apogee distance,
v_{pt1} = velocity at perigee at 1st transfer orbit,
v_{at1} = velocity at apogee at 1st transfer orbit,
v_{at2} = velocity at apogee at 2nd transfer orbit,
v_{pt2} = velocity at perigee at 2nd transfer orbit.

Since all the inclination change is done on the second and third burns, individual delta-v's for each burn becomes,

$$\Delta v_1^2=v_1^2+v_{pt1}^2-2v_1v_{pt1} \quad (32)$$

$$\Delta v_2^2=v_{at1}^2+v_{at2}^2-2v_{at1}v_{at2}\cos(f\Delta i) \quad (33)$$

The total delta-v is the sum of the three. A time of transfer can also be determined by summing the times for each time in its

$$\Delta v_3^2 = v_2^2 + v_{pt2}^2 - 2 v_2 v_{pt2} \cos((1-f)\Delta i) \quad \textbf{(34)}$$

transfer ellipse. This is given as,

$$TOF = \Pi\left(\sqrt{\frac{a_{t1}^3}{\mu}} + \sqrt{\frac{a_{t2}^3}{\mu}}\right) \quad \textbf{(35)}$$

A computer algorithm was written on MATLAB to allow iterations of f and r_t using the above equations (see Appendix E). The fraction was iterated from 0 to 1 for every r_t value from 7 to 20 DU. Inclination change was kept constant at 28.5.

Results: Overall, the total minimum energy needed for each different transfer apogees (or radius) increased as the transfer radius increased. This was expected since using the bi-elliptic for this ratio of final to initial orbits is not supposed to be energy efficient (recall ratios of 11.939 or greater will result in energy saved when using a bi-elliptic). This is probably due to the combinations of events going on. One, is that for the transfer radius' tested (from 7 to 20 DU), the thrust needed to escape the circular orbit to place the satellite into its highly elliptical orbit (burn one) and the thrust needed to finally circularize it (burn two), dominates the total thrust term. These two terms expectedly increase as the transfer radius increases. Although, the second thrust decreases with an increase in transfer radius, this term never results in a low enough thrust to compensate for the two increasing ones. Therefore, perhaps at higher transfer radius (however outrageous a distance they might be) there might be a point where the second thrust dominates the total thrust term, thereby the total delta-v would decrease. However, the total energy used would still be greater then the Hohmann Transfer, using the model problem.

In observing the effect of the inclination change (by looking at the f value), it was expected to see a similar curve to the Hohmann transfer when plotting f versus delta-v. Figure 9 shows this enlarged curve for a transfer radius of 11 DU. All other transfer radii responded the same way, giving different values of f for the minimum delta-v required for that transfer radius. Figure 10 plots the f values corresponding to the minimum delta-v needed (when considering inclination changes on the second and third burn) for each transfer radius. At lower transfer radii, it is best to do close to 98% of the inclination change at the second burn. At a transfer radius of between 13 and 14 DU, this figures drops to near 95% and then increases almost linearly. This might have occurred due to the non-dominating term of this second thrust, as addressed previously (ie., doing the entire plane change, or a large part of it, at the second burn did not seem to be crucial in lowering the total

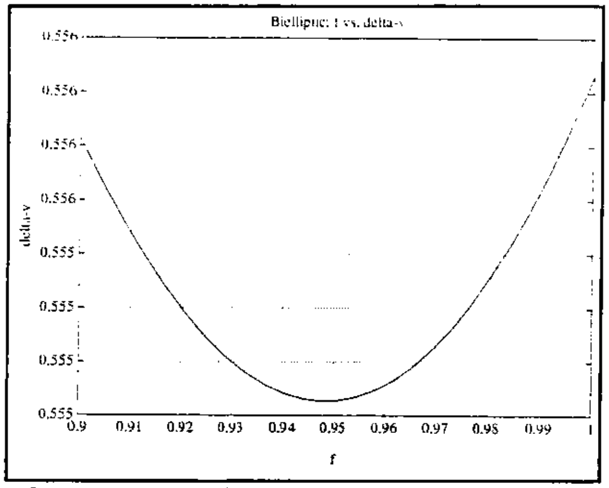

Figure 9: Bi-elliptic Inclination Fraction on Second Thrust Corresponding to Minimum Delta-V

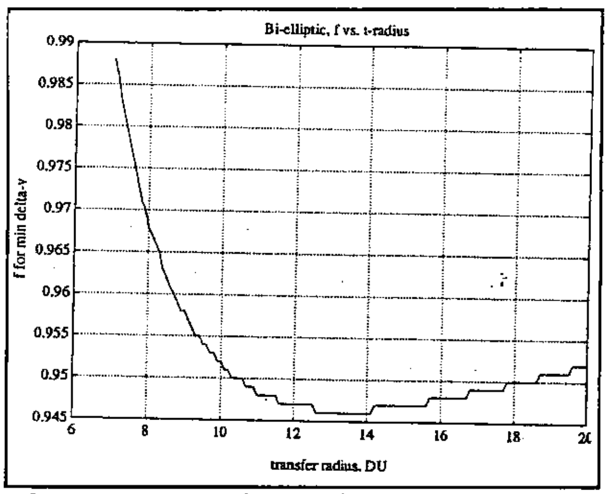

Figure 10: Bi-elliptic Transfer; Fraction of Inclination Change Accomplished on the Second Thrust Corresponding to a Transfer Radius

delta-v here, as it did in the smaller transfer radii). After 14 DU, the term seems to become more important, since the increase in f indicates a lower total delta-v when more of the inclination change is at the second burn. This indicates a possible gaining of domination by the second thrust value.

The most energy efficient run which is comparable to the Hohmann occurred at a transfer radius of 7 DU with a total delta-v of .5453 DU/TU, only slightly more then the Hohmann. However, this transfer was not time efficient. It took 81.0378 TU to complete the bi-elliptic transfer, about four times the Hohmann transfer time. Therefore, for this particular model problem, the bi-elliptic transfer was not effective in either minimizing energy or time.

Conclusion and Results.

The results and comparison of the five transfers are listed:

Transfer	Minimum Delta-V	Minimum Time to Transfer
Hohmann	.54045	23.4224
Hyperbolic:		
Case I	2.28 V_H	1.03 T_H
Case II	2.49 V_H	.371 T_H
Case III	1.95 V_H	.371 T_H
Bi-elliptic	1.01 V_H	3.45 T_H

As expected, the Hohmann transfer is the optimal type of transfer for minimizing energy, however Case III of the hyperbolic can improve upon the time to transfer. This may be more crucial then energy efficiency in some satellite placement scenarios where time to orbit is valuable. Investigation of the inclination change proved to be beneficial, in that in each case, the location of the change and the amount that is done, plays an important part in minimizing the energy of the transfer. This is necessary due to limited space for fuel on a satellite and lack of a readily available high impulse fuel. Another result of this analysis includes the easy manipulations of these basic equations, using computer algorithms, to produce "back of the envelope" results and comparisons. These can be readily used in early satellite design formulation.

There are three recommendations or extensions to this analysis. One is to investigate other transfer model problems. Only one was analyzed here, and depending on the satellite's mission (ie, interplanetary probe, earth satellite), an investigation of the energy needed and time to transfer is essential in the satellite design. Second, other methods of transfer are worth investigating, especially when collaborating these with new propulsion techniques (ie. electric, ion propulsion). Third, an investigation of other effects that might detract from an energy or time efficient transfer, that make the problem more "real" (ie. remove the assumptions and perfection of this model) can be completed. This can answer the questions of what can be done to improve satellite technology and propulsion to overcome the limitations the space environment provides. Hopefully, besides providing the author and readers some insight into optimizing transfers, this paper will provide a guide on further research.

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Bibliography.

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2. Larsen, W.J., Wertz, J.R., eds., Space Mission Analysis and Design, Kluwer Academic Publishers, Norwell, MA. 1991.

APPENDIX A

```
% This fucntion reads an input of the radii of the initial and final circular
% orbits, as well as the inclination change needed for a Hohmann Transfer.  It
% change, which is done at the first thrust.  The rest of the thrust is done on
% the second thrust.  It plots the f corresponding to a delta-v.  A look at this
% will reveal the division of the plane change betweeen the burns that will give
% the minimum energy needed.
%
function [f,dv]=deltav(r1,r2,i)
v1c=(1/r1)^.5;                          % Circular velocities for initial and
v2c=(1/r2)^.5;                          % final orbits
a=(r1+r2)/2;
v1=((2/r1)-(1/a))^.5;                   % Velocity needed for transfer perigee
v2=((2/r2)-(1/a))^.5;                   % Velocity needed for transfer apogee
inc=(i/180)*pi;
f=[0.0:.001:.1]';
  dv1=((v1c^2+v1^2)-(2*v1c*v1*cos(f(1)*inc)))^.5;     % Thrusts needed for
  dv2=((v2c^2+v2^2)-(2*v2c*v2*cos((1-f(1))*inc)))^.5; % plane change and
%                                                        transfer
  dv=dv1+dv2;
for j=2:101
  dv2=((v2c^2+v2^2)-(2*v2c*v2*cos((1-f(j))*inc)))^.5;  % Iterating equations
  dv1=((v1c^2+v1^2)-(2*v1c*v1*cos(f(j)*inc)))^.5;  % above for f=0 to 1
  dv=[dv;dv1+dv2];        % total delta-v needed
end
A=[f,dv]
plot(f,dv)
```

APPENDIX B

```
% This is a MATLAB function which will determine the delta-v and time to transfe
% needed  for Case I of a Hyperbolic Transfer.  It accepts the initial and final
% circular orbit radii in canonical units and the inclination change needed.  It
% iterates the eccentricity and outputs the delta-v and time of flight (for each
% if needed).
%
function [A]=hyper1(r1,r2,i)
f=.076;                                % Using same value as found in Hohmann
e=[1.1:.1:5.0]';                       % Iterates eccentricity rom 1.1 to 5.0
i=i*pi/180.0;
v1c=sqrt(1/r1);
v2c=sqrt(1/r2);
for j=1:40
  v1h=v1c*sqrt(1+e(j));                 % Velocity desired to put in hyperbolic orbit
  v2h=sqrt((2/r2)-((1-e(j))/r1));  % velocity on transfer corresponding to r2
  dv1(j)=sqrt(v1c^2+v1h^2-2*v1c*v1h*cos(f*i));
  a=r1/(1-e(j));                        % Semi-major axis of transfer
  nu=acos((a*(1-e(j)^2))/(e(j)*r2)-(1/e(j)));  % position at r2 on transfer
  nud=nu*180/pi;
  oe=[a;e(j);nud;180.0;0.0;i-(f*i)];   % Inputs of transfer orbit parameters to
  [r,v]=oe2rv(oe);                      % a function to determine the postion and
  rv=cross(r,v);                        % velocity vectors in ijk at intersection
  rvm=norm(rv);                         % final orbit.  "Cross" is a function
  rm=norm(r);                           % allowing the cross multiplication of
  vm=norm(v);                           % vectors.
  fpa=acos(rvm/(rm*vm));                % Flight path angle
  dv2(j)=sqrt((v2h^2+v2c^2)-(2*v2h*v2c*cos(fpa))); % thrust needed for second bu
  dv3=2*v2c*sin(((1-f)*i)/2);            % Thrust needed for third burn
  dv(j)=dv1(j)+dv2(j)+dv3;               % Total Thrust
  F=acosh((e(j)+cos(nu))/(1+e(j)*cos(nu)));  % Hyperbolic Anomaly
  t1(j)=sqrt((-a)^3)*(e(j)*sinh(F)-F);     % time to transfer from 1st to 2nd b
  tp=2*pi*((r2)^1.5);
  t2(j)=tp*(pi-nu)/(2*pi);                 % Time to transfer from 2nd to 3rd b
  tof(j)=t1(j)+t2(j);                      % Total time to transfer
end
A=[e,dv',t1',t2',tof'];
plot(A(:,1),A(:,2))
```

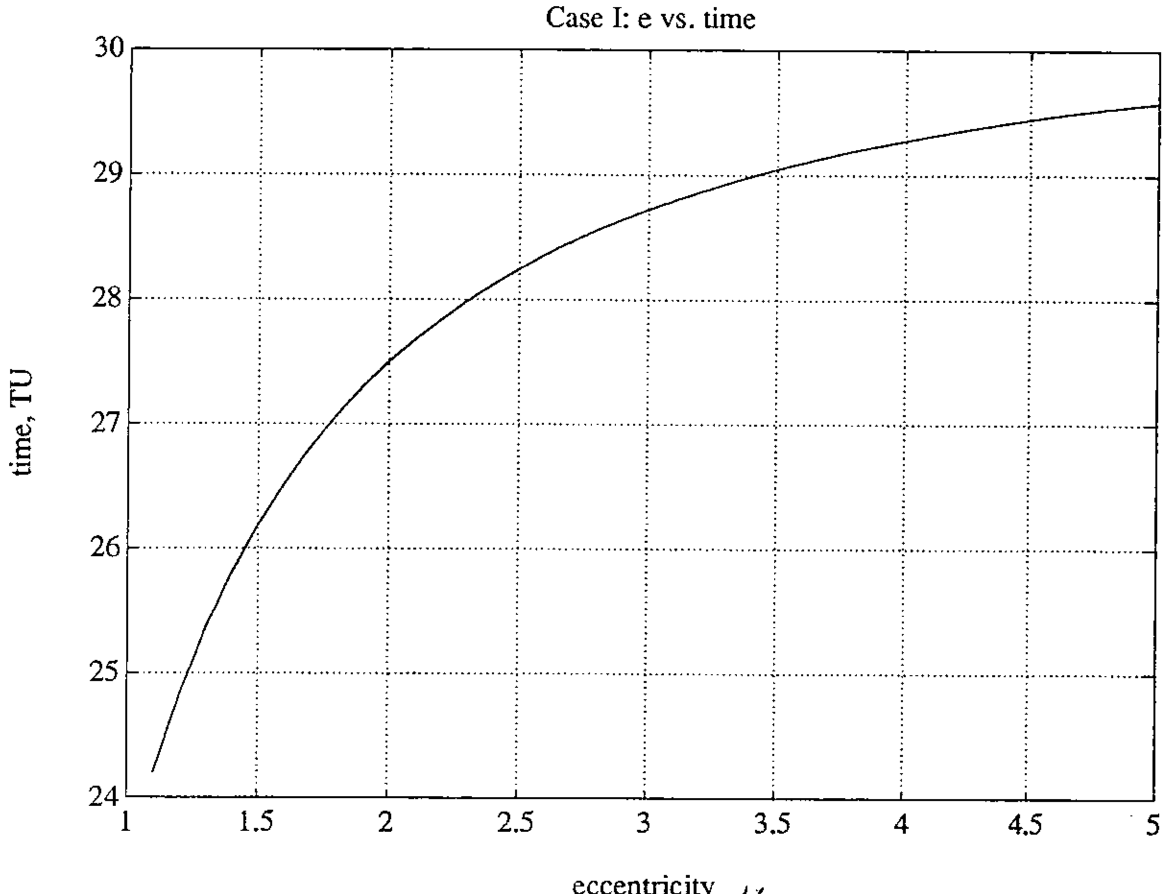

Figure B1: Hyperbolic, Case I; total time vs. eccentricity

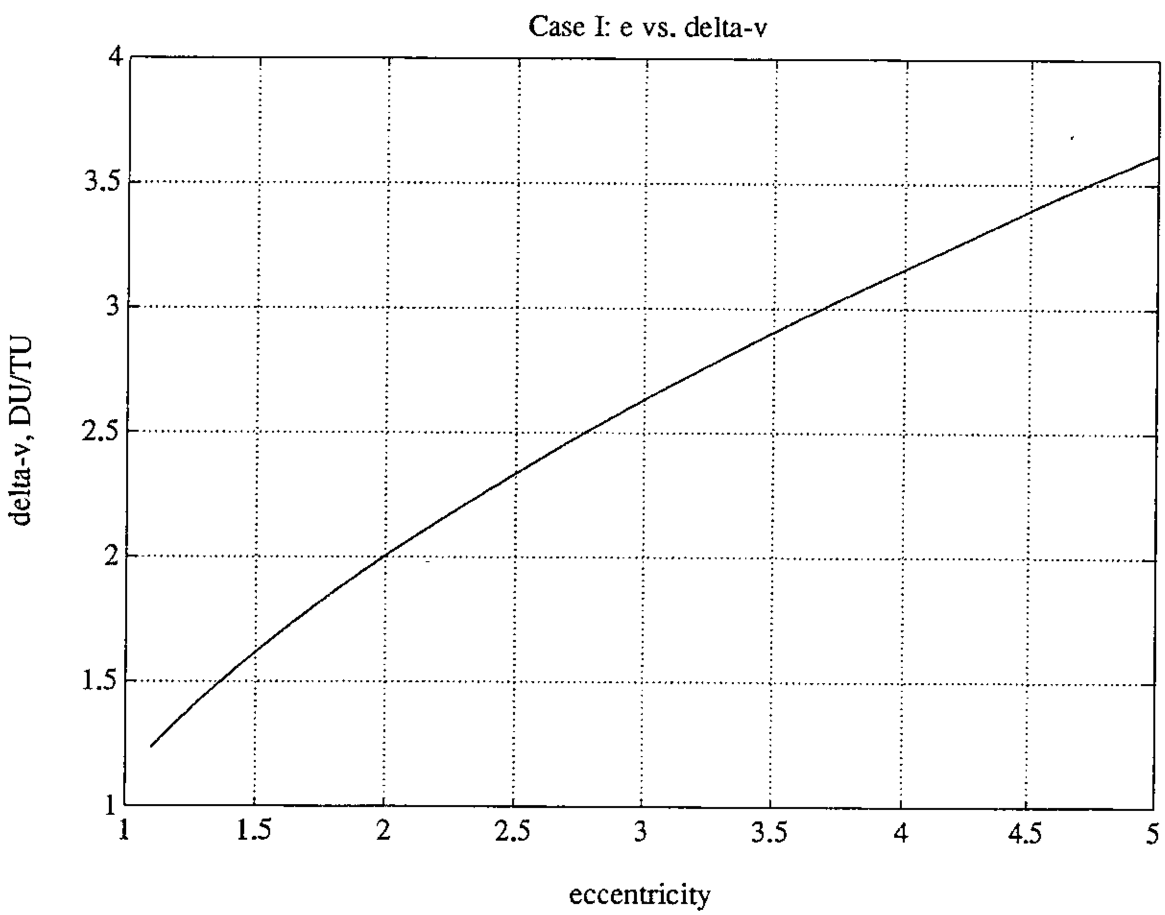

Figure B2: Hyperbolic, CaseI; total delta-v vs. eccentricity

APPENDIX C

```
 % This is a MATLAB function which will determine the delta-v and time to
% transfer needed for CAase II of a hyperbolic transfer.  It accepts the initial
% needed.  It iterates the eccentricity and outputs the delta-v and time of
% flight (for each thrust if needed).
%
function [A]=hyper2(r1,r2,i)
i=i*pi/180.0;
v1c=sqrt(1/r1);
v2c=sqrt(1/r2);
e=[1.1:.1:5.0]';                    % Iterates eccentricity from 1.1 to 5.0
for j=1:40
  v1h=v1c*sqrt(1+e(j));
  v2h=sqrt((2/r2)-((1-e(j))/r1));
  dv1(j)=sqrt(v1c^2+v1h^2-2*v1c*v1h*cos(i));  % Thrust needed for burn 1
  a = r1/(1-e(j));                            % Semi-major axis of transfer
  nu=acos((a*(1-e(j)^2))/(e(j)*r2)-(1/e(j))); % position of satellite at burn 2
  nud=nu*180/pi;
  oe=[a;e(j);nud;180.0;0.0;0];              % Inputs of transfer orbit parameters
  [r,v]=oe2rv(oe);                          % to a function to determine the
  rv=cross(r,v);                          % position and velocity vectors in ijk
  rvm=norm(rv);                           % at intersection of final orbit.
  vm=norm(v);                             % "Cross" is a function allowing the
  rm=norm(r);                             % cross-multiplication of vectors.
  fpa=acos(rvm/(rm*vm));                  % Flight path angle
  dv2(j)=sqrt((v2h^2+v2c^2)-(2*v2h*v2c*cos(fpa)));  % Thrust needed for burn 2
  dv(j)=dv1(j)+dv2(j);                              % Total thrust needed
  F=acosh((e(j)+cos(nu))/(1+e(j)*cos(nu)));
  tof(j)=sqrt((-a)^3)*(e(j)*sinh(F)-F);        % Total time to transfer
end
A=[e,dv1',dv2',dv',tof'];
plot(A(:,1),A(:,2),'-',A(:,1),A(:,3),'+')
```

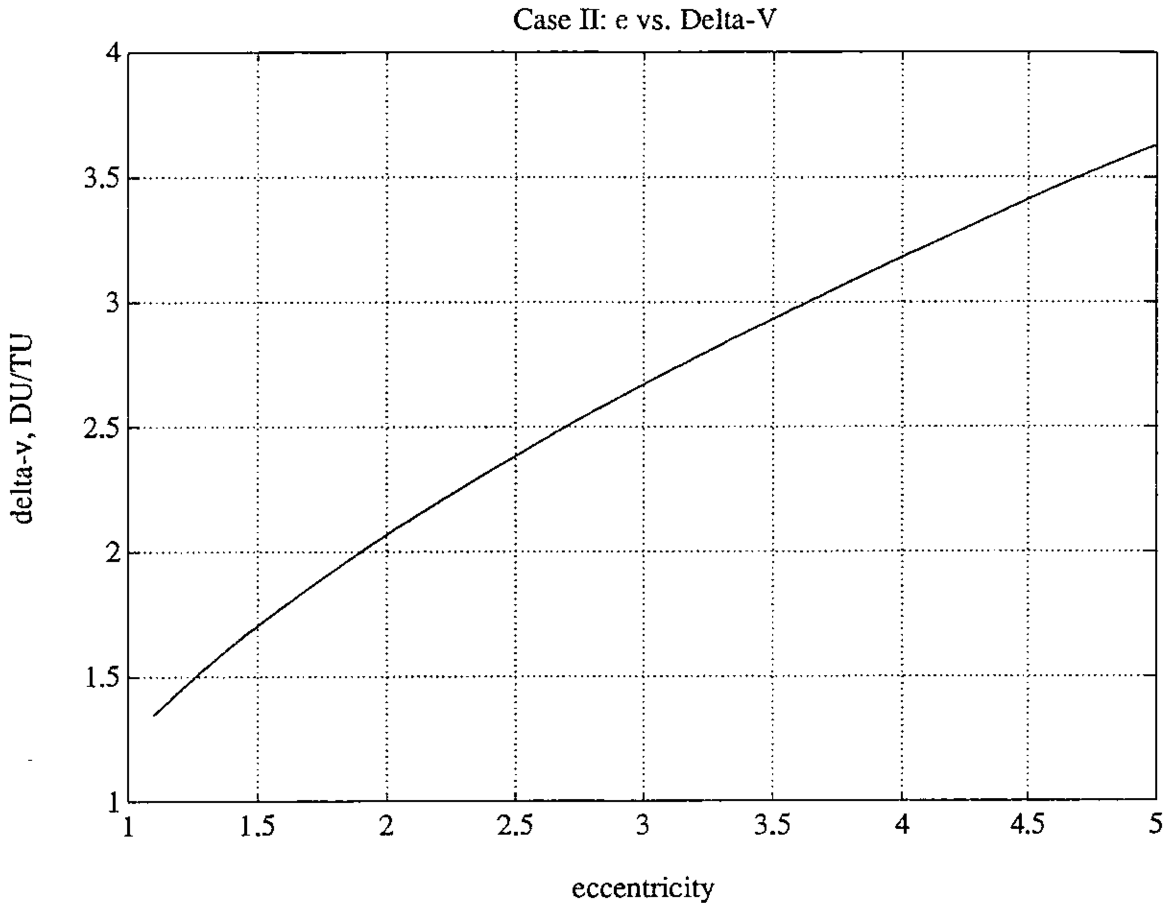

Figure C1: Hyperbolic, Case II; total delta-v vs. eccentricity

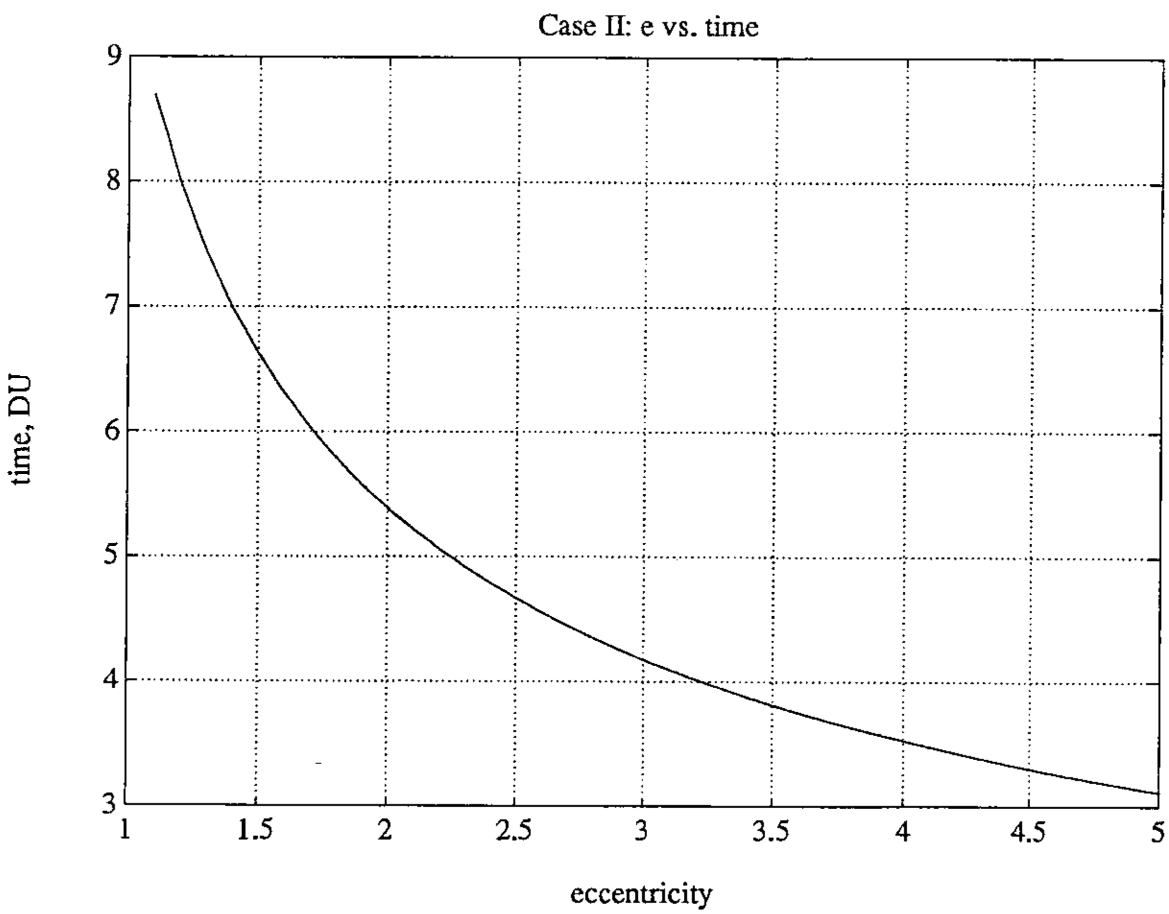

Figure C2: Hyperbolic, Case II; total time vs. eccentricity

APPENDIX D

```
% This is a MATLAB function which will determine the delta-v and time to
% transfer needed for Case III of a hyperbolic transfer.  It accepts the initial
% and final circular orbit  radii and the inclination change needed.  It
% iterates the eccentricity and outputs the delta-v and time of flight.
%
function[A]=hyper3(r1,r2,i)
e=[1.1:0.1:5.0]';
i=i*pi/180.0;
v1c=sqrt(1/r1);
v2c=sqrt(1/r2);
for j=1:40
  v1h=v1c*sqrt(1+e(j));                  % Velocity desired to put in hyperbolic orbit
  v2h=sqrt((2/r2)-((1-e(j))/r1)); % Velocity corresponding to r2
  dv1(j)=v1h-v1c;                        % Thrust needed for burn 1
  a=r1/(1-e(j));
  nu=acos((a*(1-e(j)^2))/(e(j)*r2)-(1/e(j)));  % Doing same thing as in
  nud(j)=nu*180/pi;                             % Case I and II here.
  oe=[a;e(j);nud(j);180.0;270.0;i];
  [r,v]=oe2rv(oe);
  vm=norm(v);
  v2cp=[sqrt(1/r2)*(-sin(nu));          % Calculating v2 in pqw coordinates, then
         0.0];                           % calculate the angle in between the v2 and
  rp2i=[0 -1 0;                          % final orbit velocity vector.
        1  0 0;
        0  0 1];
  v2ci=rp2i*v2cp;
  v2cm=norm(v2ci);
  al=acos((v'*v2ci)/(v2cm*vm));
  dv2(j)=sqrt((v2cm^2+vm^2)-(2*v2cm*vm*cos(al))); % Using the angle from above
  dv(j)=dv1(j)+dv2(j);                            % thrusts for burn 1 and 2
  F=acosh((e(j)+cos(nu))/(1+e(j)*cos(nu)));
  tof(j)=sqrt((-a)^3)*(e(j)*sinh(F)-F);          % Time of Flight
end
A=[e,dv1',dv2',dv',tof'];
plot(A(:,1),A(:,4))
```

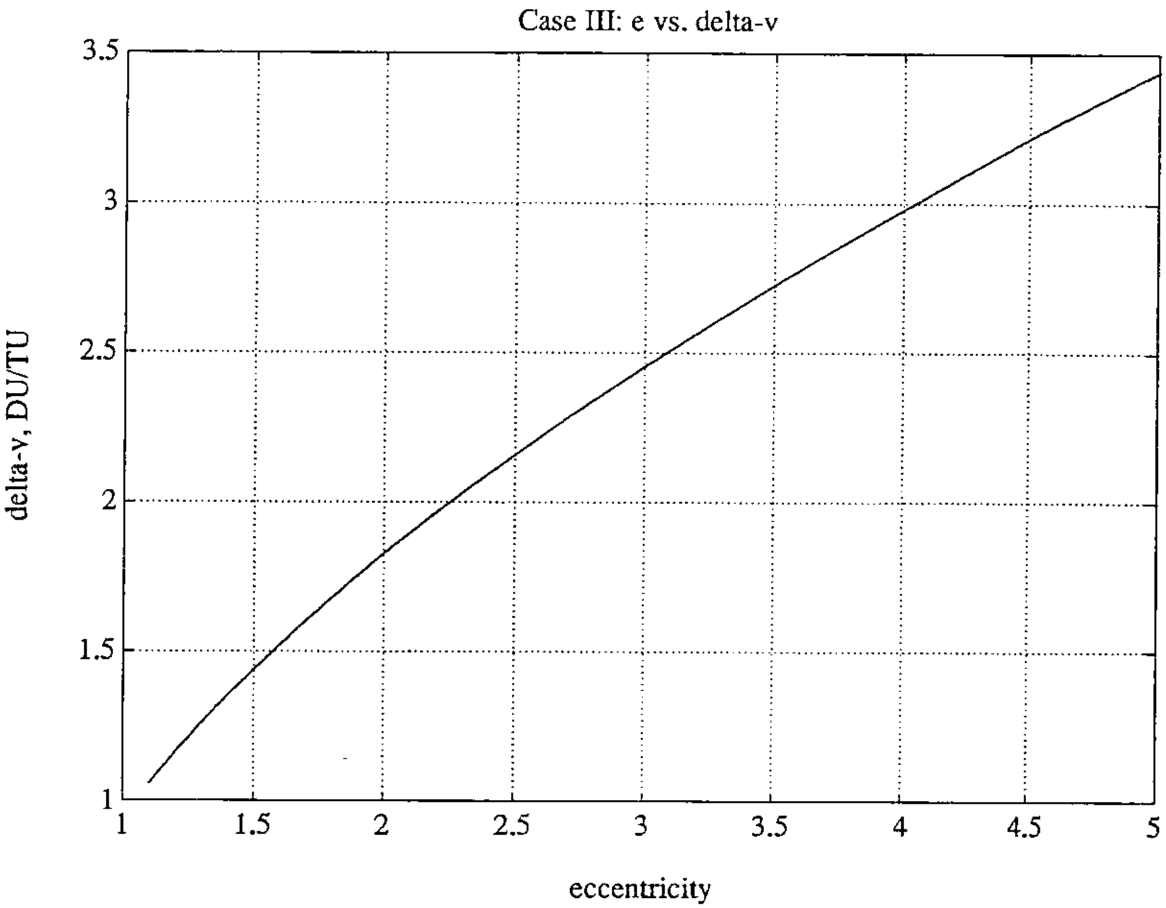

Figure D1: Hyperbolic, Case III; total delta-v vs. eccentricity

APPENDIX E

```
% This MATLAB function calculates the total delta-v and time of transfer, given
% an initial and final orbit radius, and inclination change for a Bi-elliptic
% Transfer.  It iterates at a transfer radius distance ( distance to the two
% transfers apogee) from 7 to 20 DU.  It also determines the best division of
% plane change burn, when these are done at the 2nd and 3rd burns.
%
function[A_]= bielip(r1,r2,i)
v1c=(1/r1)^.5;
v2c=(1/r2)^.5;
inc=(i/180)*pi;
rt=[7.0:.1:20.0]';              % Transfer radius iterated from 7 to 12
for k=1:131
  f=[0.90:.001:1.0]';           % Fraction iterated to find best division of
  for j=1:101                   % inclination change to give minimum delta-v
    at1=(r1+rt(k))/2;            % Semi-major axis of 1st transfer
    at2=(r2+rt(k))/2;            % Semi-major axis of 2nd transfer
    vpt1=sqrt(2*(1/r1-1/(r1+rt(k))));  % perigee velocity of 1st transfer
    vat1=sqrt(2*(1/rt(k)-1/(r1+rt(k)))); % apogee velocity of 1st transfer
    vat2=sqrt(2*(1/rt(k)-1/(r2+rt(k))));  % apogee velocity of 2nd transfer
    vpt2=sqrt(2*(1/r2-1/(r2+rt(k))));     % perigee velocity of 2nd transfer
    dv1(j)=((v1c^2+vpt1^2)-(2*v1c*vpt1))^.5; % thrust needed for 1st burn
    dv2(j)=((vat1^2+vat2^2)-(2*vat1*vat2*cos(f(j)*inc)))^.5; %thrust for burn 2
    dv3(j)=((v2c^2+vpt2^2)-(2*v2c*vpt2*cos((1-f(j))*inc)))^.5; %thrust for burn
    dv(j)=dv1(j)+dv2(j)+dv3(j);       % total thrust needed
    if j>1
      if dv(j)<dv(j-1)                % If statement, used to find minimum delta-
         fstar(k)=f(j);               % each transfer radii and the corresponding
      end
    end
    tof(j)=pi*(sqrt(at1^3)+sqrt(at2^3));     % time of flight
  end
  transfr=rt(k);
end
A=[rt,fstar'];
plot(A(:,1),A(:,2))
```