Marine vessel motion is more complicated than spacecraft motion due to the dependence of forces and moments on attitude and velocity.

Marine vessel motion is in some ways more complicated than aircraft motion.

Typical first investigations of ship dynamics and control focuses on the horizontal motion of symmetric (left-to-right) ships (surge, sway, and yaw).
Reference Frames for Ship D&C

**Inertial Frame, \( F_i \).** Typically assume flat Earth, but rotating Earth is used for some problems. The inertial frame has 3-axis *down*, though some sources have 3-axis *up*

For *maneuvering* problems, an Earth-fixed \( F_i \) is used, and for *seakeeping problems*, a frame moving with the vehicle’s nominal velocity is used

**Body Frame, \( F_b \).** A 3-2-1 rotation from \( F_i \) through yaw (\( \psi \)), pitch (\( \theta \)), and roll (\( \phi \)) angles: 

\[
R^b_i = R_1(\phi)R_2(\theta)R_3(\psi)
\]

*Figure 2.1: Body-fixed and earth-fixed reference frames.*
Reference Frames for Ship D&C (2)

Illustration (from T. Fossten’s *Guidance and Control of Ocean Vehicles*) shows relationship between intermediate axis systems and roll, pitch, and yaw, angles
Motion Equations for a Rigid Ship

\[ \mathcal{F}_b : \dot{v} = -\omega \times v + \frac{1}{m} [ f_{\text{fluid}} + f_{\text{thrust}} ] + R^b_i a_{\text{grav}} \]

\[ \mathcal{F}_b : \dot{\omega} = I^{-1} [ -\omega \times I \omega + g_{\text{fluid}} + g_{\text{thrust}} ] \]

\[ \mathcal{F}_i : \dot{r} = R^{ib} v \]
\[ \dot{\theta} = S^{-1}(\theta) \omega \]

where \( a_{\text{grav}} = [0 0 g]^T \)

Note that \( v \) and \( \omega \) are expressed in \( \mathcal{F}_b \), and \( r \) is expressed in \( \mathcal{F}_i \)

Much of the ship d & c literature follows aircraft notation:

\[ v = [u v w]^T, \quad \omega = [p q r]^T \]

Assuming left-right symmetry (but not up-down symmetry) implies that \( I_{xy} = I_{yz} = 0 \), but in general \( I_{xz} \neq 0 \).
Motion Equations for a Rigid Ship

Equations of motion are typically written in this form

\[ M \ddot{\nu} + C(\nu)\nu + D(\nu)\nu + g(\eta) = \tau_E + \tau \]

where \( \nu = [v^T \ \omega^T]^T \), \( \eta = [r^T \ \theta^T]^T \), \( M \) is the modified mass matrix, \( C \) is a matrix involving the gyroscopic terms, \( D \) is the damping matrix, \( g \) is the vector of restoring forces and moments, \( \tau_E \) is the vector of environmental forces and moments, and \( \tau \) is the vector of control forces and moments.

Determining the various matrices, forces and moments, is rather complicated and is the subject of texts such as


M. S. Triantafyllou and F. S. Hover, *Maneuvering and Control of Marine Vessels*, Department of Ocean Engineering, MIT, 2003 (available online)
Modeling Fluid Forces

Assumption: Forces and moments acting on a rigid body are a linear combination of

1. Radiation-induced forces: body is forced to oscillate with wave excitation frequency and there are no incident waves
   ★ Added mass: $-M_A \dot{\nu} - C_A(\nu)\nu$
   ★ Potential damping: $-D_P(\nu)\nu$
   ★ Weight and buoyancy: $-f_R(\eta)$
   ★ Other damping effects, including skin friction, wave drift damping, and vortex shedding damping

2. Froude-Kriloff and Diffraction Forces

3. Environmental forces: currents, waves, wind

4. Propulsion forces: thruster/propeller, control surfaces
Horizontal Motion

The state vector is

\[
x = \begin{bmatrix} v^T & \omega^T & r^T & \theta^T \end{bmatrix}^T
\]

\[
= [u \ v \ w \ p \ q \ r \ x \ y \ z \ \phi \ \theta \ \psi]^T
\]

For horizontal motion, the ship has zero heave, roll, and pitch; the motion is restricted to surge, sway, and yaw.

The control for horizontal motion typically involves thrusters and rudders; linearized equations are:

\[
(m - X_{\ddot{u}})\ddot{u} = X_u u + X'
\]

\[
(m - Y_{\ddot{v}})\ddot{v} + (mx_G - Y_{\ddot{r}})\ddot{r} = Y_v v + (Y_r - mU)r + Y'
\]

\[
(mx_G - N_{\ddot{v}})\ddot{v} + (I_{zz} - N_{\ddot{r}})\ddot{r} = N_v v - (N_r - mx_G U)r + N'
\]

The various \(X_u\ etc.\) terms are constants; the \(X'\ etc.\) terms are the control forces and moments.
Horizontal Motion (2)

Linearized equations for horizontal motion are:

\[(m - X_{\dot{u}})\dot{u} = X_u u + X'\]
\[(m - Y_{\dot{v}})\dot{v} + (mx_G - Y_{\dot{r}})\dot{r} = Y_v v + (Y_r - mU) r + Y'\]
\[(mx_G - N_{\dot{v}})\dot{v} + (I_{zz} - N_{\dot{r}})\dot{r} = N_v v - (N_r - mx_G U) r + N'\]

Clearly surge \((u)\) is decoupled from sway \((v)\) and yaw \((\psi)\)

The coupled sway-yaw system is

\[
\begin{bmatrix}
  m - Y_{\dot{v}} & mx_G - Y_{\dot{r}} \\
  mx_G - N_{\dot{v}} & I_{zz} - N_{\dot{r}}
\end{bmatrix} \begin{bmatrix} \dot{x} \end{bmatrix} = \begin{bmatrix}
  Y_v & Y_r - mU \\
  N_v & N_r - mx_G U
\end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} f \end{bmatrix}
\]

\[M\dot{x} = A'x + f\]
\[\dot{x} = M^{-1}A'x + M^{-1}f\]
\[\dot{x} = Ax + Bu\]
Horizontal Motion (3)

The coupled sway-yaw system is

\[
\begin{bmatrix}
m - Y_v & mx_G - Y_r \\
m x_G - N_v & I_{zz} - N_r
\end{bmatrix}
\begin{bmatrix}
x \\
\dot{x}
\end{bmatrix}
= \begin{bmatrix}
Y_v & Y_r - mU \\
N_v & N_r - m x_G U
\end{bmatrix}
\begin{bmatrix}
x \\
\dot{x}
\end{bmatrix}
+ \begin{bmatrix}
f
\end{bmatrix}
\]

\[
M \dot{x} = A'x + f
\]

\[
\dot{x} = M^{-1}A'x + M^{-1}f
\]

\[
\dot{x} = Ax + Bu
\]

Stability analysis leads to a simplified stability condition:

\[
C = Y_v N_r + N_v (m U Y_r) > 0
\]

The term \( C \) is called the vessel stability parameter

Ship design criteria are influenced by making \( C > 0 \) true; for example, **adding more aft surface area drives \( N_v \) positive, increasing stability**
Stability Analysis via Routh-Hurwitz

Suppose we know $A$ in terms of system parameters; i.e., $A = A(m, I, x^*, u^*, \cdots)$

How can we determine stability using system parameters?

Routh-Hurwitz stability criteria

Develop the characteristic polynomial for $A$:

$$p(\lambda) = \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_0$$

Develop the Hurwitz matrix

$$H = \begin{bmatrix}
a_{n-1} & a_{n-3} & \cdots & 0 \\
1 & a_{n-2} & \cdots & 0 \\
0 & a_{n-1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_0
\end{bmatrix}$$
Stability Analysis via Routh-Hurwitz (2)

The Hurwitz matrix is

\[ \mathbf{H} = \begin{bmatrix}
    a_{n-1} & a_{n-3} & \cdots & 0 \\
    1 & a_{n-2} & \cdots & 0 \\
    0 & a_{n-1} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a_0 \\
\end{bmatrix} \]

Define the principal minors of \( \mathbf{H} \) by

\[ \Delta_1 = a_{n-1} \]
\[ \Delta_2 = \det \begin{bmatrix} a_{n-1} & a_{n-3} \\ 1 & a_{n-2} \end{bmatrix} \]
\[ \vdots \]
\[ \Delta_n = \det \mathbf{H} \]
Stability Analysis via Routh-Hurwitz (3)

Necessary and sufficient conditions for the asymptotic stability of $\dot{x} = Ax$ are

$$\Delta_i > 0 \quad \forall \ i = 1, \ldots, n$$

Since $A$ depends on the parameters, $p$, the coefficients of $p(\lambda)$ depend on $p$, and the principal minors depend on $p$.

Thus these conditions define regions in parameter space where the linearized system is asymptotically stable.

Stable and unstable regions are separated by stability boundaries, which correspond to one or more eigenvalues crossing the imaginary axis.

If a real eigenvalue crosses the imaginary axis, then at that point in parameter space $a_0(p) = 0$ (exercise: convince yourself that this statement must be true).
Stability Analysis via Routh-Hurwitz (4)

If a real eigenvalue crosses the imaginary axis, then at that point in parameter space $a_0(p) = 0$

If a c.c. pair crosses the imaginary axis, then

$$\Delta_{n-1}(p) = 0$$

Thus the stability boundaries in parameter space are defined by one of the two conditions:

$$a_0(p) = 0$$

$$\Delta_{n-1}(p) = 0$$
Stability Analysis via Routh-Hurwitz (5)

Suppose \( \mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \). Then the characteristic polynomial is 
\[
p(\lambda) = \lambda^2 - (A_{11} + A_{22})\lambda + A_{11}A_{22} - A_{12}A_{21}
\]

The principal minors of \( \mathbf{H} \) are

\[
\Delta_1 = a_1 = -(A_{11} + A_{22}) > 0 \implies A_{11} + A_{22} < 0
\]

\[
\Delta_2 = \det \begin{bmatrix} a_{n-1} & a_{n-3} \\ 1 & a_{n-2} \end{bmatrix} = -(A_{11} + A_{22})(A_{11}A_{22} - A_{12}A_{21}) > 0
\]

These conditions were used in the sway-yaw example. Details are in § 4.2 of Triantafyllou and Hover.
Suppose \( A = \begin{bmatrix} -1 & A_{12} \\ -2 & -4 \end{bmatrix} \). Then the conditions are

\[
\Delta_1 > 0 \Rightarrow -1 - 4 < 0 \quad \checkmark \\
\Delta_2 > 0 \Rightarrow (4 + 2A_{12}) > 0 \\
2A_{12} > -4 \Rightarrow A_{12} > -2
\]

Exercise: verify numerically that the appropriate stability boundary condition is satisfied when \( A_{12} = -2 \) and that the stability condition developed here is valid.