AOE 5984: Exam 1
Wednesday, October 29, 2003
Open Book – Open Notes

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Problem	min	mean	max
1.	0	19	32
2.	22	39	35
3.	19	28	35
Exam	41	78	98

Problem 1. (35 points)

This problem concerns the steady motion of an asymmetric rigid body with a body-fixed torque. The body has moments of inertia of

$$\mathbf{I} = \mathrm{diag}[16, 15, 22]\,\mathrm{kg}\cdot\mathrm{m}^2$$

The mission requirement is for the body to rotate about a non-principal axis

$$\mathbf{a} = (0, \sin 22^\circ, \cos 22^\circ)$$

at a constant angular velocity of 60 rpm.

(a) (20 points) Compute the body-fixed torque required to maintain this steady motion. (Note that you *do not* have to design a control that will cause the body to achieve this angular velocity from arbitrary initial conditions.)

(b) (15 points) Tell me as much as you can about the stability of this steady motion.

Solution. To compute the body-fixed torque, simply set $\dot{\boldsymbol{\omega}} = \mathbf{0}$ and $\boldsymbol{\omega} = 2\pi\mathbf{a}$ in Euler's equations and solve for $\mathbf{g}$:

$$\mathbf{g} = [28\pi^2 \sin 22^\circ \cos 22^\circ, 0, 0]^\mathsf{T}$$

The easiest way to do this is to write down the scalar version of Euler's equations, so that you don't have to multiply out $\boldsymbol{\omega}^\times \mathbf{I}\boldsymbol{\omega}$. Also, since $\omega_1 = 0$, you should easily see that $g_2 = g_3 = 0$.

Stability. The first thing to do is to check the linear stability, by linearizing and inspecting the eigenvalues of the resulting linear equations. As usual, set $\boldsymbol{\omega} = \boldsymbol{\omega}_e + \delta\boldsymbol{\omega}$, where $\boldsymbol{\omega}_e = 2\pi\mathbf{a}$, multiply everything out and ignore all the products of the form $\delta\omega_i\delta\omega_j$. Doing this leads to

$$\delta\dot{\boldsymbol{\omega}} = \begin{bmatrix} 0 & k_1\omega_{3e} & k_1\omega_{2e} \\ k_2\omega_{3e} & 0 & 0 \\ k_3\omega_{2e} & 0 & 0 \end{bmatrix} \delta\boldsymbol{\omega}$$

where $k_1 = (I_2 - I_3)/I_1$, $k_2 = (I_2 - I_3)/I_1$, and $k_3 = (I_1 - I_2)/I_3$. Substituting the numbers into the $\mathbf{A}$ matrix gives:

$$\mathbf{A} = \begin{bmatrix} 0 & -2.5487 & -1.0298 \\ 2.3303 & 0 & 0 \\ 0.1070 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial is

$$-\lambda^3 - 2.5487 \times 2.3303\lambda - 1.0298 \times 0.1070\lambda = 0$$

Clearly $\lambda = 0$ is one root, so we already know we cannot prove nonlinear stability. The other two roots are found by factoring out $-\lambda$ to obtain a quadratic polynomial:

$$\lambda^2 + 2.5487 \times 2.3303 + 1.0298 \times 0.1070 = 0$$

whose roots are

$$\lambda = \pm 2.4596j$$

Thus the system is linearly stable. What about nonlinear stability?

We need to find an appropriate Lyapunov function to prove nonlinear stability. Unfortunately, neither energy nor angular momentum are conserved because the torque is not zero. I expect motion is nonlinearly stable, but I do not know of an appropriate Lyapunov function. The main point here is that the Lyapunov function used for proving stability of major or minor axis spins is not suitable for this problem.

Problem 2. (35 points)

Consider a torque-free axisymmetric satellite with moments of inertia

$$\mathbf{I} = \text{diag}[200, 200, 100]\,\text{kg} \cdot \text{m}^2$$

Four coiled antennas of uniform mass distribution with mass $m = 6$ kg each are stored so close to the satellite's mass center that their contribution to the moments of inertia can be ignored (when stored). The satellite has an initial angular momentum of $h = 1000$ kg·m^2/sec, and is in a steady rotation about the symmetry axis. Then the antennas are slowly and uniformly extended radially until they are fully extended to a length of 50 m each from the satellite center. At the end of the maneuver the satellite is in steady rotation about the symmetry axis.

(a) (10 points) Compute the final moments of inertia.

(c) (5 points) Compute the final angular momentum.

(c) (10 points) Compute the initial and final values of the rotational kinetic energy.

(d) (10 points) Discuss.

Solution. The final moments of inertia are computed by just adding the inertias of the four rods. Thus

$$\begin{aligned} I_1 &= I_{1b} - 2\frac{1}{3}mL^2 = 10,200\text{kg} \cdot \text{m}^2 \\ I_2 &= I_{2b} - 2\frac{1}{3}mL^2 = 10,200\text{kg} \cdot \text{m}^2 \\ I_3 &= I_{2b} - 4\frac{1}{3}mL^2 = 20,100\text{kg} \cdot \text{m}^2 \end{aligned}$$

The final angular momentum is the same as the initial angular momentum because there are no external torques. Since we need them for the next part, let's go 'head and calculate the initial and final angular velocities:

$$\begin{aligned} h &= I_{3b}\omega_{30} \Rightarrow \omega_{30} = 10\text{rad/s} \\ h &= I_3\omega_3 \Rightarrow \omega_3 = 10/201 \approx 0.05\text{rad/s} \end{aligned}$$

The rotational kinetic energy is given by

$$T = \frac{1}{2}\boldsymbol{\omega}^{\mathsf{T}}\mathbf{I}\boldsymbol{\omega} = \frac{1}{2}I_3\omega_3^2$$

where the simplification is due to the fact that both the initial and final motions are simple rotations about the "3" axis. Thus

$$T_0 = 5000\text{kg}\cdot\text{m}^2/\text{s}^2$$

and

$$T_f = 24.88\text{kg}\cdot\text{m}^2/\text{s}^2$$

There were several directions the discussion could take.

1. Why is the final energy less than the initial energy? It's due to the work done on the system by reeling out the antennas. There is no energy dissipation (at least no modelled dissipation). If the antennas were reeled back in, the energy would increase to its original level.

2. What happens as the moments of inertia change? Initially the spacecraft is a minor axis spinner and so would be unstable in the presence of energy dissipation. Somewhere in the middle of the maneuver, the MOIs transition from minor-axis spinner through spherically symmetric spinner to major-axis spinner. What effects might this have on the dynamics of a real spacecraft?

Problem 3. (35 points)

A rigid disk-shaped satellite is to be used in a gravity-gradient stabilized configuration, with the flat face of the disk pointing at the earth. The disk has mass m, radius d, and negligible thickness.

(a) (10 points) Compute the Smelt parameters k_1 and k_3 for this configuration and **discuss** the wisdom (from the point of view of stability) of this design.

The Acme Boom Company proposes the addition of a long boom to enhance the stability of the satellite. To save money and weight we want to use the shortest boom possible.

(b) (15 points) Assuming the boom is well-approximated by a rigid rod with density per unit length ρ, **determine** the minimum length boom to use.

We want the pitch frequency n_p to be greater than $2\omega_c$ where ω_c is the orbital angular velocity.

(c) (10 points) Is it possible to do this, and if so, **how long** would the boom have to be?

Solution. The moments of inertia are

$$
\begin{aligned}
I_1 &= md^2/4 \\
I_2 &- md^2/4 \\
I_3 &= md^2/2
\end{aligned}
$$

The Smelt parameters are

$$
\begin{aligned}
k_1 &= \frac{I_2 - I_3}{I_1} = \frac{1/4 - 1/2}{1/4} = -1 \\
k_3 &= \frac{I_2 - I_1}{I_3} = 0
\end{aligned}
$$

Since the point $(k_1, k_3) = (-1, 0)$ is neither in the Lagrange region nor in the DeBra-Delp region, the equilibrium orientation is unstable. One can also easily explain this result in terms of the inertias themselves. Recall that the Lagrange region requires the spacecraft to have its minor axis along the nadir axis, its major axis along the orbit normal, and its intermediate axis in the velocity vector direction. That is, $I_2 > I_1 > I_3$, whereas here we have $I_3 > I_1 = I_2$.

The easy way to do the boom analysis is to imagine installing the boom with its center at the center of the disk. That way the center of mass of the system remains at the center of mass of the disk. The symmetry axis inertia remains $I_3 = md^2/2$. The other inertias remain equal and satisfy

$$I_1 = I_2 = md^2/4 + 2m_R L^2/3$$

where $m_R = \rho L$. To get a (more or less) stable configuration, we need the boom length to be long enough so that

$$I_1 = I_2 > I_3 \Rightarrow md^2/4 + 2\rho L^3/3 > md^2/2 \Rightarrow L > \sqrt[3]{3md^2/(8\rho)}$$

Note that we cannot find a length that will put the Smelt parameters inside the Lagrange region, but only on the boundary of the Lagrange region, since $k_3 = 0$ for any boom length.

The pitch frequency is easily obtained from the linear pitch equation:

$$n_p^2 = \frac{3\omega_c^2(I_1 - I_3)}{I_2}$$

Substituting the moments of inertia into this expression leads to

$$n_p^2 = 3\omega_c^2(md^2/4 + 2\rho L^3/3 - md^2/2)/(md^2/4 + 2\rho L^3/3)$$

Clearly when the boom length is zero, $n_p^2 < 0$ and when the boom length is the minimum length derived above, the pitch frequency is zero. For any larger boom length, the pitch frequency increases. Stare at this expression long enough and you'll realize that in the limit as the boom length goes to infinity, the pitch frequency (squared) goes to $3\omega_c^2$. Therefore, there is no boom length that will make $n_p > 2\omega_c$. Of course, it's easiest to show this directly with the moments of inertia instead of substituting in the expressions for the moments of inertia.