OPTIMAL CONTROL OF A SOLAR SAIL SPACECRAFT FOR INTERPLANETARY MISSIONS

The theory of optimal control is applied to obtain minimum–time trajectories for solar sail spacecraft for interplanetary missions. We consider the gravitational force and torque due to the Sun as well as the solar radiation force and torque considering effects such as photon absorption, specular and diffuse reflection. The spacecraft is symmetrically modelled as a point mass $m_p$ rigidly connected to a flat, square sail of mass $m_s$. Coplanar circular orbits are assumed for the planets. We solve the optimal control problem via an indirect method using an efficient algorithm based on a cascaded computational scheme. The global optimizer is based on a technique called Adaptive Simulated Annealing. A Quasi–Newton Method performs the terminal fine tuning of the optimization parameters.

INTRODUCTION

The concept of using solar radiation pressure as a means of propulsion for space vehicles was first introduced in the 1920s by K. Tsiolkovsky\textsuperscript{1} and F. Tsander.\textsuperscript{2} About fifty years later, the effect of solar radiation on spacecraft attitude dynamics was first experienced with the Mariner 10 mission to Mercury and Venus. Mariner 10 was also the first spacecraft to use a gravity assist trajectory, accelerating as it entered the gravitational influence of Venus, then being flung by the planet’s gravity onto a slightly different course to reach Mercury. Since then, there have been several attempts to realize a solar sail mission. In his book Wright\textsuperscript{3} presents a detailed analysis on some possible solar sail applications. During his time at JPL Wright was actively involved in the planning of a rendezvous mission to comet Halley using solar sail technology. Unfortunately, in 1977 a solar electric propulsion concept was selected instead, primarily because of technology maturity. Not long thereafter, the Halley rendezvous mission was dropped by NASA.

During the development of the solar sail spaceflight concept numerous studies of the associated dynamics problem were presented. The first solar sail trajectories were calculated by Tsu\textsuperscript{4} and London.\textsuperscript{5} Tsu investigated various means of propulsion and showed...
that in many cases solar sails show superior performance when compared to chemical and ion propulsion systems. The author used approximated heliocentric motion equations to obtain spiraling trajectories for a “...fixed sail setting”. London presented similar spiral solutions for Earth–Mars transfers with constant sail orientation using the exact equations of motion. Optimal solar sail trajectories were first computed by Zhukov and Lebedev\textsuperscript{6} for interplanetary missions between coplanar circular orbits. In 1980 Jayaraman\textsuperscript{7} published similar minimum–time trajectories for transfers between the Earth and Mars. Two years later, Wood et al.\textsuperscript{8} presented an analytical proof to show that the orbital transfer times obtained in Ref. 7 were incorrect due to the incorrect application of a transversality condition of variational calculus and an erroneous control law. Interestingly, about two decades later Powers et al.\textsuperscript{9,10} obtained results identical to those reported in Wood’s paper with the same incorrect control law used in Ref. 7. The more general time–optimal control problem of three–dimensional, inclined and elliptic departure and rendezvous planet orbits was discussed by Sauer.\textsuperscript{11}

In the literature the optimal control of solar sail spacecraft is traditionally treated as a purely orbital control problem with the sail orientation angle as the control input and is “...not concerned with modeling the dynamics of rotation...”\textsuperscript{3} of the spacecraft. In this paper we use a two–step approach to attack the more general problem of controlling the orientation angle by a control force, or equivalently, the corresponding control torque. We begin with the development of an appropriate solar radiation pressure model. In the following sections we present the system model and the corresponding motion equations. Subsequently, we define the minimum–time optimal control problem for a simplified and a complex version of the optimal control problem choosing as the control inputs the sail orientation angle and the control force (torque), respectively.

**SOLAR RADIATION PRESSURE MODEL**

Gravity gradient and aerodynamic torques are the dominant torques on spacecraft in LEO. For spacecraft in GEO, HEO or interplanetary trajectories the torque resulting from the solar radiation pressure force is the major long–term disturbance. The solar radiation pressure forces are due to photons $\gamma$ impinging on the spacecraft surface, for example, the solar sail. If a fraction, $\gamma_a$, of the interacting photons is absorbed, a fraction, $\gamma_s$, is specularly reflected, and a fraction, $\gamma_d$, is diffusely reflected, then by mass conservation

$$\gamma_a + \gamma_s + \gamma_d = 1$$  

The radiation forces due to absorption, specular and diffuse reflection can be written as\textsuperscript{3}

$$f_a = \frac{\gamma_a AP}{\rho^2} (b_x S) S, \quad f_s = \frac{2\gamma_s AP}{\rho^2} (b_x S)^2 b_x, \quad f_d = \frac{\gamma_d AP}{\rho^2} (b_x S) (S + 2b_x/3)$$  

where $A$ is the solar sail surface area, $P = 4.563 \times 10^{-6} \text{ N/m}^2$ (Ref. 3) is the nominal solar radiation pressure constant at 1 AU, $\rho \triangleq \|r\|/(1 \text{ AU})$ is the normalized distance, $b_x$ is the
unit vector defining the symmetry axis in the body frame, and $\mathbf{S}^\dagger$ is the unit vector pointing from the Sun center to the spacecraft as illustrated in Figure 2. The total solar radiation pressure force may then be written as

$$f_\gamma = f_a + f_s + f_d = \frac{AP}{\rho^2} (b_T^T \mathbf{S}) \left\{ (1 - \gamma_s)\mathbf{S} + \left[ 2\gamma_s (b_T^T \mathbf{S}) + 2\gamma_d/3 \right] b_x \right\}$$

which simplifies to

$$f_\gamma = \frac{AP}{\rho^2} \cos \alpha \left\{ (1 - \gamma_s)\mathbf{S} + \left[ 2\gamma_s \cos \alpha + 2\gamma_d/3 \right] \mathbf{S} \cos \alpha + \mathbf{S}_\perp \sin \alpha \right\}$$

$$= \frac{AP}{\rho^2} \cos \alpha \left\{ (1 - \gamma_s + \cos \alpha (2\gamma_s \cos \alpha + 2\gamma_d/3)) \mathbf{S} \right. \right.$$ 

$$+ \sin \alpha (2\gamma_s \cos \alpha + 2\gamma_d/3) \mathbf{S}_\perp \right\}$$

$$= \Delta f_\gamma^\mathbf{S} \mathbf{S} + f_\gamma^\mathbf{S}_\perp \mathbf{S}_\perp \right.$$

observing that $(b_T^T \mathbf{S}) = \cos \alpha$ and introducing $\mathbf{S}_\perp$ as the unit vector orthogonal to $\mathbf{S}$ such that \{b_x, b_y\} = $R(\alpha)\{\mathbf{S}, \mathbf{S}_\perp\}$ where $R(\alpha)$ is the rotation matrix that rotates the \{\mathbf{S}, \mathbf{S}_\perp\} frame into the body frame. For the corresponding torque $g_\gamma$ we only take into account the effects of the solar radiation pressure on the sail. With the surface area of other parts of the spacecraft being negligibly small, $g_\gamma$ results

$$g_\gamma = d_s \left( -\sin \alpha f_\gamma^\mathbf{S} + \cos \alpha f_\gamma^\mathbf{S}_\perp \right) = d_s \frac{AP}{\rho^2} \cos \alpha \sin \alpha (1 - \gamma_s)$$

where $d_s$ is the distance between the spacecraft barycenter $O$ and the center of solar radiation pressure of the sail $\mathcal{C}P$. Note that we assume the solar sail to be perfectly flat and a homogeneous solar radiation pressure distribution over the entire sail surface. For the force model the radiation torque does not depend on $\gamma_a$ and is equal to zero for a perfectly reflective solar sail where $\gamma_s = 1$.

**CONTROL SYSTEM DESIGN CONSIDERATIONS**

One of the most important aspects of formulating an optimal control problem is choosing a particular set of control variables. An imprudent decision in that respect can lead overly complicated and even ill–posed problem statements. Figure 1 illustrates several feasible control system designs. Probably the most straightforward way to control the attitude of the spacecraft is to apply a control torque $g_u$ about the center of mass $O$. Applying a control force $f_u$ instead slightly complicates the analysis since the force appears in the orbital equations, as well. However, note that in both cases the control variable appears linearly in the motion equations which yields to bang–type control laws which in turn could

\[\text{We point out that even though this notation seems to be standard in engineering it should not be confused with its counterpart in the sciences the so-called Poynting vector } \mathbf{S}, \text{ which is defined in cgs units as } \mathbf{S} = \frac{c^4}{4\pi} \mathbf{E} \times \mathbf{B}. \text{ } \mathbf{E} \text{ and } \mathbf{B} \text{ are the electric and magnetic fields, respectively. The Poynting vector gives the energy flux associated with the electromagnetic wave, that is, the two definitions of } \mathbf{S} \text{ discussed here basically differ only by their magnitudes!}\]
potentially render the system uncontrollable for a plain minimum–time control problem. Using a combination of control force and torque or the implementation of symmetrically placed control vanes at the solar sail tips with control force $f_1$ and $f_2$ are possible back doors. One way to avoid the usage of thrusters is to take advantage of the solar radiation pressure via control panels and choosing as the control variables the panel deflection angles $\zeta_1$ and $\zeta_2$. The major drawback of this design probably concerns the structural integrity of the spacecraft. Furthermore, the achievable control torques are not only rather small but are also a function of the distance of the spacecraft from the Sun. An elegant approach to control the attitude of the spacecraft considers control masses which are displaced symmetrically from the center of mass by a distance $\Delta(t)$. Note that the moments of Inertia $I_1(\Delta(t)) = I_2(\Delta(t))$ are functions of the control mass displacement and the gravity gradient torque is $g_G \propto (I_1(\Delta(t)) - I_3)$. Therefore, to be effectively controllable the spacecraft would have to be designed about an inertially symmetric operating point $\Delta_{\text{ref}}$ such that an increase (decrease) of $\Delta(t)$ results a negative (positive) torque $g_G$ for a corresponding fixed orientation angle. Similar to the control panel concept the control mass approach suffers from a strongly varying control effectiveness ($\propto 1/r^3$). Also the structural maturity of the design is a major concern not to mention the increased spacecraft mass.

**Figure 1:** Spacecraft control system design comparison.

**SYSTEM MODEL AND MOTION EQUATIONS**

We consider a symmetric spacecraft system as illustrated in Figure 2. The payload is modelled as a point mass $m_p$ connected rigidly to a perfectly flat, square solar sail of mass $m_s$ and surface area $A$. We define the body–fixed reference frame $\{b_x, b_y\}$ with the $b_x$ axis identifying the system symmetry axis and passing through the spacecraft system barycenter $O$ and the center of pressure $CP$ of the solar sail. The distance between $O$ and $CP$ is $d_s$. The sail orientation angle $\alpha$ is defined as the angle between the sail normal ($= b_x$) and the solar flux direction $S$. A positive angle rotates the $b_x$ anti–clockwise into $S$. The sail orientation angle is controlled via a control force $f_u$ – or equivalently – the corresponding control torque $g_u$. For convenience, we define $S_\perp$ as the unit vector orthogonal to $S$ such that $S_\perp$ and $b_y$ are aligned for $\alpha = 0$. The environmental forces and torques acting on the
spacecraft system are due to the gravitational field and the solar radiation of the Sun using the solar radiation pressure model described in the previous section. The initial and target spacecraft trajectories are modelled as heliocentric, circular, and coplanar orbits.

![Image of spacecraft and solar radiation](image.png)

Figure 2: Interplanetary missions using solar sail spacecraft.

We define the generalized coordinate vector as \( \mathbf{r} \equiv (r, \theta)^T \) and the corresponding velocity vector as \( \mathbf{v} \equiv (v_r, v_\theta)^T \), where \( v_r = \dot{r} \) and \( v_\theta = r \dot{\theta} \). In the inertial reference frame the velocity and acceleration of the spacecraft are obtained as

\[
\mathbf{v} = \mathbf{i} \mathbf{S} + r \dot{\theta} \mathbf{S}_\perp
\]

\[
\mathbf{a} = (\ddot{r} - r \dot{\theta}^2) \mathbf{S} + (2\dot{r} \dot{\theta} + r \ddot{\theta}) \mathbf{S}_\perp = (\dot{v}_r - \frac{v_\theta^2}{r}) \mathbf{S} + (v_r \dot{v}_\theta / r + \dot{v}_\theta) \mathbf{S}_\perp
\]

With the angular rate of the spacecraft defined as \( \omega \equiv \dot{\alpha} \) and using the solar radiation pressure model introduced in the previous section we obtain for the dimensional motion equations

\[
\dot{v}_r = \ddot{v}_r
\]

\[
\dot{\theta} = \frac{\ddot{v}_\theta}{r}
\]

\[
\ddot{v}_r = \dddot{v}_r / m + \dddot{v}_\theta^2 / r - \ddot{\mu} / r^2
\]

\[
\ddot{v}_\theta = \dddot{v}_\theta / m - \dddot{v}_r \ddot{v}_\theta / r
\]

\[
\dot{\omega} = \frac{-3\dddot{\mu}(\dddot{I}_1 - \dddot{I}_3) \cos \dddot{\alpha} \sin \dddot{\alpha} / \rho^2 + \dddot{g}_\gamma + \dddot{g}_u}{\dddot{I}_2}
\]

Note that we temporarily introduced the \( \dddot{\cdot} \) notation and redefined the dimensional time derivative as \( d/dt (\cdot) \equiv (\cdot)' \) to simplify the subsequent nondimensional analysis. In equations (11–16) \( \dddot{m} = \dddot{m}_s + \dddot{m}_p \) is the total spacecraft mass, \( \dddot{\mu} \) is the gravitational constant of
the Sun, and the moments of inertia $I_i$ are

$$I_1 = I_2 = \tilde{m}_s \tilde{A}/12 + \tilde{m}_s \tilde{d}_s^2 + \tilde{m}_p \tilde{d}_p^2 \quad \text{and} \quad I_3 = \tilde{m}_s \tilde{A}/6$$  \hspace{1cm} (17)

Equations (11–16) represent the motion equations for a solar sail spacecraft subject to environmental forces and torques due to solar gravity and radiation. The attitude dynamics of the system is described by equations (15,16), the first term on the right-hand side of equation (16) is the gravity gradient torque as derived in Ref. 12.

Due to the numerical complexity of the optimization problem we restrict the analysis in this paper to the case of a perfectly reflective solar sail with $\gamma_s = 1$. Once an optimal solution has been identified for this limiting case the simplification can be removed by introducing a bookkeeping parameter $\epsilon \in [0,1]$ such that $\epsilon = 0$ corresponds to $\gamma_s = 1$. By letting $\epsilon \to 1$ optimal solutions can then be obtained for the general case of a non-optimal solar sail with $\gamma_s < 1$.

To nondimensionalize the motion equations we choose the Sun–Earth distance as the distance unit, that is, 1 DU = 1 AU. The time unit 1 TU is chosen such that $\tilde{\mu} \frac{TU^2}{AU^3} = 1$.

The dimensionless parameters $\beta$ and $\sigma$ are defined as follows:

$$2 \tilde{A} \tilde{P}/\tilde{m} = \tilde{\beta} \triangleq \beta \text{AU}/\text{TU}^2 \quad \text{and} \quad -3 \tilde{\mu} (\tilde{I}_1 - \tilde{I}_3)/\tilde{I}_2 = \tilde{\sigma} \triangleq \sigma \text{AU}^3/\text{TU}^2$$  \hspace{1cm} (18)

The nondimensional motion equations for a perfectly reflective solar sail spacecraft then result

$$\begin{align*}
\dot{r} &= v_r \\
\dot{\theta} &= v_{\theta}/r \\
\dot{v}_r &= \beta \cos^3 \alpha/r^2 + v_{\theta}^2/r - 1/r^2 \\
\dot{v}_{\theta} &= \beta \sin \alpha \cos^2 \alpha/r^2 - v_r v_{\theta}/r \\
\dot{\alpha} &= \omega \\
\dot{\omega} &= \sigma \sin \alpha \cos \alpha/r^3 + g_u
\end{align*}$$  \hspace{1cm} (19–22)

Note that the angle $\theta$ is an ignorable coordinate and can therefore be eliminated from the analysis. The motion equations can be rewritten in condensed form as $\dot{x} = f(x)$ where the state vector is defined as $x = (r, \theta, v_r, v_{\theta}, \alpha, \omega)^T$. The normalized boundary conditions for the system (19–22) are

$$\begin{align*}
r(t_0) &= 0 \quad \theta(t_0) = \text{free} \quad v_r(t_0) = 0 \quad v_{\theta}(t_0) = 1 \\
r(t_f) &= r_f \quad \theta(t_f) = \text{free} \quad v_r(t_f) = 0 \quad v_{\theta}(t_f) = 1/\sqrt{r_f}
\end{align*}$$  \hspace{1cm} (25–26)

and for the equations defining the attitude dynamics

$$\begin{align*}
\alpha(t_0) &= \pi/2 \quad \omega(t_0) = 0 \quad \alpha(t_0) = \alpha_{0\text{opt}} \quad \omega(t_0) = 0 \\
\alpha(t_f) &= \pi/2 \quad \omega(t_f) = 0 \quad \alpha(t_f) = \alpha_{f\text{opt}} / \text{free} \quad \omega(t_f) = 0 / \text{free}
\end{align*}$$  \hspace{1cm} (27)

The first set of boundary conditions in equation (27) seems to be a natural way to set up the optimal control problem for a transfer between coplanar circular orbits. However, a different
set of boundary conditions might provide more valuable information when comparing the transfer characteristics of the spacecraft for the two cases when attitude dynamics is taken into account and when only the orbital problem is considered. That is, using the optimal initial and terminal control angles $\alpha_{0}^{\text{opt}}$ and $\alpha_{f}^{\text{opt}}$ as obtained from the orbital control problem offers a fair approach to determine the effect of the attitude dynamics on, for example, the minimum transfer time. Alternatively, it might prove advantageous not to prescribe the final orientation angle and/or angular velocity. A matter of common knowledge, complex performance indices significantly complicate the numerical analysis and therefore we chose as the constraint at $t = t_{f}$

$$\psi(x(t_{f}), t_{f}) = (r(t_{f}) - r_{f}, v_{r}(t_{f}), v_{\theta}(t_{f}) - 1/\sqrt{r_{f}})^{T} = 0$$

(28)

and allow $\alpha(t_{f})$ and $\omega(t_{f})$ to vary freely. In the next section we formulate the optimal control problem for a reduced and the full system model.

**OPTIMAL CONTROL PROBLEM FORMULATION**

The optimal control problem is to find an optimal control input $u^{*}$ for a generally nonlinear system $\dot{x} = f(x, u, t; p)$ such that the associated performance index

$$J = \phi(x(t_{f}), t_{f}) + \int_{t_{0}}^{t_{f}} L(x, u, t; p) dt$$

(29)

is minimized, and such that the constraint at final time $t_{f}$

$$\psi(x(t_{f}), t_{f}) = 0$$

(30)

is satisfied. In equations (29,30) $x$ is the $n$–dimensional state vector, $u$ is the $m$–dimensional control input, $p$ is the $k$-dimensional parameter vector, and $\phi$ and $L$ are the final and intermediate weighting functions, respectively. Instead of solving a constrained optimization problem it is usually advantageous to consider the corresponding unconstrained optimization problem using the augmented performance index

$$J^{+} = \phi(x(t_{f}), t_{f}) + \nu^{T} \psi(x(t_{f}), t_{f}) \int_{t_{0}}^{t_{f}} \{ L(x, u, t; p) + \lambda^{T} (f(x, u, t; p) - \dot{x}) \} dt$$

(31)

Defining the *Hamiltonian function* $\mathcal{H}$ as

$$\mathcal{H} = L(x, u, t; p) + \lambda^{T} f(x, u, t; p)$$

(32)

the state and costate equations are obtained as

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial \lambda} = f(x, u, t; p) \quad \text{and} \quad \dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x} = -\frac{\partial L(x, u, t; p)}{\partial x} - \frac{\partial f(x, u, t; p)}{\partial x} \lambda$$

(33)

In the following we present the optimality conditions for the two different system models developed in the previous section. The *reduced system model* considers only the orbital
dynamics of the spacecraft and is described by equations (19–22). The control variable for
the reduced system model is the solar sail orientation angle $\alpha$. The full system model also
accounts for the spacecraft attitude dynamics which is governed by equations (23,24). To
simplify the numerical analysis a control torque $g_u$ rather than a control force $f_u$ is chosen
as the control input. As pointed out previously we are only interested in the case where
$\gamma_s = 1$, that is, the perfectly reflective solar sail.

Case 1: Optimality Conditions for the Reduced System Model

For the minimum–time transfer problem using the Lagrange formulation the Hamiltonian
for the reduced system model is given by

$$H_r = \lambda_1 v_r + \lambda_3 \left( \beta \cos^3 \alpha/r^2 + v_\theta^2/r - 1/r^2 \right) + \lambda_4 \left( \beta \sin \alpha \cos^2 \alpha/r^2 - v_r v_\theta/r \right)$$

(34)

The corresponding costate equations are defined by $\dot{\lambda} = -\partial H_r/\partial x$ and are given as

$$\dot{\lambda}_1 = \lambda_3 \left( 2 \beta \cos^3 \alpha/r^3 + v_\theta^2/r^2 - 2/r^3 \right) + \lambda_4 \left( 2 \beta \sin \alpha \cos^2 \alpha/r^3 - v_r v_\theta/r^2 \right)$$

(35)

$$\lambda_2 = \text{const.} = 0$$

(36)

$$\dot{\lambda}_3 = -\lambda_1 + \lambda_4 v_\theta/r$$

(37)

$$\dot{\lambda}_4 = -2\lambda_3 v_\theta/r + \lambda_4 v_r/r$$

(38)

Applying Pontryagin’s Minimum Principle,13 the optimal control $u^* \equiv \alpha^*$ is chosen such
that the Hamiltonian $H \equiv H_r$ is minimized, that is,

$$u^* = \arg \min_{u \in U} H (x^*, \lambda^*, u), \quad \forall t \geq 0$$

(39)

where $x^*$ and $\lambda^*$ denote the optimal state and costate vector. Therefore the stationary
condition yields

$$\partial H_r/\partial \alpha = 0 = -3\lambda_3 \beta \sin \alpha \cos^2 \alpha/r^2 + \lambda_4 \beta \left( \cos^3 \alpha - 2 \sin^2 \alpha \cos \alpha \right)/r^2$$

(40)

which is satisfied if

$$\begin{cases}
\cos \alpha^* = 0 \\
\cos \alpha^* \neq 0 \quad \text{and} \quad \tan^2 \alpha^* + \frac{3\lambda_3}{2\lambda_4} \tan \alpha^* - \frac{1}{2} = 0
\end{cases}$$

(41)

The optimal control angle $\alpha^*$ which minimizes the Hamiltonian in equation (34) is given by

$$\alpha^* = \begin{cases}
\tan^{-1} \left\{ \left( -3\lambda_3 - \sqrt{9\lambda_3^2 + 8\lambda_4^2} \right)/(4\lambda_4) \right\} & \text{if } \lambda_4 \neq 0 \\
0 & \text{if } \lambda_4 = 0, \lambda_3 < 0 \\
\pm \pi/2 & \text{if } \lambda_4 = 0, \lambda_3 > 0
\end{cases}$$

(42)

The control law (42) depends on the unknown costates $\lambda_1, \lambda_3$ and $\lambda_4$. The standard ap-
proach in the literature to solve the optimization problem at hand has been to estimate
independently the initial values of the costates (and the transfer time). There is, however,
an intimate relationship between two of the costates, $\lambda_3$ and $\lambda_4$, at $t = t_0$ which can be exploited to great advantage to simplify drastically the optimization problem and therefore increase both the radius and rate of convergence. Note, that the final time $t_f$ is unspecified; therefore, the transversality condition $H_r(t_f) = -1$ is satisfied. Moreover, since the motion equations are autonomous

\[ \frac{d}{dt} H_r = \frac{\partial}{\partial t} H_r = 0 \quad \Rightarrow \quad H_r = \text{const.} = -1 \]  

(43)

As a result, the Hamiltonian at $t_0$ can be written using equations (34,43) and the boundary conditions (25) as

\[ H_r(t_0) = -1 = \{ \lambda_3 \beta \cos^3 \alpha^* + \lambda_4 \beta \sin \alpha^* \cos^2 \alpha^* \} \big|_{t=t_0} \]  

(44)

which can be rewritten cumbersomely as

\[ h(\alpha^*, \lambda_3, \lambda_4) = 0 \]

where $h(\alpha^*, \lambda_3, \lambda_4) = 0$ is satisfied. Moreover, since the motion equations are autonomous, the Hamiltonian at $t_0$ can be written using equations (34,43) and the boundary conditions (25) as

\[ H_r(t_0) = -1 = \{ \lambda_3 \beta \cos^3 \alpha^* + \lambda_4 \beta \sin \alpha^* \cos^2 \alpha^* \} \big|_{t=t_0} \]  

(44)

which can be rewritten cumbersomely as

\[ h(\alpha^*, \lambda_3, \lambda_4) = 0 \]

(45)

Substituting the control law (42) for $\lambda_4(t_0)$ for a given $\lambda_4(t_0)$ into equation (45) finally yields an equation of the form $h(\lambda_3(t_0), \lambda_4(t_0)) = 0$ which can be solved numerically for $\lambda_4(t_0)$ for a given $\lambda_4(t_0)$. The reduction in dimensionality has proved to be extremely efficient, particularly due to the good convergence characteristics (radius and rate) of the zero–point problem $\hat{h} = 0$.

Case 2: Optimality Conditions for the Full System Model

For the full system model the Hamiltonian is defined by

\[ H_f = H_r + \lambda_5 \omega + \lambda_6 \left( \sigma \sin \alpha \cos \alpha / r^3 + g_u \right) \]  

\[ + \kappa g_u^2 \]  

(46)

Differentiating $H_f$ with respect to the state vector we obtain for the costate equations

\[ \dot{\lambda}_1 = \lambda_3 \left( 2 \beta \cos^3 \alpha / r^3 + v_6^2 / r^2 - 2 / r^3 \right) + \lambda_4 \left( 2 \beta \sin \alpha \cos^2 \alpha / r^3 - v_r v_6 / r^2 \right) \]  

(47)

\[ + \lambda_6 \left( 3 \sigma \cos \alpha \sin \alpha / r^4 \right) \]  

(48)

\[ \lambda_2 = \text{const.} = 0 \]  

(49)

\[ \dot{\lambda}_3 = - \lambda_1 + \lambda_4 v_6 / r \]  

(50)

\[ \dot{\lambda}_4 = - 2 \lambda_3 v_6 / r + \lambda_4 v_r / r \]  

(51)

\[ \dot{\lambda}_5 = 3 \lambda_3 \beta \sin \alpha \cos^2 \alpha / r^2 - \lambda_4 \beta \left( 3 \cos^3 \alpha - 2 \cos \alpha \right) / r^2 - \lambda_6 \sigma \cos 2 \alpha / r^3 \]  

(52)

\[ \dot{\lambda}_6 = - \lambda_5 \]  

(53)

Note that in equation (46) we have formulated two different minimization problems. The term in square brackets in the Hamiltonian generalizes the “pure” minimum–time problem to a minimum–time minimum–cost control problem. The weighing constant $\kappa$ allows one
to penalize excessive control cost relative to increased transfer time. Also, by choosing \( \kappa \ll 1 \) the minimum–time problem can easily be recovered. As a matter of fact, the bang–type control we obtain for the “pure” minimum–time problem renders the spacecraft uncontrollable (this conjecture requires a solid mathematical proof which seems to be highly nontrivial). Therefore, the minimum–time problem can only be solved using the more general approach choosing \( \kappa \) appropriately.

Omitting the control cost term the control torque appears only linearly in the Hamiltonian and Pontryagin’s Minimum Principle yields for the optimal control law and logic

\[
g_u^* = \begin{cases} 
g_u^{\text{max}} & \text{if } \lambda_6 < 0 \\
g_u^{\text{min}} & \text{if } \lambda_6 > 0 \\
\text{singular} & \text{if } \lambda_6 = 0
\end{cases}
\]  

(54)

Standard minimum–time problem

For the combined minimum–time minimum control cost problem the optimal control law results

\[
g_u^* = -\lambda_6/(2\kappa) \quad \text{Minimum–time minimum–cost problem}
\]  

(55)

which is considerably less complex than the control law (54).

As for the reduced system model the transversality condition for the full system model yields \( \mathcal{H}_f(t_f) = -1 \) and since the Hamiltonian is not explicitly dependent on time, \( \mathcal{H}_f = \text{const} \) is satisfied. In the following section we analyze the possibility of singular control arcs for the minimum–time problem.

**Singular control arc analysis**

The switching function \( S \) is defined as

\[
S \equiv \frac{\partial \mathcal{H}}{\partial u} \equiv \frac{\partial \mathcal{H}_f}{\partial g_u} = \lambda_6 \equiv S_f
\]  

(56)

Using the control logic (54) the control is singular whenever \( S_f \equiv 0 \) during a finite time interval. For a singular control arc, \( g_u \) is determined by successive differentiation of the switching function until the control variable appears explicitly. Furthermore, it is required that \( S_f \) be differentiated an even number of times for \( g_u \) to be optimal. Hence

\[
\hat{g}_u^* = \arg \left\{ \left( \frac{d^{2j} S_f}{dt^{2j}} \right) = 0 \right\}, \quad j \in \mathbb{N}
\]  

(57)

where we used the hat–notation \( \hat{\cdot} \) to denote the singular control arc. In addition, Kelley’s optimality condition has to be satisfied along an optimal singular subarc, that is,

\[
(-1)^j \frac{\partial}{\partial g_u} \left( \frac{d^{2j} S_f}{dt^{2j}} \right) \geq 0
\]  

(58)
The first time derivative of the switching function yields $\dot{S}_f = \dot{\lambda}_6 = -\lambda_5 \equiv 0$. It is also straightforward to show that the second time derivative results

$$\ddot{S}_f \equiv S_f^{(2)} = 0 \quad \rightarrow \quad [\lambda_4(3\cos(2\alpha) - 1) - 3\lambda_3\sin(2\alpha)]\cos \alpha = 0 \quad (59)$$

At this point the algebra becomes decently involved. We obtain for the third time derivative of the switching function

$$S_f^{(3)} = 0 \quad \rightarrow \quad -12\lambda_1 r \cos \alpha^2 \sin \alpha$$

$$+ \lambda_3(3\omega r + 2v_\theta)[\cos \alpha + 3 \cos(3\alpha)]$$

$$+ \lambda_4\{\omega r[\sin \alpha + 9 \sin(3\alpha)] + 2 \cos \alpha[v_r - 3v_r \cos(2\alpha) + 3v_\theta \sin(2\alpha)]\} = 0 \quad (60)$$

Finally, taking the fourth time derivative of $S_f$ the control variable appears linearly.

$$S_f^{(4)} = 0 \quad \rightarrow \quad -8\lambda_1 r^2 \cos \alpha\{6\omega r[3\cos(2\alpha) - 1] + 3v_r \sin(2\alpha) + 2v_\theta[3\cos(2\alpha) - 1]\}$$

$$+ 4\lambda_3\{2 \cos \alpha^2 \sin \alpha[12 - 3\sigma - 8\beta \cos \alpha + 9\sigma \cos(2\alpha)]\}$$

$$+ 3r^3\{g_u[\cos \alpha + 3 \cos(3\alpha)] - \omega^2[\sin \alpha + 9 \sin(3\alpha)]\} - 36\omega^2 \cos \alpha^2 \sin \alpha$$

$$+ \omega r^2\{3v_r[\cos \alpha + 3 \cos(3\alpha)] - 4v_\theta[\sin \alpha + 9 \sin(3\alpha)]\}$$

$$+ \lambda_1\{-2 \cos \alpha[4 - \sigma + 6\beta \cos \alpha - 4(3 + 2\sigma) \cos(2\alpha) + 2\beta \cos(3\alpha)]$$

$$+ 9\sigma \cos(4\alpha)] + 4\omega^2 r^3[\cos \alpha + 27 \cos(3\alpha)] + 4g_u r^3[\sin \alpha + 9 \sin(3\alpha)]\}$$

$$+ 12\omega r^2\{v_r[\sin \alpha + 9 \sin(3\alpha)] + 2v_\theta[\cos \alpha + 3 \cos(3\alpha)]\}$$

$$- 8r \cos \alpha\{v_r^2[3 \cos(2\alpha) - 1] - 3v_r v_\theta \sin(2\alpha) + v_\theta^2[1 - 3 \cos(2\alpha)]\}\} = 0 \quad (61)$$

Therefore, solving equation (61) for $g_u \rightarrow \dot{g}_u^*$ and using equations (59,60) the optimal singular control $\dot{g}_u^*$ is of fourth order and is given by

$$\dot{g}_u^* = \{36\omega^2 r^3[3 - 12 \cos(2\alpha) + \cos(4\alpha)]$$

$$- 3 \sin(2\alpha)^2\{13\beta \cos \alpha + 3[6 - 6\sigma + 2(\sigma - 9) \cos(2\alpha) + 5\beta \cos(3\alpha)]\}\}$$

$$+ 8r^2v_\theta^2[19 - 36 \cos(2\alpha) + 9 \cos(4\alpha)]$$

$$- 72\omega r^2[\cos(2\alpha) - 3][v_r \sin(2\alpha) + v_\theta - 3v_\theta \cos(2\alpha)]\} / \{36 \omega^3 \cos \alpha[\sin(2\alpha) + \sin(3\alpha)]\} = 0 \quad (62)$$

Note, that the optimal control on a singular control arc does not depend on any of the costates. Also, for orientation angles of $\alpha = n \pi / 2 , n \in \mathbb{Z}$ the singular control law becomes, again, undefined.

**SPACECRAFT DESIGN PARAMETERS**

The control performance of solar sail spacecraft depends critically on the specific design parameters. An important design criteria and also the most common performance metric is the characteristic acceleration $\tilde{\beta}$ which was defined in a previous section. To simplify the design process for our analysis we adopt the one-third rule as used in Ref. 14 which
Table 1: Spacecraft design parameters for a 50 kg payload and a square solar sail.

<table>
<thead>
<tr>
<th>$\tilde{\beta}$ in mm/s$^2$</th>
<th>$\beta$ nondimensional</th>
<th>$\tilde{\rho}$ in g/m$^2$</th>
<th>$\tilde{A}$ in m$^2$</th>
<th>$\tilde{a}$ in m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000</td>
<td>0.16892</td>
<td>3.04200</td>
<td>16,436.5</td>
<td>128.205</td>
</tr>
<tr>
<td>2.00000</td>
<td>0.33784</td>
<td>1.52100</td>
<td>32,873.1</td>
<td>181.309</td>
</tr>
</tbody>
</table>

allocates one-third of the total system mass to the payload. We further assume that the remaining two-thirds of the spacecraft mass (solar sail, sail structures, deployment mechanism) are evenly distributed over the sail surface. With these assumptions the characteristic acceleration yields

$$\tilde{\beta} = \frac{2\tilde{A} \tilde{P}}{\tilde{m}} = \frac{2\tilde{P}}{3\tilde{\rho}}$$

or, given $\tilde{\beta}$:

$$\tilde{\rho} = \frac{2\tilde{P}}{3\tilde{\beta}}$$

(63)

where $\tilde{\rho}$ is the mass density of the solar sail. For a payload mass of $\tilde{m}_p = 50$ kg ($\tilde{m} = 3\tilde{m}_p = 150$ kg) and for typical characteristic accelerations of $\tilde{\beta} = 1$ mm/s$^2$ and 2 mm/s$^2$ the required sail mass densities result $\tilde{\rho} = 3.042$ g/m$^2$ and 1.5241 g/m$^2$ which is technically feasible according to McInnes.\textsuperscript{14} Table 1 summarizes the spacecraft design parameters for a 150 kg spacecraft and a square solar sail with surface area $\tilde{A} = \tilde{a}^2$.

The design parameter $\sigma$ in equation (24) controls the geometry of the spacecraft system and therefore the sign of the gravity gradient torque for a fixed orientation angle. With the one-third/two-thirds mass allocation between the payload and the solar sail the dimensions of the spacecraft result $\tilde{d}_s = \tilde{d}/3$, $\tilde{d}_p = 2\tilde{d}/3$, the moments of inertia become

$$\tilde{I}_1 = \tilde{I}_2 = 2\tilde{m}_s\tilde{A}/36 + 2\tilde{m}_s\tilde{d}^2/27 + 4\tilde{m}_s\tilde{d}^2/27 = \tilde{m}_s\tilde{A}/18 + 2\tilde{m}_s\tilde{d}^2/9$$

and

$$\tilde{I}_3 = \tilde{m}_s\tilde{A}/9$$

(64)

which yields for $\sigma$

$$\sigma = -3 \frac{I_1 - I_3}{I_2} = -3 \frac{4\tilde{d}^2 - \tilde{A}}{4\tilde{d}^2 + \tilde{A}} \rightarrow \sigma \in [\sigma_{\text{min}}, \sigma_{\text{max}}] = [-3, +3]$$

(65)

For $\sigma < 0$ the symmetry axis is the minor axis and the gravity gradient torque acts stabilizing for attitudes $\alpha = n\pi$, $n \in \mathbb{Z}$. The case of $\sigma = 0$ corresponds to inertially symmetric spacecraft where $\tilde{I}_i = \tilde{I}$, $i = 1, 2, 3$; the gravity gradient torque is equal to zero. To take into account the effect of the gravity gradient torque on the control performance we choose $\sigma_{\text{ref}} = +1$ as the reference value.

**NUMERICAL APPROACH**

The inherent difficulty of global optimization problems lies in finding the very best optimum from a multitude of local optima. Using indirect methods to solve optimum control problems the globality stems from the fact that the initial conditions of the Lagrange multipliers of the associated two-point boundary value problem cannot be estimated – not even approximately – without extensive analysis. Stryk and Bulirsch\textsuperscript{15} and later Seywald and Kumar\textsuperscript{16}
introduced the idea of combining direct and indirect methods to obtain approximate solutions with the direct method to generate accurate solutions with the indirect method. An obvious drawback of this approach is that using two different solution methodologies the control problem has to be formulated twice, as well. Also, an interface is necessary to communicate between the two algorithms (Lagrange multipliers). To circumvent the development of an extensive software package the optimal control problem in this paper is solved using a cascaded computational scheme using an indirect method.

The Computational Scheme

Figure 3 illustrates the computational scheme. Simulated Annealing (SA) is a global, statistical optimization algorithm which was first introduced by Kirkpatrick et al.\textsuperscript{17} to solve discrete optimization problems such as computer chip packing and wiring and to analyze classical problems, the travelling salesmen problem being one of the most well known one.\textsuperscript{18} We use a variant of the SA algorithm, namely Adaptive Simulated Annealing (ASA), as the initial optimization tool to obtain approximate estimates for the costates and the optimal transfer time $T = t_f - t_0$. Since statistical algorithms are in general neither efficient nor accurate the global algorithm is merely used to find out of the set of local minima the region in the parameter space which contains the true global minimum.

Once the algorithm has located a set of parameters in the close vicinity of the optimal set a Quasi–Newton method is used to further refine the parameter set. A crucial aspect of Quasi–Newton methods is the computation of second–order derivative information. Rather than calculating the Hessian of the objective function accurately at every iteration step or even just every so often we found that approximate Hessian information obtained using update formulas can significantly increase the algorithm effectiveness. In particular the Inverse Rank–One update\textsuperscript{19} and the Inverse–Broyden–Fletcher–Goldfarb–Shanno update\textsuperscript{19} (IBFGS) provide satisfactory optimization performance.

The Newton–Raphson method is known to be the most efficient zero–finding algorithm provided the starting guess of the unknowns lies within the region of attraction of the algorithm. For the reduced system model Newton’s method presents a superb approach to obtain highly accurate solutions. The full system model is presented as a problem with three equations in five unknowns; Newton’s method is not applicable.

Simulated Annealing – A Global, Statistical Optimization Algorithm

Statistical optimization methods such as Simulated Annealing differ from deterministic techniques in that the iteration procedure need not get stuck since transitions out of a local optimum are always possible. Another feature is that an adaptive divide–and-conquer occurs: coarse features of the optimal parameter set appear at higher temperatures, fine details develop at lower temperatures. In general, the method consists of the three functional relationships

- $g(\xi(i))$: Probability density of parameter–space. $\xi(i) = (\xi_1(i), \ldots, \xi_l(i))^T$ is the $l$–dimensional parameter vector at the $i$th iteration step.
• $h(\delta E)$: Probability for accepting a new performance index given just the previous value thereof.

• $\vartheta(j)$: Annealing schedule for temperature $\vartheta(j)$ at the $j$th annealing level.

Putting into words, the basic features of SA are as follows: Starting from a randomly generated initial point $\xi(i = i_0) = \xi_0$ in the parameter domain of interest and with an assigned initial temperature $\vartheta(j = j_0)$ the algorithm generates a new point $\xi(i + 1)$ in the neighborhood of $\xi_0$ and evaluates the performance index, the “energy” of the system $E(\xi(i + 1))$. If the energy change between the two points $\delta E(i) = E(\xi(i + 1)) - E(\xi(i))$ represents a decrease in the performance index, the new point is accepted right away. Otherwise, the new point is accepted with a probability $h(\delta E) \propto \exp(\delta E(i)/\vartheta(j))$. The sequence of generated points with probabilistic acceptance is often referred to in the literature as a Markov chain. After a sufficient number of trial points have been generated a new Markov chain is generated at a lower temperature level $\vartheta(j + 1)$.

The cooling schedule $\vartheta(j)$ critically effects the tendency of the algorithm to find the region enclosing the global minimum. Initially the temperature is chosen relatively high. Most trial points are accepted and there is little chance of the algorithm zooming in on a local minimum in the early stages of the search. As the temperature is decreased for later generations of Markov chains the trial points are accumulating more regionally and the search becomes localized. However, unlike for deterministic optimization techniques with SA statistical temperature fluctuations can always cause uphill steps out of a local minimum. Another important aspect of the SA algorithm is trial point generation. Various approaches have been proposed, for example, using a uniform distribution on the domain of interest or a mix of a uniformly distributed draw and deterministic steps into a descent direction from the current point. One of the more popular ways to generate trial points is

Figure 3: Cascaded numerical algorithm for solving the optimal control problem.
based on the Metropolis acceptance probability\textsuperscript{20}

\[ A(\xi(i), \vartheta(j)) = \min \{ 1, \exp \left[ -(\xi^+(i) - \xi(i))/\vartheta(j) \right] \} \]  \hspace{1cm} (66)

where \( A(\xi(i), \vartheta(j)) \) is the probability of accepting a point \( \xi^+(i) \) if \( \xi(i) \) is the current point and \( \xi^+(i) \) is generated as a possible new point.

SIMULATION RESULTS

For the reduced system model the results obtained for Earth–to–Mars transfers were in excellent agreement with data published by Wood et al.\textsuperscript{8} Table 2 shows transfer times and initial and final costates for two different characteristic accelerations. We obtained high-accuracy results with \(|\psi(x(t_f), t_f)| < 10^{-14}\) which result in slightly improved transfer times in the order of 10 to 15 hours compared to reported results in Ref. 8. Note that the nondimensional characteristic accelerations of \( \tilde{\beta} = 0.16892 \) and \( \tilde{\beta} = 0.33784 \) correspond to nominal values of \( \tilde{\beta} = 1\text{mm/s}^2 \) and \( \tilde{\beta} = 2\text{mm/s}^2 \). The minimum transfer times correspond to 323.87 and 407.62 days, respectively (1 TU = 365.25/(2\pi) days = 58.1313 days).

Table 2: Minimum transfer times and corresponding costates for Earth–to–Mars transfers for reduced system model.

<table>
<thead>
<tr>
<th>Analysis Author</th>
<th>Characteristic acceleration</th>
<th>Transfer time</th>
<th>Initial costates</th>
<th>Final costates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wood, et al.\textsuperscript{8}</td>
<td>0.16892</td>
<td>7.02232</td>
<td>(-7.40581)</td>
<td>(-21.5481)</td>
</tr>
<tr>
<td></td>
<td>0.33784</td>
<td>5.57911</td>
<td>(-2.59597)</td>
<td>+10.1662</td>
</tr>
<tr>
<td>Kim</td>
<td>0.16892</td>
<td>7.01204</td>
<td>(-7.90591)</td>
<td>(-46.5737)</td>
</tr>
<tr>
<td></td>
<td>0.33784</td>
<td>5.57134</td>
<td>(-2.59741)</td>
<td>+10.1619</td>
</tr>
<tr>
<td>Powers, et al.\textsuperscript{9,10}</td>
<td>0.33784</td>
<td>5.57 – –</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Figure 4 shows orbit transfer time as a function of final orbit radius for \( \beta = 0.16892 \) and \( \beta = 0.33784 \). The initial orbit radius is 1 AU. As expected transfers times are significantly lower the higher the characteristic acceleration, especially for outbound trajectories where \( r(t_f) > r(t_0) \). Initial orientation angle \( \alpha(0) \) versus target orbit radius is shown in Figure 5. For inbound (outbound) trajectories the initial control angles are negative (positive) to
decrease (increase) the tangential velocity component \( v_\theta \). Also, increasing the characteristic accelerations allows the spacecraft to start with relatively smaller orientation angles which is used favorably to gain radial distance. This trend holds up to an evident turning point at a target radius of about \( r_f \approx 2.7 \) AU. The turning point behavior of course appears also in Figure 6 which shows the initial costates as a function of target orbit radius for \( \beta = 0.16892 \). The analogous plot for high characteristic acceleration is shown in Figure 7.

Figures 8-11 show transfer trajectories and solar sail orientation angle time histories for various transfer scenarios originating from the Earth. Figures 8 and 9 show simulation results for Earth–to–Mars transfers for low and high characteristic accelerations. Similar plots can also be found in Ref.8. High characteristic acceleration transfer orbits from Earth to 0.5 AU and 2.0 AU are shown in Figure 10. Note that for these two transfers the ratio \( \Xi = r_f/r_0 \) is inversely equal, that is, \( 0.5 = \Xi_{\oplus\rightarrow0.5\text{–AU}} = 1/\Xi_{\oplus\rightarrow2.0\text{–AU}} = 1/2 \). Comparing the control angle histories we notice perfect symmetry considering the different time scales. Further analysis is necessary, however, to confirm this observation in general. Nevertheless, it turns out that for the specific cases of transfers to 0.5 AU and to 2.0 AU the final orientation angle for the Earth–to–0.5 AU transfer is exactly negative equal the initial orientation angle for the Earth–to–2.0 AU transfer and vice versa.

The convergence rate of the optimization tools for the full system model is significantly lower than for the reduced system model. Furthermore, using the problem statement discussed in a previous section Newton–Raphson methods are not applicable. Nevertheless, we obtained optimized solutions with \( |\psi (x(t_f), t_f)| \lesssim 10^{-5} \) which corresponds to a mismatch, for example, in radial distance in the order of some hundreds of kilometers. Figures 11 and 12 show a typical simulation result for the full system model for a high characteristic acceleration Earth–to–Mars transfer. Note that since \( \kappa = 1 \) we consider a mixed minimum–time minimum–cost control problem which yields an increased transfer time of \( T = 6.83254 \) as compared to \( T = 5.57134 \) for the minimum–time problem obtained using the reduced system model. The difference corresponds to approximately 73.32 days. Furthermore, note that since the final orientation angle and angular velocity are allowed to vary freely the spacecraft is rotating after completing the transfer. Comparing Figure 11 and the corresponding Figure for the reduced system model the results are quite different, as can be expected since with \( \kappa = 1 \) we do not solve a true minimum–time problem.

**SUMMARY AND CONCLUSIONS**

The optimal control problem of a solar sail spacecraft for interplanetary missions has been studied in detail. This very same subject has been addressed in the literature numerous times, however considering only what we refer to as the reduced system model which does not take into account the rotational dynamics of the spacecraft (full system model). We solve both control problems using an indirect method. The cascaded computational scheme is divided into two optimization levels. On the first level a global statistical algorithm based on Adaptive Simulated Annealing is used to find an approximate guess for the Lagrange multipliers and the transfer time. The optimization parameters are then refined using a Quasi–Newton method. The composite algorithm proofs extremely efficient finding highly
accurate solutions to the minimum–time problem for the reduced system model. For the full system model the convergence rate is significantly lower, also finding a working set of optimizer parameters for the SA algorithm turns out to be quite tedious for an unexperienced SA novice.

Several questions remain unanswered: Using a control torque the minimum–time problem for the full system model results a bang–type control law which might render the system uncontrollable. Also, simulation results indicate that singular control arcs are rare which leaves for the control law plain square–wave functions. Which naturally poses the question under which circumstances a certain type of control law/logic render a in general nonlinear system uncontrollable. Another interesting question in that respect is: What can be said about the existence of singular control arcs for the case when the control on such a subarc does not depend on any of the costates but is only a function of (some of) the states?

Clearly the analysis presented in this paper is far from being complete. Future work will include the verification of simulation results using available optimization tools such as EZopt21 and DIDO.22 Both software packages have been used successfully by several researchers to solve a variety of optimal control problems. EZopt and DIDO solve optimization problems using a direct method, nevertheless, one of the most important features of DIDO is its ability to provide estimates for the Lagrange multipliers. Also further analysis is necessary to investigate the influence of the spacecraft geometry (parameter $\sigma$) and the control parameter $\kappa$ on optimal transfer time and control torque. Ideally, for small $\kappa$ it should be possible to recover the control angle histories for the reduced system model.

Figure 4: Transfer time as a function of target orbit radius for $\beta = 0.16892$ (solid lines) and $\beta = 0.33784$ (dashed lines) for reduced system model. The initial orbit radius is 1 AU.
Figure 5: Initial solar sail orientation angle $\alpha(0)$ as a function of target orbit radius for $\beta = 0.16892$ (solid lines) and $\beta = 0.33784$ (dashed lines) for reduced system model. The initial orbit radius is 1 AU.

Figure 6: Initial values of costates $\lambda_i(0)$ as a function of target orbit radius for $\beta = 0.16892$ for reduced system model. The initial orbit radius is 1 AU.
Lagrange multipliers $\lambda_i(0)$ versus target orbit radius for $\beta = 0.33784$

Figure 7: Initial values of costates $\lambda_i(0)$ as a function of target orbit radius for $\beta = 0.33784$ for reduced system model. The initial orbit radius is 1 AU.

Transfer trajectory for $\beta = 0.16892$

Solar sail orientation angle for $\beta = 0.16892$

Figure 8: Transfer trajectory and solar sail orientation angle history for an Earth–to–Mars transfer with $\beta = 0.16892$ for reduced system model.
Figure 9: Transfer trajectory and solar sail orientation angle history for an Earth–to–Mars transfer with $\beta = 0.33784$ for reduced system model.

Figure 10: Comparison of transfer trajectories and solar sail orientation angle histories for Earth–to–0.5 AU and Earth–to–2.0 AU transfers with $\beta = 0.33784$ for reduced system model.
Figure 11: Transfer trajectory and solar sail orientation angle history for an Earth–to–Mars transfer for full system model. Simulation parameters are $\beta = 0.33784$, $\sigma = 1$ and $\kappa = 1$.

Figure 12: Lagrange multipliers $\lambda_i$, nondimensional angular velocity $\omega$ and control torque $g_u$ as a function of time for an Earth–to–Mars transfer for full system model. Simulation parameters are $\beta = 0.33784$, $\sigma = 1$ and $\kappa = 1$. 
REFERENCES


