Chapter 5

Linear Systems Analysis and Control

In this chapter we show how to linearize the equations of motion. Subsequently we show how to perform stability analysis and modal analysis for small motions near equilibrium states. Control theory is developed using proportional-integral-derivative (PID) control and the linear quadratic regulator (LQR) control synthesis. We also develop the concepts of controllability, observability, stabilizability, and detectability.

5.1 A “Standard” Nonlinear System

Throughout this chapter, we study linear systems that are approximations of nonlinear systems. In particular, we are interested in nonlinear systems that are in the form of

\[ \dot{x} = f(x, u, t) \]  \hspace{1cm} \text{(5.1)}

\[ y = g(x, u, t) \]  \hspace{1cm} \text{(5.2)}

where \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \), and \( u \in \mathbb{R}^p \). The vector \( x \) is the state vector, the vector \( u \) is the input vector, and the vector \( y \) is the output vector. The vector \( f \) is sometimes called the vector field. Often \( f \) will not depend on \( t \), and in that case the system is called autonomous. Furthermore, \( y \) usually does not depend on \( u \); if \( y \) does depend on \( u \), then the system is said to have a feedforward connection between the input and the output.

Note well that the \( f \) and \( g \) in Eqs. (5.1) and (5.2) are distinct from the \( f \) and \( g \) denoting force and torque in the principles of linear and angular momentum. The meaning of these symbols will always be evident from the context.

As an example, suppose we are interested in the dynamics of a rigid body with two single-axis rate gyros and a single attitude control thruster. The state vector would comprise the angular velocity vector \( \omega \) and the quaternion \( \mathbf{q} \), the output vector would...
be a $2 \times 1$ matrix of the two angular rates, and the input vector would be the single attitude control torque. Thus $n = 7$, $m = 2$, and $p = 1$.

Example 5.1. Consider the nonlinear, second-order differential equation
\[ \ddot{x} + c(x, \dot{x}) + k(x) = f \]

Put the differential equation into the standard form [Eq. (5.1)].
Define $x_1 = x, x_2 = \dot{x}, x = [x_1 \ x_2]^T$, then $\dot{x}_1 = x_2$, and $\dot{x}_2 = \ddot{x} = -c(x, \dot{x}) - k(x) + f = -c(x_1, x_2) - k(x_1) + u$. The state vector form of the equation is thus
\[ \dot{x} = \begin{bmatrix} x_2 \\ -c(x_1, x_2) - k(x_1) + u \end{bmatrix} = f(x, u) \]
Clearly $n = 2$ and $p = 1$.

5.2 Linearization

Suppose that there is some constant input, $u^*$, and a constant state vector, $x^*$, such that
\[ \dot{x} = f(x^*, u^*, t) = 0 \] (5.3)

This combination of input and state is called an equilibrium, since $\dot{x} = 0$ implies that the state remains constant.

For a conceptual example, consider a pendulum, where $\theta$ describes the angle between the pendulum and the vertical, with $\theta = 0$ in the “down” position and $\theta = \pi$ in the “up” position. The pendulum is in equilibrium for either of these two values of $\theta$. However, the $\theta = 0$ position is stable, whereas the $\theta = \pi$ position is unstable. One of the major goals of this chapter is to develop the tools to describe these equilibrium and stability concepts rigorously.

Example 5.2. Consider the nonlinear, second-order differential equation
\[ \ddot{x} + c(x, \dot{x}) + k(x) = f \]
with $c(x, \dot{x}) = (a - x^2) \dot{x}$, $k(x) = x + \alpha x^2 + \beta x^3$. Determine $u^*$ such that $x^* = [a \ 0]^T$ is an equilibrium.

As developed in Ex. 5.1, the nonlinear form of the equations is
\[ \dot{x} = \begin{bmatrix} x_2 \\ -c(x_1, x_2) - k(x_1) + u \end{bmatrix} = f(x, u) \]
\[ \dot{x} = 0 \text{ and } x^* = [a \ 0]^T \Rightarrow \]
\[ 0 = x_2^* - x_1^* - \alpha x_1^{*2} - \beta x_1^{*3} + u^* \Rightarrow \]
\[ u^* = x_1^* + \alpha x_1^{*2} + \beta x_1^{*3} = a + \alpha a^2 + \beta a^3 \]
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Thus \((x^*, u^*) = ([a\ 0]T, a + a\alpha^2 + \beta a^3)\) is an equilibrium for this system. Note that if the requirement were to determine \(u^*\) such that \(x^* = [a\ b]^T\) is an equilibrium, then there would be no possible solution. This system is an example of an uncontrollable system, a topic that we treat in §5.5.

\(\square\)

Stability is generally characterized by asking what happens when a system’s initial conditions are close to, but not exactly equal to, an equilibrium state. We ask this question mathematically by expanding the equations of motion in a Taylor series about the equilibrium, truncating the resulting series, and then determining whether the motions of the resulting approximate solution remain “small” or not.

We carry out the Taylor series expansion as follows. First, we rewrite the state vector and input vector as

\[
x = x^* + \delta x
\]

\[
u = u^* + \delta u
\]

Recall that for a scalar function \(f(x)\), the Taylor series is

\[
f(x) = f(x^* + \delta x) = f(x^*) + f'(x^*)\delta x + \frac{1}{2!}f''(x^*)\delta x^2 + \cdots
\]

For the vector functions of vector arguments here, the Taylor series is expressed similarly:

\[
f(x) = f(x^* + \delta x) = f(x^*) + \frac{\partial f}{\partial x}(x^*, u^*, t)\delta x + \cdots
\]

where the \(\cdots\) represent the higher order terms in the Taylor series. Thus, since the vector field depends on \(x\) and \(u\), we apply the Taylor series expansion to \(f(x, u, t)\) as

\[
f(x, u, t) = f(x^*, u^*, t) + \frac{\partial f}{\partial x}(x^*, u^*, t)\delta x + \frac{\partial f}{\partial u}(x^*, u^*, t)\delta u + \cdots
\]

By definition, the term \(f(x^*, u^*, t) = 0\), so that the differential equation, Eq. (5.1) is approximated as

\[
\dot{x} = \frac{\partial f}{\partial x}(x^*, u^*, t)\delta x + \frac{\partial f}{\partial u}(x^*, u^*, t)\delta u
\]

The terms \(\frac{\partial f}{\partial x}(x^*, u^*, t)\) and \(\frac{\partial f}{\partial u}(x^*, u^*, t)\) are \(n \times n\) and \(n \times p\) matrices, respectively. These matrices are determined by taking the indicated derivatives of the function \(f(\cdot, \cdot, \cdot)\).

First consider \(\frac{\partial f}{\partial x}(x^*, u^*, t)\). The elements of this \(n \times n\) matrix are the derivatives of the components of \(f\) with respect to the components of \(x\), evaluated at the specific values \(x^*\) and \(u^*\):

\[
\frac{\partial f}{\partial x}(x^*, u^*, t) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}_{(x^*, u^*)} \equiv A(t)
\]
where the notation \( |_{(x^*,u^*)} \) denotes that the terms in the matrix are evaluated at \((x, u) = (x^*, u^*)\).

Similarly, the matrix \( \frac{\partial f}{\partial u}(x^*, u^*, t) \) is

\[
\frac{\partial f}{\partial u}(x^*, u^*, t) = \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n}
\end{bmatrix} \equiv B(t) \quad (5.11)
\]

Thus, the linearized equations of motion can be written generally as

\[
\dot{\delta x} = A(t)\delta x + B(t)\delta u \quad (5.12)
\]

In the event that \( A \) and \( B \) do not depend on time, the linear system is called time-invariant.

Frequently, we drop the “\( \delta \)” notation and write the linear system as

\[
\dot{x} = A(t)x + B(t)u \quad (5.13)
\]

One should always be aware that the state \( x \) in the linearized equations is not necessarily the same as the state \( x \) in the nonlinear equations. Of course, if the equilibrium state is the origin, then the two states are approximately equal, differing only because of the nonlinear terms dropped in the truncation of the Taylor series.

**Example 5.3.** Consider the nonlinear system

\[
\dot{x} = \begin{bmatrix} x_2 - (a - x_1^2)x_2 - x_1 - \alpha x_1^2 - \beta x_1^3 + u \end{bmatrix}^T
\]

Linearize about the equilibrium \((x^*, u^*) = ([a 0]^T, a + \alpha a^2 + \beta a^3)\).

As developed in Ex. 5.2, the proposed equilibrium does in fact satisfy \( f(x^*, u^*) = 0 \).

Here \( f_1 = x_2 \) and \( f_2 = -(a - x_1^2)x_2 - x_1 - \alpha x_1^2 - \beta x_1^3 + u \). Taking the partial derivatives and evaluating at the equilibrium leads to

\[
A = \begin{bmatrix} 0 & 1 \\ -1 = 2\alpha a - 3\beta a^2 & 1 \end{bmatrix}
\]

\[
B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

The \( A \) and \( B \) matrices define the linearized system dynamics.

\[ \square \]

**Example 5.4.** Consider the motion of a spinning rigid body with a constant “environmental” torque. The desired motion of spinning body is \( \omega^* = [\Omega_1 \Omega_2 \Omega_3] \). We want to linearize the equations of motion about the desired motion.
Expressed in a principal frame, the environmental torque is $g_e$. The “control” torque, $g^*$, required to maintain the desired motion is easily computed using Euler’s equations:

$$
\dot{\omega} = -\Gamma^{-1}\omega^* I\omega + \Gamma^{-1}g_e + \Gamma^{-1}g = f(x, u) \quad (5.14)
$$

$$
0 = -\Gamma^{-1}\omega^* I\omega^* + \Gamma^{-1}g_e + \Gamma^{-1}g^* = f(x^*, u^*) \quad (5.15)
$$

$$
g^* = \omega^* I\omega^* - g_e \quad (5.16)
$$

Given the desired motion, $\omega^*$ and the environmental torque $g_e$, we can easily compute $g^*$, which is clearly constant.

In this problem, $x = \omega$, $u = g$, $x^* = \omega^*$, and $u^* = g^*$. For this example, we write out the equations in scalar form so that the derivatives are easy to see.

$$
\dot{\omega}_1 = \frac{I_2 - I_3}{I_1} \omega_2 \omega_3 + \frac{g_{e1}}{I_1} + \frac{g_1}{I_1} = f_1(x, u) \quad (5.17)
$$

$$
\dot{\omega}_2 = \frac{I_3 - I_1}{I_2} \omega_1 \omega_3 + \frac{g_{e2}}{I_2} + \frac{g_2}{I_2} = f_2(x, u) \quad (5.18)
$$

$$
\dot{\omega}_3 = \frac{I_1 - I_2}{I_3} \omega_1 \omega_2 + \frac{g_{e3}}{I_3} + \frac{g_3}{I_3} = f_3(x, u) \quad (5.19)
$$

The matrix, $\frac{\partial f}{\partial x}(x^*, u^*, t)$, is

$$
\frac{\partial f}{\partial x}(x^*, u^*, t) = \begin{bmatrix}
0 & \frac{I_3 - I_1}{I_1} \Omega_3 & \frac{I_2 - I_3}{I_1} \Omega_2 \\
\frac{I_3 - I_1}{I_2} \Omega_3 & 0 & \frac{I_1 - I_3}{I_2} \Omega_1 \\
\frac{I_2 - I_3}{I_3} \Omega_2 & \frac{I_1 - I_3}{I_2} \Omega_1 & 0
\end{bmatrix} \equiv A \quad (5.20)
$$

Clearly $A$ depends only on the inertias and the desired steady motion $\omega^*$.

Similarly, the matrix $\frac{\partial f}{\partial u}(x^*, u^*, t)$ is

$$
\frac{\partial f}{\partial u}(x^*, u^*, t) = \begin{bmatrix}
\frac{1}{I_1} & 0 & 0 \\
0 & \frac{1}{I_2} & 0 \\
0 & 0 & \frac{1}{I_3}
\end{bmatrix} \equiv B \quad (5.21)
$$

As with $A$, the matrix $B$ is constant and depends only on the inertias. Note that $B = I^{-1}$.

Thus, the linearized equations of motion are

$$
\delta \dot{\omega} = A \delta \omega + B \delta g \quad (5.22)
$$

where $A$ and $B$ are as developed above, $\delta \omega = \omega - \omega^*$, and $\delta g = g - g^*$. A typical control problem is to determine the control $\delta g$ that will maintain the desired motion in the presence of initial condition errors and other disturbances.

A recommended exercise is to choose values of $I$, $g_e$, and $\omega^*$, and then compare the results of integrating the nonlinear differential equations with the results of integrating the linear equations.

□
A similar approach is taken to obtain the linearized output equation for Eq. (5.2):

\[ y = g(x, u, t) \]  
(5.23)

\[ y^* + \delta y = g(x^* + \delta x, u^* + \delta u, t) \]  
(5.24)

\[ \delta y = \frac{\partial g}{\partial x}(x^*, u^*, t)\delta x + \frac{\partial g}{\partial u}(x^*, u^*, t)\delta u \]  
(5.25)

\[ \delta y = C\delta x + D\delta u \]  
(5.26)

The general form for a linear system is then

\[ \dot{x} = Ax + Bu \]  
(5.27)

\[ y = Cx + Du \]  
(5.28)

where the four matrices, \( A, B, C, \) and \( D \) may be constant or time-dependent. These four matrices are called the plant, input, output, and feedforward matrices, respectively, and much of linear control theory is based on the study of equations in this form.

A typical control application of the linearized system involves a feedback control in the form \( u = -Kx \), which leads to

\[ \dot{x} = Ax - BKx \]  
(5.29)

\[ = [A - BK]x \]  
(5.30)

\[ \dot{\tilde{x}} = \tilde{A}x \]  
(5.31)

This type of control is called full-state feedback, and \( K \) is called the gain matrix. We revisit this application later, but point out here that the stability analysis of \( \dot{x} = Ax \) is equally applicable to \( \dot{x} = \tilde{A}x \).

### 5.3 Stability Analysis

Before developing a control law for a dynamic system, we should first decide whether any control is required. For example, if the system will remain near the desired motion without expending any control effort, then perhaps no control is needed. Thus, we want to determine the stability of particular motions. For linear systems, we are generally interested in the stability of the origin, as the process of linearizing about a particular motion includes em shifting the origin so that the origin is the desired motion.

In this section, we explain how to determine the stability of the origin of an uncontrolled, time-invariant, linear system:

\[ \dot{x} = Ax \]  
(5.32)

where \( A \) has been obtained using the linearization process defined in the previous section. The solution to this differential equation is well-known to be

\[ x(t) = e^{At}x(0) \]  
(5.33)
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Furthermore, the stability conditions are well-known, and are usually expressed in terms of the eigenvalues of $A$, $\lambda_j(A)$:

\[
\begin{align*}
\text{Re } \lambda_j < 0 \quad &\Rightarrow \quad \text{stable} \\
\text{Re } \lambda_j > 0 \quad &\Rightarrow \quad \text{unstable} \\
\text{Re } \lambda_j = 0 \quad &\Rightarrow \quad \text{inconclusive}
\end{align*}
\]

In the rest of this section, we develop these results.

5.3.1 Diagonal Systems

Stability analysis of this problem is based on the eigenvalue decomposition of the plant matrix $A$. Consider the case where $A$ is diagonal; i.e., all of the off-diagonal elements of $A$ are zero. In that case, the differential equation, Eq. (5.32) decouples into the $n$ first-order differential equations

\[
\dot{x}_j = A_{jj}x_j, \quad j = 1, \cdots, n
\]

and each solution is immediately integrable as

\[
x_j(t) = e^{A_{jj}t}x_j(0)
\]

Clearly, if $A_{jj} < 0$, then $\lim_{t \to \infty} x_j(t) = 0$, and if $A_{jj} > 0$, then $\lim_{t \to \infty} x_j(t) = \infty$. This observation motivates the statement that $A_{jj} < 0 \Rightarrow$ stability, whereas $A_{jj} > 0 \Rightarrow$ instability, where we are referring to the stability or instability of the $j$th state. If all of the states are stable, then the system is stable. If any of the states are unstable, then the system is unstable.

Furthermore, the diagonal elements of a diagonal matrix are the eigenvalues of the matrix, typically denoted $\lambda_j(A), j = 1, \ldots, n$. Thus, we can state that, for a diagonal matrix, $\lambda_j < 0 \forall j \Rightarrow$ stability, whereas $\lambda_j > 0$ for any $j \Rightarrow$ instability.

5.3.2 Eigenvalue Decomposition

In general, $A$ is not diagonal, and its eigenvalues are not real, but rather are either real or are in complex conjugate pairs. We denote the real and imaginary parts of a particular $\lambda_j$ by Re $\lambda_j$ and Im $\lambda_j$, respectively. The generalization of our stability statements for a general $A$ matrix is as follows: $\text{Re } \lambda_j < 0 \forall j \Rightarrow$ stability, and $\text{Re } \lambda_j > 0$ for any $j \Rightarrow$ instability. If $\text{Re } \lambda_j = 0$ for any $j$, then the stability of the system requires further study, which we discuss later.

For the general constant-coefficient system, we can find solutions to the differential equation by seeking solutions of the form

\[
x(t) = e^{\lambda t}e
\]
where $\lambda$ and $e$ are unknown constants ($\lambda \in \mathbb{C}, e \in \mathbb{C}^n$). If we substitute this assumed form in the differential equation, we obtain

$$\lambda e^{\lambda t} = Ae^{\lambda t}$$ (5.40)

$$e^{\lambda t} [\lambda 1 - A] e = 0$$ (5.41)

The scalar $e^{\lambda t} \neq 0$ for any value of $\lambda t$, so we can divide the expression above by this value and obtain:

$$[\lambda 1 - A] e = 0$$ (5.42)

We are only interested in non-trivial solutions, so $e \neq 0$, from which we conclude that the matrix $[\lambda 1 - A]$ must be singular in order for solutions to exist. Thus

$$\det [\lambda 1 - A] = 0$$ (5.43)

The determinant is an $\text{nd}$ order polynomial in $\lambda$, and in general has $n$ roots, which comprise the eigenvalues of the matrix $A$. In general, there are only a few possibilities for the eigenvalues: all $n$ eigenvalues are distinct; some of the eigenvalues are repeated; all $n$ eigenvalues are real; some of the eigenvalues are complex conjugate pairs.

The characteristic polynomial, denoted $p(\lambda)$ contains significant information as regards the system stability, and there are useful methods of extracting that stability information without actually solving for the roots of the polynomial. Of course, these methods were devised well before the widespread use of the modern computational tools now available. However, these methods are still useful for determining stability in terms of the parameters in specific problems.

**Example 5.5.** Compute the eigenvalues of a $2 \times 2$ matrix.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 6 \end{bmatrix} \Rightarrow [\lambda 1 - A] = \begin{bmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 6 \end{bmatrix}$$

$$\det [\lambda 1 - A] = 0 \Rightarrow p(\lambda) = (\lambda - 1)(\lambda - 6) + 2 = 0$$

$$\lambda^2 - 7\lambda + 8 = 0 \Rightarrow \lambda = 5.5616, 1.4384$$

Matlab: `eig([1 2; -1 6])` and `roots([1 -7 8])` both give this result.

Suppose we have found $n$ distinct eigenvalues, $\lambda_j, j = 1, \ldots, n$. For each eigenvalue, there is a distinct eigenvector $e_j$, such that

$$A e_j = \lambda_j e_j, j = 1, \ldots, n \ (\text{cf. Eq. (5.42)})$$ (5.44)

If we combine these $n$ relationships, we can form a matrix equation:

$$A E = E \Lambda$$ (5.45)
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where $E = [e_1 \cdots e_n]$ and $\Lambda = \text{diag} [\lambda_1 \cdots \lambda_n]$. The eigenvectors are linearly independent, so $E$ is non-singular, and $A$ can be written as

$$A = E\Lambda E^{-1} \quad (5.46)$$

Recall the assumed form of the solution,

$$x_j(t) = e^{\lambda_j t} e_j \quad (5.47)$$

Since there are $n$ linearly independent eigenvectors, the general solution for arbitrary initial conditions can be written as a linear combination of the $n$ linearly independent solution:

$$x(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} e_j \quad (5.48)$$

where the $c_j$ are determined from the initial data $x(0)$. The summation can be rewritten as

$$x(t) = E \text{ diag } [e^{\lambda_1 t} \cdots e^{\lambda_j t} \cdots e^{\lambda_n t}] \ c = E e^{At} c = E e^{At} E^{-1} x(0) \quad (5.49)$$

Comparison of Eq. (5.49) with Eq. (5.33) indicates that

$$e^{At} = E e^{At} E^{-1} \quad (5.50)$$

which should be compared with the similar relationship in Eq. (5.46).

Also, we can use the eigenvalue decomposition to decouple the dynamics. Premultiplying both sides of Eq. (5.49) by $E^{-1}$ leads to

$$E^{-1} x(t) = e^{At} E^{-1} x(0) \quad (5.51)$$

and by defining $z(t) = E^{-1} x(t)$, we have effectively changed variables so that the new system is completely decoupled:

$$z(t) = e^{At} z(0) \quad (5.52)$$

This application is one of the more important applications of the eigenvalue decomposition of the plant matrix $A$.

Note that just as with the case of a diagonal $A$ matrix, the solution turns out to involve constants multiplying terms with $e^{\lambda_j t}$. Thus the stability criteria described above apply: $\text{Re } \lambda_j < 0 \forall j \Rightarrow$ stability, and $\text{Re } \lambda_j > 0$ for any $j \Rightarrow$ instability.
Example 5.6. Calculate the eigensystem and develop the decoupled system for a $3 \times 3$ system with real eigenvalues.

$$
A = \begin{bmatrix}
-2 & -1 & 1 \\
-0.5 & -2.5 & 0.5 \\
0.5 & -0.5 & -1.5 \\
\end{bmatrix}
$$

Matlab: \([\Lambda, E] = \text{eig}(A)\)

$$
\Lambda = \begin{bmatrix}
-3 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -2 \\
\end{bmatrix}
$$

$$
E = \begin{bmatrix}
0.7071 & -0.7071 & 0 \\
0.7071 & 0 & 0.7071 \\
0 & -0.7071 & 0.7071 \\
\end{bmatrix}
$$

Matlab check: $E \Lambda \text{inv}(E)$ should $= A$

Initial conditions

$$
x(0) = [\begin{array}{c} 2 \\ -3 \\ 1 \end{array}]^T
$$

$$
z(0) = E^{-1} [\begin{array}{c} 2 \\ -3 \\ 1 \end{array}]^T
$$

$z(0)$

$$
[\begin{array}{c}
-1.4142 \\
-4.2426 \\
-2.8284
\end{array}]^T
$$

Now we can write the solution as

$$
z(t) = z_1(0)e^{-3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z_2(0)e^{-1t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z_3(0)e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

or

$$
x(t) = c_1e^{-3t} \begin{bmatrix} 0.7071 \\ 0.7071 \\ 0 \end{bmatrix} + c_2e^{-1t} \begin{bmatrix} -0.7071 \\ 0 \\ -0.7071 \end{bmatrix} + c_3e^{-2t} \begin{bmatrix} 0 \\ 0.7071 \\ 0.7071 \end{bmatrix}
$$

□

Example 5.7. Calculate the eigensystem and develop the decoupled system for a $3 \times 3$ system with 1 real eigenvalue and a complex conjugate pair of eigenvalues.

$$
A = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 3 & -2 \\
0 & 5 & -3 \\
\end{bmatrix}
$$

$$
\Lambda = \begin{bmatrix}
i & 0 & 0 \\
0 & -i & 0 \\
0 & 0 & -1 \\
\end{bmatrix}
$$

$$
E = \begin{bmatrix}
0.5071 + 0.1690i & 0.5071 - 0.1690i & 0 \\
0.5071 - 0.1690i & 0.5071 + 0.1690i & 0 \\
0.8452 & 0.8452 & 0 \\
\end{bmatrix}
$$
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Λ is diagonal, but it is also complex. We can create a real-valued block-diagonal matrix and block-diagonal eigenvalue matrix as follows

\[
E_b = \begin{bmatrix} Re\{e_1\} & Im\{e_1\} & e_3 \\ 0 & 0 & 1 \\ 0.5071 & 0.1690 & 0 \\ 0.8452 & 0 & 0 \end{bmatrix}
\]

\[
\Lambda_b = E_b^{-1} A E_b
\]

\[
\Lambda_b = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\]

Thus the system is block-diagonalized. This factorization is not quite as good as diagonalization, but it partially decouples the states. The solution is written as

\[
x(t) = E_b e^{\Lambda_b t} E_b^{-1} x(0)
\]

where

\[
e^{\Lambda_b t} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}
\]

The partially decoupled states are

\[
z(t) = E_b^{-1} x(t)
\]

\[
z(t) = e^{\Lambda_b t} z(0)
\]

\[
z(t) = z_1(0) \begin{bmatrix} \cos t \\ -\sin t \\ 0 \end{bmatrix} + z_2(0) \begin{bmatrix} \sin t \\ \cos t \\ 0 \end{bmatrix} + z_3(0) \begin{bmatrix} 0 \\ 0 \\ e^{-t} \end{bmatrix}
\]

The block-diagonalization procedure works for complex eigenvalues of the form \( \sigma + i\omega \), and makes use of Euler’s identity:

\[
e^{\sigma + i\omega} = e^\sigma e^{i\omega} = e^\sigma (\cos \omega + i \sin \omega)
\]

The matrix exponential, \( e^{At} \), arises in the solution of linear systems, and we need to see how to evaluate this function. The matrix exponential function is defined so that it has the same Taylor series form as the scalar exponential. Here are the two Taylor series expansions:

\[
e^{at} = 1 + at + \frac{1}{2!} a^2 t^2 + \frac{1}{3!} a^3 t^3 + \cdots + \frac{1}{n!} a^n t^n + \cdots
\]

\[
e^{At} = 1 + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots + \frac{1}{n!} A^n t^n + \cdots
\]

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In the development of the eigenvalue decomposition, we claim that
\[ e^{At} = \text{diag} [e^{\lambda_1 t} \ldots e^{\lambda_j t} \ldots e^{\lambda_n t}] \] (5.55)
for the diagonal \( \Lambda \). Here we expand a simple \( 2 \times 2 \) example to make the case that our claim is true.

**Example 5.8.** Compute the exponential of a \( 2 \times 2 \) diagonal matrix.

\[
A = \begin{bmatrix}
a_1 & 0 \\
0 & a_2 \\
\end{bmatrix}
\]

\[
e^{At} = 1 + A t + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots + \frac{1}{n!} A^n t^n + \cdots
\]

\[
= \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} + \begin{bmatrix}
a_1 & 0 \\
0 & a_2 \\
\end{bmatrix} t + \frac{1}{2!} \begin{bmatrix}
a_1 & 0 \\
0 & a_2 \\
\end{bmatrix} t^2 + \frac{1}{3!} \begin{bmatrix}
a_1 & 0 \\
0 & a_2 \\
\end{bmatrix} t^3 + \cdots
\]

\[
= \begin{bmatrix}
1 + a_1 t + \frac{1}{2!} a_1^2 t^2 + \frac{1}{3!} a_1^3 t^3 + \cdots & 0 \\
0 & 1 + a_2 t + \frac{1}{2!} a_2^2 t^2 + \frac{1}{3!} a_2^3 t^3 + \cdots
\end{bmatrix}
\]

\[
= \begin{bmatrix}
e^{a_1 t} & 0 \\
0 & e^{a_2 t}
\end{bmatrix}
\]

While this example does not constitute a proof, it does make it clear that the exponential of a diagonal matrix is the diagonal matrix of exponentials.

\[
\square
\]

In Ex. 5.7, we claim that the matrix exponential of block-diagonal \( \Lambda_b \) involves sines and cosines. Here we complete two examples that illustrate the calculation of the matrix exponential of a \( 2 \times 2 \) non-diagonal matrix, such as arises when we have complex eigenvalues. The first example uses a simple \( 2 \times 2 \) block in the form of Ex. 5.7, which involves a system with a pure imaginary complex conjugate pair of eigenvalues.

**Example 5.9.** Compute the exponential of a \( 2 \times 2 \) block-diagonal matrix.

\[
A = \begin{bmatrix}
0 & \omega \\
-\omega & 0 \\
\end{bmatrix}
\]

\[
e^{At} = 1 + A t + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots + \frac{1}{n!} A^n t^n + \cdots
\]

\[
= \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} + \begin{bmatrix}
0 & \omega \\
-\omega & 0 \\
\end{bmatrix} t + \frac{1}{2!} \begin{bmatrix}
0 & \omega \\
-\omega & 0 \\
\end{bmatrix} t^2 + \frac{1}{3!} \begin{bmatrix}
0 & \omega \\
-\omega & 0 \\
\end{bmatrix} t^3 + \cdots
\]

\[
= \begin{bmatrix}
1 + \frac{1}{2!} \omega^2 t^2 + \cdots & \omega t - \frac{1}{3!} \omega^3 t^3 + \cdots \\
-\omega t + \frac{1}{3!} \omega^3 t^3 + \cdots & 1 - \frac{1}{2!} \omega^2 t^2 + \cdots
\end{bmatrix}
\]

\[
= \begin{bmatrix}
cos \omega t & \sin \omega t \\
-\sin \omega t & \cos \omega t
\end{bmatrix}
\]

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5.3. STABILITY ANALYSIS

As with the previous example, this example does not constitute a proof. However, it does make it clear that the exponential of a block-diagonal matrix of this form is a simple $2 \times 2$ rotation matrix.

The next example involves the more general case where the eigenvalues are a complex conjugate pair of the form $\lambda_{1,2} = \sigma \pm i\omega$. In this case, the $2 \times 2$ block of $\Lambda_b$ is

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

In this case, the most straightforward derivation is to use the eigenvalue decomposition of $A$, and then apply $e^{At} = E e^{\Lambda t} E^{-1}$.

**Example 5.10.** The $2 \times 2$ block-diagonal matrix associated with a complex conjugate eigenvalue pair is

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

The eigenvalues are $\lambda_{1,2} = \sigma \pm i\omega$, and one easily finds the eigenvectors to be

$$E = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

Note that we have not normalized these eigenvectors, a standard procedure that is unnecessary, and in this case, would make the matrix $E$ more complicated than it is in the simple form given. The inverse of $E$ is

$$E^{-1} = \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix}$$

Then we can form $e^{At}$ as

$$e^{At} = \exp \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{(\sigma+i\omega)t} & 0 \\ 0 & e^{(\sigma-i\omega)t} \end{bmatrix} \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix}$$

Recall that $e^{(\sigma+i\omega)t} = e^{\sigma t} (\cos \omega t + i \sin \omega t)$. With this identity, we can carry out the algebra to obtain

$$e^{At} = \exp \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix} e^{\sigma t}$$

Clearly, if $\sigma < 0$, then the state goes to the origin as $t \to \infty$, and if $\sigma > 0$, the state goes to infinity as $t \to \infty$, thus justifying the stability claim made in Eqs. (5.34) and (5.35).

□
We summarize by providing an example with five eigenvalues: one real, $\lambda_1$; two pure imaginary, $\lambda_{2,3} = \pm i\omega_1$; and two complex conjugates, $\lambda_{4,5} = \sigma \pm i\omega_2$.

**Example 5.11.** Compute the matrix exponential $e^{\Lambda_b t}$ for the $5 \times 5$ matrix

$$\Lambda_b = \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & 0 & \omega_1 & 0 & 0 \\
0 & -\omega_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma & \omega_2 \\
0 & 0 & 0 & -\omega_2 & \sigma
\end{bmatrix}$$

The matrix exponential of a block-diagonal matrix is a block-diagonal matrix, with the same basic structure of $\Lambda_b$. Thus, using the previously developed simple matrix exponentials, we can write down that

$$e^{\Lambda_b t} = \begin{bmatrix}
\begin{array}{ccccc}
e^{\lambda_1 t} & 0 & 0 & 0 & 0 \\
0 & \cos \omega_1 t & \sin \omega_1 t & 0 & 0 \\
0 & -\sin \omega_1 t & \cos \omega_1 t & 0 & 0 \\
0 & 0 & 0 & e^{\sigma t} \cos \omega_2 t & e^{\sigma t} \sin \omega_2 t \\
0 & 0 & 0 & -e^{\sigma t} \sin \omega_2 t & e^{\sigma t} \cos \omega_2 t
\end{array}
\end{bmatrix}$$

Note that for any matrix $A$ that has the five eigenvalues given in this example, $\Lambda_b$ is a block-diagonalization of that $A$ matrix.

□

There is one more important property of $A$ that we need to know: the matrix $A$ satisfies its own characteristic polynomial. This property is known as the Cayley-Hamilton Theorem. Specifically, if the characteristic polynomial of $A$ is

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

then $A$ satisfies the matrix polynomial

$$p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 1 = 0$$

A useful application of the Cayley-Hamilton Theorem is the straightforward proof that all powers of $A$ can be written as linear combinations of $\{1, A, A^2, \cdots A^{n-1}\}$. Thus, any infinite series of $A$ can be written in terms of these $n$ matrices. Specifically, the matrix exponential $e^{At}$ can be written as

$$e^{At} = a_0(t)1 + a_1(t)A + a_2(t)A^2 + \cdots + a_{n-1}(t)A^{n-1}$$

This application is especially important in developing the conditions for controllability and observability.

Another useful application is the analytical computation of the inverse of $A$. Specifically, if one multiplies $p(A)$ by $A^{-1}$, one can solve for $A^{-1}$ in terms of the polynomial coefficients and powers of $A$.  

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5.4 Modal Analysis

In the preceding section, we developed the eigenvalue decomposition, and used it for stability analysis, state vector solution, and state vector decoupling. The state vector decoupling can be developed further into the useful concept of modal analysis. We begin with an example.

**Example 5.12.** The $3 \times 3$ system with plant matrix

$$ A = \begin{bmatrix} 4 & -6 & -1 \\ 2 & -1 & -1 \\ 4 & -4 & -4 \end{bmatrix} $$

has eigenvalues $(-3.6252, 1.3126 \pm i1.9478)$ (so the system is unstable), and eigenvectors

$$ E = \begin{bmatrix} 0.2565 & -0.8154 & -0.8154 \\ 0.1673 & -0.2967 + i0.2740 & -0.2967 - i0.2740 \\ 0.9520 & -0.4109 - i0.0556 & -0.4109 + i0.0556 \end{bmatrix} $$

Thus the block-diagonalization, $A = E_b \Lambda_b E_b^{-1}$, is

$$ E_b = \begin{bmatrix} 0.2565 & -0.8154 & 0 \\ 0.1673 & -0.2967 + 0.2740 & 0.2740 \\ 0.9520 & -0.4109 - 0.0556 & -0.0556 \end{bmatrix} $$

$$ \Lambda_b = \begin{bmatrix} -3.6252 & 0 & 0 \\ 0 & 1.3126 & 1.9478 \\ 0 & -1.9478 & 1.3126 \end{bmatrix} $$

We have three different state vectors that we can use to study the dynamics of this system: $x$, $z$, and $z_b$. Each state vector has a different plant matrix, but all three have the same eigenvalues. The $A$ matrix is the plant matrix for the $x$ state vector, the $\Lambda$ matrix is the plant matrix for the $z$ state vector, and the $\Lambda_b$ matrix is the plant matrix for the $z_b$ state vector. The $z$ state vector has the advantage that the states are completely decoupled, but the problem with the decoupled state vector is that its components are complex-valued, which makes plotting and other visualization techniques problematic. The $z_b$ vector has the advantage over $x$ that its states are partially decoupled, and the advantage over $z$ that its components are real-valued.

Consider the initial condition $x(0) = [1 \ 0 \ 0]^T$. If we plot the solution for one period of the oscillatory part of the motion ($T = 2\pi / \text{Im}(\lambda_{2,3})$), using $x(t) = e^{A^t} x(0)$, we obtain the plot in Fig. 5.1. Even though the initial condition has $x_2 = x_3 = 0$, and one of the eigenvalues has a negative real part, we still find that all three states experience exponential growth due to the system being unstable.

If we transform the state from $x$ to $z_b$, we find that $z_1 \to 0$, since it is the state associated with the stable eigenvalue, $\lambda_1 = -3.6252$ (see Fig. 5.2). The other two
Figure 5.1: State Vector with $\mathbf{x}(0) = [1 \ 0 \ 0]^T$
5.5 CONTROLLABILITY, OBSERVABILITY, STABILIZABILITY, AND DETECTABILITY

Thus far we have worked almost exclusively with the zero-input state dynamics; that is, we have worked with $\dot{x} = Ax$. In this section, we investigate various properties
Figure 5.3: State Vector with $\mathbf{z}(0) = [-0.6897 \ 0 \ 0]^T$
Figure 5.4: State Vector with $\mathbf{z}(0) = [0 \quad -1.4433 \quad -1.1421]^T$
associated with the complete linear system:

\[
\begin{align*}
x & = Ax + Bu \\
y & = Cx + Du
\end{align*}
\]  

(5.59)

(5.60)

We begin by developing the solution to the state dynamics equation with input, and then proceed with the definitions of and tools for analysis of the concepts of controllability, observability, stabilizability, and detectability.

### 5.5.1 State Vector Solution

Consider the \( n = p = 1 \) differential equation

\[
\begin{align*}
\dot{x} & = ax + u \\
\dot{x} - ax & = u
\end{align*}
\]  

(5.61)

(5.62)

where \( a \) is a constant scalar, and \( u = u(t) \) is the time-varying input. We begin by multiplying both sides of Eq. (5.62) by an unknown function \( y(t) \), and determine what value of \( y(t) \) makes the left-hand side a total derivative, \( d(xy)/dt \):

\[
\begin{align*}
y\dot{x} - axy & = uy \\
y\dot{x} - axy & = \frac{d}{dt}(xy) = \dot{x}y + x\dot{y}
\end{align*}
\]  

(5.63)

(5.64)

Subtracting \( y\dot{x} \) from both sides, we are left with

\[-axy = xy \Rightarrow -ay = \dot{y} \Rightarrow y(t) = e^{-at}y(0)\]

(5.65)

Taking \( y(0) = 1 \), we have

\[
\begin{align*}
y(t) & = e^{-at} \\
\frac{d}{dt}(xy) & = uy \\
d(xy) & = uy \, dt
\end{align*}
\]  

(5.66)

(5.67)

(5.68)

\[
\int_0^t d(xy) = \int_0^t uy(\tau) \, d\tau
\]

(5.69)

\[
x(t)y(t) - x(0)y(0) = \int_0^t uy(\tau) \, d\tau
\]

(5.70)

\[
x(t)e^{-at} - x(0) = \int_0^t ue^{-a\tau} \, d\tau
\]

(5.71)

\[
x(t)e^{-at} = x(0) + \int_0^t ue^{-a\tau} \, d\tau
\]

(5.72)

\[
x(t) = e^{at}x(0) + e^{at} \int_0^t ue^{-a\tau} \, d\tau
\]

(5.73)
The form of the solution for general $n$ and $p$ is the same as Eq. (5.73):

$$x(t) = e^{At}x(0) + e^{At} \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$  (5.74)

Evaluation of the solution of Eqs. (5.73) and (5.74) depends on the form of the input $u$. 

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5.9 Summary of Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
<td>( f(x, u, t) ), vector field, maps ( \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \mapsto \mathbb{R}^n )</td>
</tr>
<tr>
<td>( g )</td>
<td>( g(x, u, t) ), output function, maps ( \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \mapsto \mathbb{R}^m )</td>
</tr>
<tr>
<td>( u )</td>
<td>input vector, element of ( \mathbb{R}^p )</td>
</tr>
<tr>
<td>( u^* )</td>
<td>constant input vector, element of ( \mathbb{R}^p )</td>
</tr>
<tr>
<td>( \delta u )</td>
<td>perturbation from constant input vector, element of ( \mathbb{R}^p )</td>
</tr>
<tr>
<td>( x )</td>
<td>state vector, element of ( \mathbb{R}^n )</td>
</tr>
<tr>
<td>( x^* )</td>
<td>constant state vector, element of ( \mathbb{R}^n )</td>
</tr>
<tr>
<td>( \delta x )</td>
<td>perturbation from constant state vector, element of ( \mathbb{R}^n )</td>
</tr>
<tr>
<td>( y )</td>
<td>output vector, element of ( \mathbb{R}^m )</td>
</tr>
<tr>
<td>( z )</td>
<td>decoupled state vector, element of ( \mathbb{C}^n )</td>
</tr>
<tr>
<td>( z_b )</td>
<td>partially decoupled state vector, element of ( \mathbb{R}^n )</td>
</tr>
<tr>
<td>( A )</td>
<td>plant matrix, element of ( \mathbb{R}^{n\times n} )</td>
</tr>
<tr>
<td>( B )</td>
<td>input matrix, element of ( \mathbb{R}^{n\times p} )</td>
</tr>
<tr>
<td>( C )</td>
<td>output matrix, element of ( \mathbb{R}^{m\times n} )</td>
</tr>
<tr>
<td>( D )</td>
<td>feedforward matrix, element of ( \mathbb{R}^{m\times p} )</td>
</tr>
<tr>
<td>( \lambda_j )</td>
<td>eigenvalues of ( A ), ( j = 1, \ldots, n ), element of ( \mathbb{C} )</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>diagonal matrix of eigenvalues of ( A ), element of ( \mathbb{C}^{n\times n} )</td>
</tr>
<tr>
<td>( \Lambda_b )</td>
<td>block diagonal matrix of eigenvalues of ( A ), element of ( \mathbb{R}^{n\times n} )</td>
</tr>
<tr>
<td>( e_j )</td>
<td>eigenvector of ( A ), ( j = 1, \ldots, n ), element of ( \mathbb{C}^n )</td>
</tr>
<tr>
<td>( E )</td>
<td>eigenvector matrix of ( A ), element of ( \mathbb{C}^{n\times n} )</td>
</tr>
<tr>
<td>( E_b )</td>
<td>modified eigenvector matrix of ( A ), element of ( \mathbb{R}^{n\times n} )</td>
</tr>
</tbody>
</table>

5.10 References and Further Reading

The literature associated with linear systems, analysis, and control theory is quite rich. An excellent textbook on modern control theory is Brogan.\(^1\) Elementary feedback control theory is treated effectively in Franklin \textit{et al.}\(^2\) Reid\(^3\) covers a variety of linear systems concepts, with especially useful treatment of the \( z \) transform. Stengel\(^4\) provides a thorough treatment of linear systems with uncertainty, including the Kalman filter and Extended Kalman filter. Strang\(^5\) is a linear algebra textbook that is especially succinct in its treatment of linear algebra topics that are useful in linear systems theory.

Bibliography

5.11 Exercises

1. Derive the equation of motion for a simple pendulum. Determine the equilibrium motions and linearize about each of them. Characterize the stability properties for each equilibrium.

2. Find the eigenvalues and eigenvectors of the matrix

\[
A = \begin{bmatrix}
1 & 7 & 0 \\
-1 & 2 & 3 \\
-6 & 1 & 3
\end{bmatrix}
\]

*Matlab Hint: [evecs,evals]=eig(A) returns the eigenvectors and eigenvalues of A.*

3. Find the eigenvalues and eigenvectors of the matrix

\[
A = \begin{bmatrix}
0.2750 & -0.0610 & 0.2880 \\
-0.0690 & 0.6170 & 0.0420 \\
-0.2970 & -0.1330 & 0.9600
\end{bmatrix}
\]

4. Develop the characteristic polynomial for the matrix

\[
A = \begin{bmatrix}
1 & 7 & 0 \\
-1 & 2 & 3 \\
-6 & 1 & 3
\end{bmatrix}
\]

What are the roots of this polynomial? *Matlab Hint: roots([c(1) c(2) \cdots c(n+1)]) returns the roots of the polynomial \(c_1 x^n + c_2 x^{n-1} + \cdots + c_n x + c_{n+1} = 0\).*

5. Determine the equilibrium motions for the system

\[
\begin{align*}
\dot{x}_1 &= x_1(1 + x_2) \\
\dot{x}_2 &= x_2(1 - x_1)
\end{align*}
\]

Linearize about the equilibria and determine their stability.
5.12. PROBLEMS

5.12 Problems

1. Consider the motion of a spinning rigid body with a constant “environmental” torque. The desired motion of the spinning body is $\omega^* = [\Omega_1 \, \Omega_2 \, \Omega_3]^T$. Linearize the equations of motion about the desired motion.

   For your numerical work, use $I = \text{diag}[30 \, 40 \, 50] \text{kg} \cdot \text{m}^2$. The environmental torque is $g_e = [1 \, 1 \, 1]^T \text{Nm}$. For the desired angular velocity, use two different values:

   $$\omega^* = [10 \, 0 \, 6]^T \text{ and } [10 \, 1 \, 6]^T \text{ rad/s}$$

   (a) What are the eigenvalues of the linearized system?

   (b) What is the nominal torque $g^*$?

   (c) What can you say about the stability of the nonlinear system based on your analysis of the linearized system?

   (d) Make a graph of the perturbed angular velocity $\delta \omega$ for initial conditions $\delta \omega = [0.1 \, 0.1 \, 0.1]^T \text{rad/s}$. The timespan of the graph should be one period of the oscillatory part of the solution. *Hint: the imaginary part of the complex conjugate pair of eigenvalues is the frequency of oscillation.*

2. Using the linear system of Prob. 1, add a feedback control torque ($\delta g$) with

   $$u = [-k_1 \delta \omega_1 \, -k_2 \delta \omega_2 \, -k_3 \delta \omega_3]^T$$

   with $k_1 = k_2 = k_3 = 1$. What are the eigenvalues of this system with the feedback control? Make a graph of the perturbed angular velocity $\delta \omega$ for initial conditions $\delta \omega = [0.1 \, 0.1 \, 0.1]^T \text{rad/s}$. The timespan of the graph should be one period of the oscillatory part of the solution.

3. Repeat the previous two problems using the nonlinear system and compare your results.

4. Consider the $5 \times 5$ matrix

   $$A = \begin{bmatrix}
   -5 & 2 & 5 & 8 & 1 \\
   4 & -7 & 9 & 4 & 1 \\
   6 & 3 & -6 & 8 & 6 \\
   1 & 8 & 6 & -7 & 1 \\
   5 & 4 & 2 & 8 & -1
   \end{bmatrix}$$

   Determine the eigenvalues and eigenvectors of $A$; i.e., determine $\Lambda$ and $E$. Determine the real-valued block-diagonalization of $A$; i.e., determine $\Lambda_b$ and $E_b$. 

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Using the initial state $x(0) = [1 \ 2 \ 3 \ 4 \ 5]^T$, determine the initial value of the completely decoupled state $z$ and the block-decoupled state $z_b$. Using the matrix exponential, show that the state at $t = 1$ for each of the three state representations are equivalent.

1. Programming Project.