

## Linstedt-Poincaré

We are interested in approximate solution to the nonlinear constant-coefficient differential equation

$$\ddot{x} + \omega_o^2 x + \sum_{i=2}^n \alpha_i x^i = 0 \quad (1)$$

One standard approach is to use the perturbation method known as Linstedt-Poincaré, also known as the method of strained coordinates.

We assume that the initial conditions are  $x(0) = a$ ,  $\dot{x} = 0$ . These conditions can be made more general, but the usual reason for applying Linstedt-Poincaré is to obtain the relationship between the amplitude and frequency of nonlinear oscillations.

For this analysis, we will simplify the basic equation to a quadratic nonlinearity:

$$\ddot{x} + \omega_o^2 x + \varepsilon x^2 = 0 \quad (2)$$

where  $\varepsilon$  is usually a “small” parameter. Note that, for  $\varepsilon = 0$ , the differential equation is linear and the solution may be easily written as

$$x_0(t) = a_0 \cos(\omega_o t + \phi_0) \quad (3)$$

where the amplitude  $a_0$  and phase  $\phi_0$  depend on the initial conditions.

We assume that the general solution to the nonlinear equation is “close” to the solution to the linear equation, but may have different amplitude, phase, and frequency. We seek an approximation to the solution by expanding the unknown solution  $x(t)$  in a Taylor series in  $\varepsilon$  as

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \varepsilon^3 x_3(t) + \dots \quad (4)$$

This assumption leads to

$$\dot{x} = \dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \varepsilon^3 \dot{x}_3 + \dots \quad (5)$$

$$\ddot{x} = \ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \varepsilon^3 \ddot{x}_3 + \dots \quad (6)$$

$$x^2 = x_0^2 + \varepsilon (2x_0 x_1) + \varepsilon^2 (x_1^2 + 2x_0 x_2) + \varepsilon^3 (2x_1 x_2 + 2x_0 x_3) + \dots \quad (7)$$

$$(8)$$

where we have omitted the  $(t)$  notation for simplicity.

Another element of the L-P method involves assuming that the frequency of the solution depends on the amplitude, and defining a new frequency  $\omega$  by

$$\omega = \omega_o + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \varepsilon^3 \omega_3 + \dots \quad (9)$$

and a new time  $\tau$  by

$$\tau = \omega t \quad (10)$$

This latter term is sometimes referred to as “stretched” or “strained” time, hence the term “strained coordinates.” Applying the chain rule, we find that

$$\frac{d^2}{d\tau^2}(\cdot) = \frac{1}{\omega^2} \frac{d^2}{dt^2}(\cdot) \quad (11)$$

which we write as

$$\omega^2(\cdot)'' = (\ddot{\cdot}) \quad (12)$$

Making the appropriate substitutions into the original differential equation, and grouping terms according to the powers of  $\varepsilon$ , we obtain

$$\varepsilon^0 (\omega_0^2 x_0'' + \omega_0^2 x_0) + \quad (13)$$

$$\varepsilon (\omega_0^2 x_1'' + \omega_0^2 x_1 + x_0^2 + 2\omega_0 \omega_1 x_0'') + \quad (14)$$

$$\varepsilon^2 (\omega_0^2 x_2'' + \omega_0^2 x_2 + 2x_0 x_1 + 2\omega_0 \omega_2 x_0'' + 2\omega_0 \omega_1 x_1'') = \mathcal{O}(\varepsilon^3) \quad (15)$$

where the expression  $\mathcal{O}(\varepsilon^3)$  means that the remaining terms are of “order”  $\varepsilon^3$ . We can of course carry out this series to arbitrary order.

Ignoring the higher order terms, the expression can be recognized as a polynomial in  $\varepsilon$ :

$$\varepsilon^0 (\omega_0^2 x_0'' + \omega_0^2 x_0) + \quad (16)$$

$$\varepsilon (\omega_0^2 x_1'' + \omega_0^2 x_1 + x_0^2 + 2\omega_0 \omega_1 x_0'') + \quad (17)$$

$$\varepsilon^2 (\omega_0^2 x_2'' + \omega_0^2 x_2 + 2x_0 x_1 + 2\omega_0 \omega_2 x_0'' + 2\omega_0 \omega_1 x_1'') = 0 \quad (18)$$

The only way this equation can be satisfied for all values of  $\varepsilon$  is for the coefficients of each power of  $\varepsilon$  to all vanish. Thus we obtain a sequence of linear differential equations, which we write as:

$$\varepsilon^0 : \omega_0^2 x_0'' + \omega_0^2 x_0 = 0 \quad (19)$$

$$\varepsilon^1 : \omega_0^2 x_1'' + \omega_0^2 x_1 = -x_0^2 - 2\omega_0 \omega_1 x_0'' \quad (20)$$

$$\varepsilon^2 : \omega_0^2 x_2'' + \omega_0^2 x_2 = -2x_0 x_1 - 2\omega_0 \omega_2 x_0'' - 2\omega_0 \omega_1 x_1'' \quad (21)$$

If we had kept terms of higher-order in  $\varepsilon$ , we would have more equations in this sequence. Note well that the first equation is a homogeneous equation and we already know its solution,  $x_0(\tau) = a_0 \cos(\tau + \phi_0)$ , whereas the subsequent equations are non-homogeneous, with forcing terms depending on previous terms in the solution. That is, the equation for  $x_1$  is forced by terms depending on  $x_0$ , the equation for  $x_2$  is forced by terms depending on  $x_0$  and  $x_1$ , and so forth.

One question that arises is how to deal with the initial conditions. One straightforward thing to do is to let the two unknown constants in the zeroth-order solution capture the initial conditions, and thus only retain the particular solutions for the subsequent equations.

Another important point in the L-P technique is the selection of the frequency correction terms,  $\omega_i, i > 0$ . These terms are chosen to ensure that the resonant forcing terms in the higher-order equations vanish.

For example, the right-hand side of the  $x_1$  differential equation turns out to be

$$-2a_0 \omega_0 \omega_1 \cos(\tau + \phi_0) + a_0^2 \cos^2(\tau + \phi_0) \quad (22)$$

The  $\cos^2$  term expands to  $(1 + \cos(2\tau + 2\phi_0))/2$  and does not produce any resonance. However, the  $\cos$  term is a resonant term and will lead to a secular term in the solution.

By choosing  $\omega_1 = 0$ , we can ensure that the approximate solution is an oscillatory solution rather than a solution with growing amplitude.

Choosing  $\omega_1 = 0$  leads to

$$x_1 = -\frac{a_0^2}{2\omega_0^2} + \frac{a_0^2}{6\omega_0^2} \cos(2\tau + 2\phi_0) \quad (23)$$

which is used, along with  $x_0$ , to form the forcing term for the  $x_2$  equation:

$$-2x_0x_1 - 2\omega_0\omega_2x_0'' = -\frac{a(5a_0^2 + 12\omega_0^3\omega_2)}{6\omega_0^2} \cos(\tau + \phi_0) + \frac{a^3}{6\omega_0^2} \cos(3\tau + 3\phi_0) \quad (24)$$

The  $\cos^3$  term does not include any resonant term, but the  $\cos$  term does. Therefore, we choose  $\omega_2$  to make that term vanish:

$$5a_0^2 + 12\omega_0^3\omega_2 = 0 \Rightarrow \omega_2 = -\frac{5a_0^2}{12\omega_0^3} \quad (25)$$

Thus the approximate natural frequency is

$$\omega = \omega_0 + \varepsilon^2\omega_1 \quad (26)$$

$$= \omega_0 - \varepsilon^2 \frac{5a_0^2}{12\omega_0^3} \quad (27)$$

Finally, the approximate solution to the nonlinear differential equation is

$$x(t) = x_0(t) + \varepsilon x_1(t) \quad (28)$$

$$= a_0 \cos(\omega t + \phi_0) + \varepsilon \left( -\frac{a_0^2}{2\omega_0^2} + \frac{a_0^2}{6\omega_0^2} \cos(2\omega t + 2\phi_0) \right) \quad (29)$$

where  $\omega$  is defined at Eq. (27).

To match the initial conditions, set  $x(0) = a$  and  $\dot{x}(0) = 0$ , which leads to  $\phi_0 = 0$  and a quadratic for  $a_0$ :

$$\varepsilon \frac{a_0^2}{3\omega_0^2} - a_0 + a = 0 \quad (30)$$

It turns out the correct root is the smaller of the two roots, and it can be found easily using the Matlab statement

```
min(roots([ep/3/wo^2 -1 a]))
```

where `ep` is  $\varepsilon$ , `wo` is  $\omega_0$ , and `a` is  $a$ . Another way to approximate the root is to apply a Taylor series expansion to the quadratic, setting  $a_0 = a + \varepsilon a_1 + \dots$ , which leads to  $a_1 = a^2/(3\omega_0^2)$ , so an approximation for  $a_0$  is

$$a_0 = a + \varepsilon \frac{a^2}{3\omega_0^2} \quad (31)$$

In the two figures, we plot three solutions: the “exact” nonlinear solution obtained using `ode45`; the linear approximation; and the Linstedt-Poincaré approximation. The

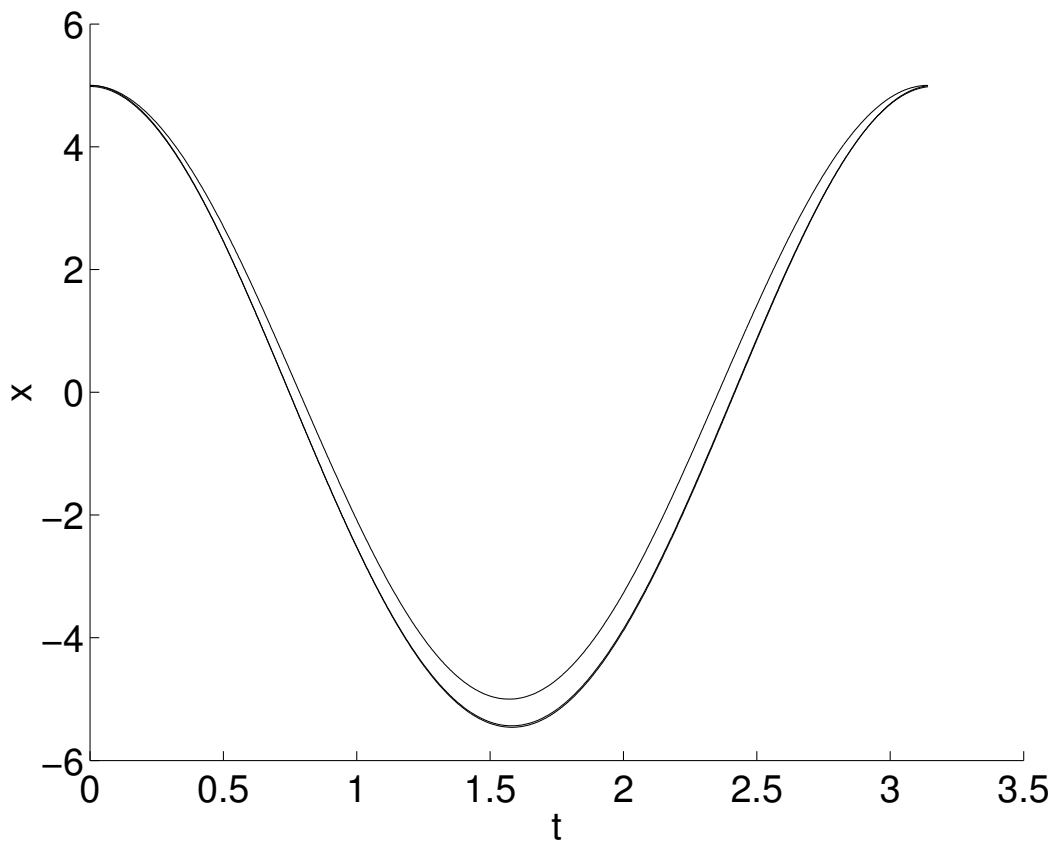


Figure 1: Comparison for  $a = 5$

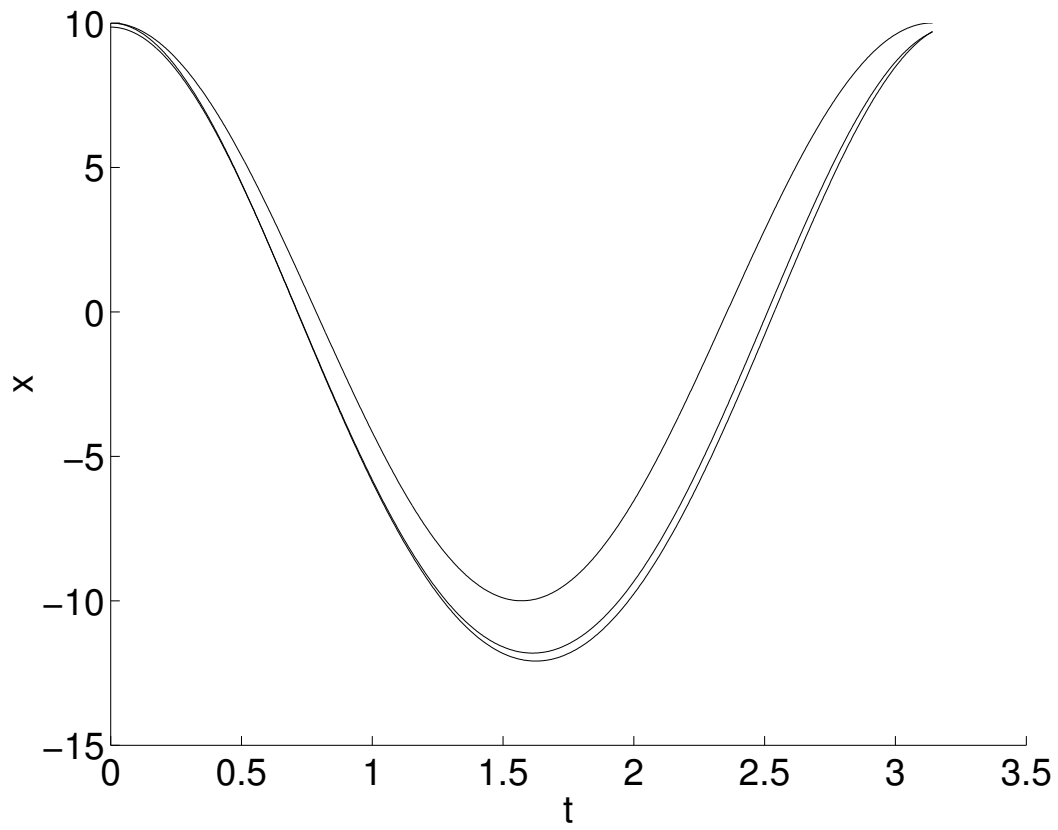


Figure 2: Comparison for  $a = 10$

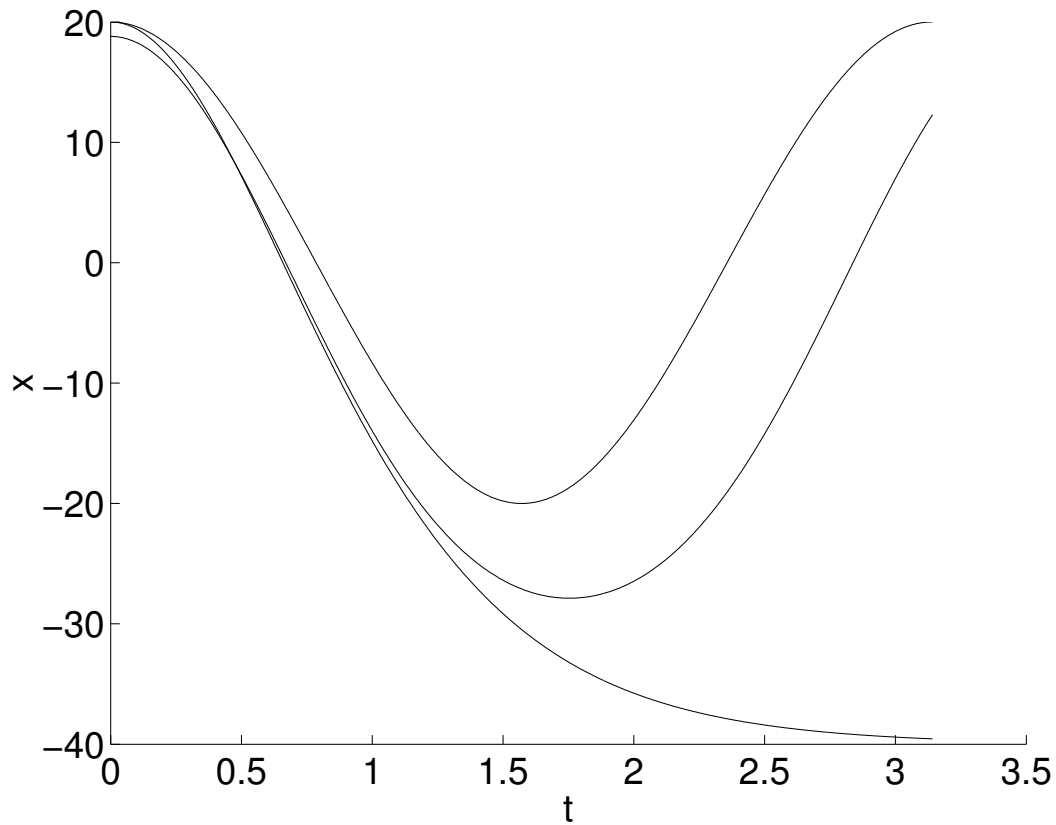


Figure 3: Comparison for  $a = 20$

first plot has initial amplitude  $a = 5$  and the second plot has initial amplitude  $a = 10$ . In both cases the “exact” and Linstedt-Poincaré solutions are quite close, but the linear solution is not very accurate.

Another interesting thing to do is to see what happens as the amplitude is further increased. In Fig. 3, the same three graphs are plotted for  $a = 20$ . The linear and L-P approximations are both still periodic, but the “exact” solution is no longer oscillatory.

The Matlab code is given below.

Driver file:

```
% perturbation example
```

```
wo=2;
```

```
ep=0.1;
```

```
a=20;
```

```

x0=[a;0];

options = odeset('abstol',1e-10, 'reltol',1e-10);
[t,x] = ode45('quadnl',[0 2*pi/wo], x0, options, wo, ep);

figure(1); hold on
plot(t,x(:,1))

figure(2); hold on
plot(x(:,1),x(:,2))

[t1,x1] = ode45('quadnl',[0 2*pi/wo], x0, options, wo, 0);

figure(1);
plot(t1,x1(:,1),'r')

figure(2);
plot(x1(:,1),x1(:,2))

ta=t;
om=wo-ep^2*5*a^2/12/wo^3;

a0=min(roots([ep/3/wo^2 -1 a]))
a0=a+ep*a^2/3/wo^2
x0=a0*cos(om*ta);
x1=-a0^2/2/wo^2 + a0^2*cos(2*om*t)/6/wo^2;
xa=x0+ep*x1;

figure(1);
plot(ta,xa,'g')

fh=xlabel('t');
set(fh,'fontsize',18);
fh=ylabel('x');
set(fh,'fontsize',18);
set(gca,'fontsize',18);

```

Righthandside file:

```

function xdot = quadnl(t,x,opt,wo,ep)

xdot = [x(2); -wo^2*x(1) - ep*x(1)^2];

```