Chapter 3

Kinematics

As noted in the Introduction, the study of dynamics can be decomposed into the study of kinematics and kinetics. For the translational motion of a particle of mass m, this decomposition amounts to expressing Newton's second law,

$$m\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{f}} \tag{3.1}$$

a 2nd-order vector differential equation, as the two 1st-order vector differential equations

$$\dot{\vec{\mathbf{r}}} = \vec{\mathbf{p}}/m \tag{3.2}$$
$$\dot{\vec{\mathbf{p}}} = \vec{\mathbf{f}} \tag{3.3}$$

Here $\vec{\mathbf{r}}$ is the position vector of the particle relative to an inertial origin $\mathbf{O}$, $\vec{\mathbf{p}} = m\dot{\vec{\mathbf{r}}}$ (or $m\vec{\mathbf{v}}$) is the linear momentum of the particle, and $\vec{\mathbf{f}}$ is the sum of all the forces acting on the particle. Equation (3.2) is the kinematics differential equation, describing how position changes for a given velocity; *i.e.*, integration of Eq. (3.2) gives $\vec{\mathbf{r}}(t)$. Equation (3.3) is the kinetics differential equation, describing how velocity changes for a given force. It is also important to make clear that the "dot", $\dot{()}$, represents the rate of change of the vector as seen by a fixed (inertial) observer (reference frame).

In the case of rotational motion of a reference frame, the equivalent to Eq. (3.2) is not as simple to express. The purpose of this chapter is to develop the kinematic equations of motion for a rotating reference frame, as well as the conceptual tools for visualizing this motion. In Chapter 4 we describe how the attitude of a spacecraft is determined. In Chapter 5 we develop the kinetic equations of motion for a rigid body.

This chapter begins with the development of attitude representations, including reference frames, rotation matrices, and some of the variables that can be used to describe attitude motion. Then we develop the differential equations that describe attitude motion for a given angular velocity. These equations are equivalent to Eq. (3.2), which describes translational motion for a given translational velocity.

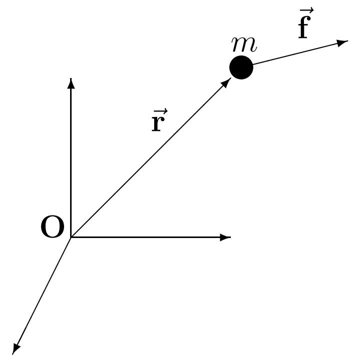

Figure 3.1: Dynamics of a particle

3.1 Attitude Representations

In this section we discuss various representations of the attitude or orientation of spacecraft. We begin by discussing reference frames, vectors, and their representations in reference frames. The problem of representing vectors in different reference frames leads to the development of rotations, rotation matrices, and various ways of representing rotation matrices, including Euler angles, Euler parameters, and quaternions.

3.1.1 Reference Frames

A *reference frame*, or *coordinate system*, is generally taken to be a set of three unit vectors that are mutually perpendicular. An equivalent definition is that a reference frame is a *triad* of *orthonormal* vectors. Triad of course means three, and orthonormal means *ortho*gonal and *normal*. The term orthogonal is nearly synonymous with the term perpendicular, but has a slightly more general meaning when dealing with other sorts of vectors (which we do not do here). The fact that the vectors are normalized means that they are *unit* vectors, or that their lengths are all unity (1) in the units of choice. We also usually use *right-handed* or *dextral* reference frames, which simply means that we order the three vectors in an agreed-upon fashion, as described below.

The reason that reference frames are so important in attitude dynamics is that following the orientation of a reference frame is completely equivalent to following the orientation of a rigid body. Although no spacecraft is perfectly rigid, the rigid body model is a good first approximation for studying attitude dynamics. Similarly, no spacecraft (or planet) is a point mass, but the point mass model is a good first approximation for studying orbital dynamics.

We normally use a triad of unit vectors, denoted by the same letter, with subscripts 1,2,3. For example, an *inertial* frame would be denoted by $\left\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\right\}$, an *orbital* frame by $\{\hat{\mathbf{o}}_1, \hat{\mathbf{o}}_2, \hat{\mathbf{o}}_3\}$, and a body- (or spacecraft-) fixed frame by $\left\{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\right\}$. The hats are used to denote that these are unit vectors. We also use the notation $\mathcal{F}_i$, $\mathcal{F}_o$, and $\mathcal{F}_b$, to represent these and other reference frames.

The orthonormal property of a reference frame's base vectors is defined by the dot products of the vectors with each other. Specifically, for a set of orthonormal base vectors, the dot products satisfy

$$\begin{array}{lll} \hat{\mathbf{i}}_1 \cdot \hat{\mathbf{i}}_1 = 1 & \hat{\mathbf{i}}_1 \cdot \hat{\mathbf{i}}_2 = 0 & \hat{\mathbf{i}}_1 \cdot \hat{\mathbf{i}}_3 = 0 \\ \hat{\mathbf{i}}_2 \cdot \hat{\mathbf{i}}_1 = 0 & \hat{\mathbf{i}}_2 \cdot \hat{\mathbf{i}}_2 = 1 & \hat{\mathbf{i}}_2 \cdot \hat{\mathbf{i}}_3 = 0 \\ \hat{\mathbf{i}}_3 \cdot \hat{\mathbf{i}}_1 = 0 & \hat{\mathbf{i}}_3 \cdot \hat{\mathbf{i}}_2 = 0 & \hat{\mathbf{i}}_3 \cdot \hat{\mathbf{i}}_3 = 1 \end{array} \tag{3.4}$$

which may be written more concisely as

$$\hat{\mathbf{i}}_i \cdot \hat{\mathbf{i}}_j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{3.5}$$

or even more concisely as

$$\hat{\mathbf{i}}_i \cdot \hat{\mathbf{i}}_j = \delta_{ij} \tag{3.6}$$

where δ_{ij} is the Kronecker delta, for which Eq. (3.5) may be taken as the definition.

We often find it convenient to collect the unit vectors of a reference frame into a 3×1 column matrix of vectors, and we denote this object by

$$\left\{\hat{\mathbf{i}}\right\} = \left\{ \begin{array}{c} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 \\ \hat{\mathbf{i}}_3 \end{array} \right\} \tag{3.7}$$

This matrix is a rather special object, as its components are unit vectors instead of scalars. Hughes[1] introduced the term *vectrix* to describe this vector matrix. Using this notation, Eq. (3.4) can be written as

$$\left\{\hat{\mathbf{i}}\right\} \cdot \left\{\hat{\mathbf{i}}\right\}^T = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = \mathbf{1} \tag{3.8}$$

which is the 3×3 identity matrix. The superscript T on $\left\{\hat{\mathbf{i}}\right\}$ transposes the matrix from a column matrix (3×1) to a row matrix (1×3).

The *right-handed* or *dextral* property of a reference frame's base vectors is defined by the cross products of the vectors with each other. Specifically, for a right-handed set of *orthonormal* base vectors, the cross products satisfy

$$\begin{array}{lll} \hat{\mathbf{i}}_1 \times \hat{\mathbf{i}}_1 = \vec{\mathbf{0}} & \hat{\mathbf{i}}_1 \times \hat{\mathbf{i}}_2 = \hat{\mathbf{i}}_3 & \hat{\mathbf{i}}_1 \times \hat{\mathbf{i}}_3 = -\hat{\mathbf{i}}_2 \\ \hat{\mathbf{i}}_2 \times \hat{\mathbf{i}}_1 = -\hat{\mathbf{i}}_3 & \hat{\mathbf{i}}_2 \times \hat{\mathbf{i}}_2 = \vec{\mathbf{0}} & \hat{\mathbf{i}}_2 \times \hat{\mathbf{i}}_3 = \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_3 \times \hat{\mathbf{i}}_1 = \hat{\mathbf{i}}_2 & \hat{\mathbf{i}}_3 \times \hat{\mathbf{i}}_2 = -\hat{\mathbf{i}}_1 & \hat{\mathbf{i}}_3 \times \hat{\mathbf{i}}_3 = \vec{\mathbf{0}} \end{array} \tag{3.9}$$

This set of rules may be written more concisely as

$$\hat{\mathbf{i}}_i \times \hat{\mathbf{i}}_j = \varepsilon_{ijk}\hat{\mathbf{i}}_k \tag{3.10}$$

where ε_{ijk} is the *permutation symbol*, defined as

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for } i,j,k \text{ an even permutation of 1,2,3} \\ -1 & \text{for } i,j,k \text{ an odd permutation of 1,2,3} \\ 0 & \text{otherwise } (i.e., \text{ if any repetitions occur}) \end{cases} \tag{3.11}$$

Equation (3.9) can also be written as

$$\left\{\hat{\mathbf{i}}\right\} \times \left\{\hat{\mathbf{i}}\right\}^T = \left\{ \begin{array}{ccc} \vec{\mathbf{0}} & \hat{\mathbf{i}}_3 & -\hat{\mathbf{i}}_2 \\ -\hat{\mathbf{i}}_3 & \vec{\mathbf{0}} & \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 & -\hat{\mathbf{i}}_1 & \vec{\mathbf{0}} \end{array} \right\} = -\left\{\hat{\mathbf{i}}\right\}^\times \tag{3.12}$$

The $\times$ superscript is used to denote a skew-symmetric 3×3 matrix associated with a 3×1 column matrix. Specifically, if $\mathbf{a}$ is a 3×1 matrix of scalars a_i, then

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Rightarrow \mathbf{a}^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{3.13}$$

Note that $\mathbf{a}^\times$ satisfies the skew-symmetry property $(\mathbf{a}^\times)^T = -\mathbf{a}^\times$.

3.1.2 Vectors

A *vector* is an abstract mathematical object with two properties: direction and length (or magnitude). Vector quantities that are important in this course include, for example, angular momentum, $\vec{\mathbf{h}}$, angular velocity, $\vec{\omega}$, and the direction to the sun, $\hat{\mathbf{s}}$. Vectors are denoted by a bold letter, with an arrow (hat if a unit vector), and are usually lower case.

Vectors can be expressed in any reference frame. For example, a vector, $\vec{\mathbf{v}}$, may be written in the inertial frame as

$$\vec{\mathbf{v}} = v_1\hat{\mathbf{i}}_1 + v_2\hat{\mathbf{i}}_2 + v_3\hat{\mathbf{i}}_3 \tag{3.14}$$

The scalars, v_1, v_2, and v_3, are the *components* of $\vec{\mathbf{v}}$ expressed in $\mathcal{F}_i$. These components are the dot products of the vector $\vec{\mathbf{v}}$ with the three base vectors of $\mathcal{F}_i$. Specifically,

$$v_1 = \vec{\mathbf{v}} \cdot \hat{\mathbf{i}}_1, \quad v_2 = \vec{\mathbf{v}} \cdot \hat{\mathbf{i}}_2, \quad v_3 = \vec{\mathbf{v}} \cdot \hat{\mathbf{i}}_3 \tag{3.15}$$

Since the $\hat{\mathbf{i}}$ vectors are unit vectors, these components may also be written as

$$v_1 = v\cos\alpha_1, \quad v_2 = v\cos\alpha_2, \quad v_3 = v\cos\alpha_3 \tag{3.16}$$

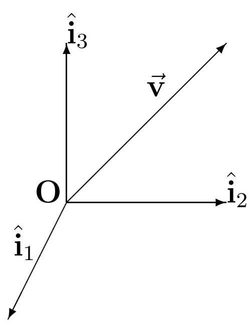

Figure 3.2: Components of a vector

where $v = \|\vec{\mathbf{v}}\|$ is the magnitude or length of $\vec{\mathbf{v}}$, and α_j is the angle between $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}_j$ for $j = 1, 2, 3$. These cosines are also called the *direction cosines* of $\vec{\mathbf{v}}$ with respect to $\mathcal{F}_i$. We frequently collect the components of a vector $\vec{\mathbf{v}}$ into a column matrix $\mathbf{v}$, with three rows and one column:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{3.17}$$

A bold letter without an overarrow (or hat) denotes such a matrix. Sometimes it is necessary to denote the appropriate reference frame, in which case we use $\mathbf{v}_i$, $\mathbf{v}_o$, $\mathbf{v}_b$, *etc.*

A handy way to write a vector in terms of its components and the base vectors is to write it as the product of two matrices, one the component matrix, and the other a matrix containing the base unit vectors. For example,

$$\vec{\mathbf{v}} = [v_1 \quad v_2 \quad v_3] \left\{ \begin{array}{c} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 \\ \hat{\mathbf{i}}_3 \end{array} \right\} = \mathbf{v}_i^T \left\{ \hat{\mathbf{i}} \right\} \tag{3.18}$$

Recall that the subscript i denotes that the components are with respect to $\mathcal{F}_i$.

Using this notation, we can write $\vec{\mathbf{v}}$ in terms of different frames as

$$\vec{\mathbf{v}} = \mathbf{v}_i^T \left\{ \hat{\mathbf{i}} \right\} = \mathbf{v}_o^T \left\{ \hat{\mathbf{o}} \right\} = \mathbf{v}_b^T \left\{ \hat{\mathbf{b}} \right\} \tag{3.19}$$

and so forth.

There are two types of reference frame problems we encounter in this course. The first involves determining the components of a vector in one frame (say $\mathcal{F}_i$) when the components in another frame (say $\mathcal{F}_b$) are known, and the relative orientation of the two frames is known. The second involves determining the components of a vector in a frame that has been reoriented or rotated. Both problems involve *rotations*, which are the subject of the next section.

3.1.3 Rotations

Suppose we have a vector $\vec{\mathbf{v}}$, and we know its components in $\mathcal{F}_b$, denoted $\mathbf{v}_b$, and we want to determine its components in $\mathcal{F}_i$, denoted $\mathbf{v}_i$. Since

$$\vec{\mathbf{v}} = \mathbf{v}_i^T \left\{\hat{\mathbf{i}}\right\} = \mathbf{v}_b^T \left\{\hat{\mathbf{b}}\right\} \tag{3.20}$$

we seek a way to express $\left\{\hat{\mathbf{i}}\right\}$ in terms of $\left\{\hat{\mathbf{b}}\right\}$, say

$$\left\{\hat{\mathbf{i}}\right\} = \mathbf{R}\left\{\hat{\mathbf{b}}\right\} \tag{3.21}$$

where $\mathbf{R}$ is a 3×3 *transformation* matrix. Then we can write

$$\vec{\mathbf{v}} = \mathbf{v}_i^T \left\{\hat{\mathbf{i}}\right\} = \mathbf{v}_i^T \mathbf{R}\left\{\hat{\mathbf{b}}\right\} = \mathbf{v}_b^T \left\{\hat{\mathbf{b}}\right\} \tag{3.22}$$

Comparing the last two terms in this equation, we see that

$$\mathbf{v}_i^T \mathbf{R} = \mathbf{v}_b^T \tag{3.23}$$

Transposing both sides*, we get

$$\mathbf{R}^T \mathbf{v}_i = \mathbf{v}_b \tag{3.24}$$

Thus, to compute $\mathbf{v}_i$, we just need to determine $\mathbf{R}$ and solve the linear system of equations defined by Eq. (3.24).

If we write the components of $\mathbf{R}$ as R_{ij}, where i denotes the row and j denotes the column, then Eq. (3.21) may be expanded to

$$\begin{aligned}
\hat{\mathbf{i}}_1 &= R_{11}\hat{\mathbf{b}}_1 + R_{12}\hat{\mathbf{b}}_2 + R_{13}\hat{\mathbf{b}}_3 && (3.25)\\
\hat{\mathbf{i}}_2 &= R_{21}\hat{\mathbf{b}}_1 + R_{22}\hat{\mathbf{b}}_2 + R_{23}\hat{\mathbf{b}}_3 && (3.26)\\
\hat{\mathbf{i}}_3 &= R_{31}\hat{\mathbf{b}}_1 + R_{32}\hat{\mathbf{b}}_2 + R_{33}\hat{\mathbf{b}}_3 && (3.27)
\end{aligned}$$

Comparing these expressions with Eqs. (3.14–3.16), it is evident that $R_{11} = \hat{\mathbf{i}}_1 \cdot \hat{\mathbf{b}}_1$, $R_{12} = \hat{\mathbf{i}}_1 \cdot \hat{\mathbf{b}}_2$, and in general, $R_{ij} = \hat{\mathbf{i}}_i \cdot \hat{\mathbf{b}}_j$. Using direction cosines, we can write $R_{11} = \cos\alpha_{11}$, $R_{12} = \cos\alpha_{12}$, and in general, $R_{ij} = \cos\alpha_{ij}$, where α_{ij} is the angle between $\hat{\mathbf{i}}_i$ and $\hat{\mathbf{b}}_j$. Thus $\mathbf{R}$ is a matrix of direction cosines, and is frequently referred to

*Recall that to transpose a product of matrices, you reverse the order and transpose each matrix. Thus, $(\mathbf{A}\mathbf{B}^T\mathbf{C})^T = \mathbf{C}^T\mathbf{B}\mathbf{A}^T$.

as the DCM (direction cosine matrix). As with Eq. (3.8), where we have $\left\{\hat{\mathbf{i}}\right\}\cdot\left\{\hat{\mathbf{i}}\right\}^T = \mathbf{1}$, we can also write $\mathbf{R}$ as the dot product of $\left\{\hat{\mathbf{i}}\right\}$ with $\left\{\hat{\mathbf{b}}\right\}^T$, *i.e.*,

$$\mathbf{R} = \left\{\hat{\mathbf{i}}\right\} \cdot \left\{\hat{\mathbf{b}}\right\}^T \tag{3.28}$$

If we know the relative orientation of the two frames, then we can compute the matrix $\mathbf{R}$, and solve Eq. (3.24) to get $\mathbf{v}_i$. As it turns out, it is quite simple to solve this linear system, because the inverse of a direction cosine matrix is simply its transpose. That is,

$$\mathbf{R}^{-1} = \mathbf{R}^T \tag{3.29}$$

To discover this fact, note that it is simple to show that

$$\left\{\hat{\mathbf{b}}\right\} = \mathbf{R}^T \left\{\hat{\mathbf{i}}\right\} \tag{3.30}$$

using the same $\mathbf{R}$ as in Eq. (3.21). Comparing this result with Eq. (3.21), it is clear that Eq. (3.29) is true. A matrix with this property is said to be *orthonormal*, because its rows (and columns) are orthogonal to each other and they all represent unit vectors. This property applied to Eq. (3.24) leads to

$$\mathbf{v}_i = \mathbf{R}\mathbf{v}_b \tag{3.31}$$

Thus $\mathbf{R}$ is the transformation matrix that takes vectors expressed in $\mathcal{F}_b$ and transforms or rotates them into $\mathcal{F}_i$, and $\mathbf{R}^T$ is the transformation that takes vectors expressed in $\mathcal{F}_i$ and transforms them into $\mathcal{F}_b$. We use the notation $\mathbf{R}^{bi}$ to represent the rotation matrix from $\mathcal{F}_i$ to $\mathcal{F}_b$, and $\mathbf{R}^{ib}$ to represent the rotation matrix from $\mathcal{F}_b$ to $\mathcal{F}_i$. Thus

$$\mathbf{v}_b = \mathbf{R}^{bi}\mathbf{v}_i \quad \text{and} \quad \mathbf{v}_i = \mathbf{R}^{ib}\mathbf{v}_b \tag{3.32}$$

The intent of the ordering of b and i in the superscripts is to place the appropriate letter closest to the components of the vector in that frame. The ordering of the superscripts is also related to the rows and columns of $\mathbf{R}$. The first superscript corresponds to the reference frame whose base vector components are in the rows of $\mathbf{R}$, and the second superscript corresponds to the frame whose base vector components are in the columns of $\mathbf{R}$. Similarly the superscripts correspond directly to the dot product notation of Eq. (3.28); *i.e.*, $\mathbf{R}^{ib} = \left\{\hat{\mathbf{i}}\right\} \cdot \left\{\hat{\mathbf{b}}\right\}^T$, and $\mathbf{R}^{bi} = \left\{\hat{\mathbf{b}}\right\} \cdot \left\{\hat{\mathbf{i}}\right\}^T$.

Looking again at Eqs. (3.25–3.27), it is clear that the rows of $\mathbf{R}$ are the components of the corresponding $\hat{\mathbf{i}}_i$, expressed in $\mathcal{F}_b$, whereas the columns of $\mathbf{R}$ are the components of the corresponding $\hat{\mathbf{b}}_j$, expressed in $\mathcal{F}_i$. To help remember this relationship, we write the rotation matrix $\mathbf{R}^{ib}$ as follows:

$$\mathbf{R}^{ib} = \begin{bmatrix} \mathbf{i}_{1b}^T \\ \mathbf{i}_{2b}^T \\ \mathbf{i}_{3b}^T \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{1i} & \mathbf{b}_{2i} & \mathbf{b}_{3i} \end{bmatrix} \tag{3.33}$$

Although the discussion here has centered on frames $\left\{\hat{\mathbf{b}}\right\}$ and $\left\{\hat{\mathbf{i}}\right\}$, the development is the same for any two reference frames.

3.1.4 Euler Angles

Computing the nine direction cosines of the DCM is one way to construct a rotation matrix, but there are many others. One of the easiest to visualize is the Euler angle approach. Euler† reasoned that any rotation from one frame to another can be visualized as a sequence of three *simple rotations* about base vectors. Let us consider the rotation from $\mathcal{F}_i$ to $\mathcal{F}_b$ through a sequence of three angles θ_1, θ_2, and θ_3.

We begin with a simple rotation about the $\hat{\mathbf{i}}_3$ axis, through the angle θ_1. We denote the resulting reference frame as $\mathcal{F}_{i'}$, or $\{\hat{\mathbf{i}}'\}$. Using the rules developed above for constructing $\mathbf{R}^{i'i}$, it is easy to show that the correct rotation matrix is

$$\mathbf{R}^{i'i} = \mathbf{R}_3(\theta_1) = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.34}$$

so that

$$\mathbf{v}_{i'} = \mathbf{R}_3(\theta_1)\mathbf{v}_i \tag{3.35}$$

The subscript 3 in $\mathbf{R}_3(\theta_1)$ denotes that this rotation matrix is a "3" rotation about the "3" axis. Note that we could have performed the first rotation about $\hat{\mathbf{i}}_1$ (a "1" rotation) or $\hat{\mathbf{i}}_2$ (a "2" rotation). Thus there are three possibilities for the first simple rotation in an Euler angle sequence. For the second simple rotation, we cannot choose $\hat{\mathbf{i}}'_3$, since this choice would amount to simply adding to θ_1. Thus there are only two choices for the second simple rotation.

We choose $\hat{\mathbf{i}}'_2$ as the second rotation axis, rotate through an angle θ_2, and call the resulting frame $\mathcal{F}_{i''}$, or $\{\hat{\mathbf{i}}''\}$. In this case, the rotation matrix is

$$\mathbf{R}^{i''i'} = \mathbf{R}_2(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \tag{3.36}$$

so that

$$\mathbf{v}_{i''} = \mathbf{R}_2(\theta_2)\mathbf{v}_{i'} = \mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_1)\mathbf{v}_i \tag{3.37}$$

Now $\mathbf{R}^{i''i} = \mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_1)$ is the rotation matrix transforming vectors from $\mathcal{F}_i$ to $\mathcal{F}_{i''}$.

For the third, and final, rotation, we can use either a "1" rotation or a "3" rotation. We choose a "1" rotation through an angle θ_3, and denote the resulting reference frame

†Leonhard Euler (1707–1783) was a Swiss mathematician and physicist who was associated with the Berlin Academy during the reign of Frederick the Great and with the St Petersburg Academy during the reign of Catherine II. In addition to his many contributions on the motion of rigid bodies, he was a major contributor in the fields of geometry and calculus. Many of our familiar mathematical notations are due to Euler, including e for the natural logarithm base, $f()$ for functions, i for $\sqrt{-1}$, π for π, and Σ for summations. One of my favorites is the special case of Euler's formula: $e^{i\pi}+1=0$, which relates 5 fundamental numbers from mathematics.

$\mathcal{F}_b$, or $\{\hat{\mathbf{b}}\}$. The rotation matrix is

$$\mathbf{R}^{bi''} = \mathbf{R}_1(\theta_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{bmatrix} \quad (3.38)$$

so that

$$\mathbf{v}_b = \mathbf{R}_1(\theta_3)\mathbf{v}_{i''} = \mathbf{R}_1(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_1)\mathbf{v}_i \quad (3.39)$$

Now the matrix transforming vectors from $\mathcal{F}_i$ to $\mathcal{F}_b$ is $\mathbf{R}^{bi} = \mathbf{R}_1(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_1)$.

For a given rotational motion of a reference frame, if we can keep track of the three Euler angles, then we can track the changing orientation of the frame.

As a final note on Euler angle sequences, recall that there were three axes to choose from for the first rotation, two to choose from for the second rotation, and two to choose from for the third rotation. Thus there are twelve $(3 \times 2 \times 2)$ possible sequences of Euler angles. These are commonly referred to by the axes that are used. For example, the sequence used above is called a "3-2-1" sequence, because we first rotate about the "3" axis, then about the "2" axis, and finally about the "1" axis. It is also possible for the third rotation to be of the same type as the first. Thus we could use a "3-2-3" sequence. This type of sequence (commonly called a symmetric Euler angle set) leads to difficulties when θ_2 is small, and so is not widely used in vehicle dynamics applications.

Example 3.1 *Let us develop the rotation matrix relating the Earth-centered inertial (ECI) frame $\mathcal{F}_i$ and the orbital frame $\mathcal{F}_o$. We consider the case of a circular orbit, with* right ascension of the ascending node *(or RAAN),* Ω, inclination, i, *and* argument of latitude, u. *Recall that argument of latitude is the angle from the ascending node to the position of the satellite, and is especially useful for circular orbits, since* argument of periapsis, ω, *is not defined for circular orbits.*

We denote the ECI frame ($\mathcal{F}_i$) by $\{\hat{\mathbf{i}}\}$, and the orbital frame ($\mathcal{F}_o$) by $\{\hat{\mathbf{o}}\}$. Intermediate frames are designated using primes, as in the Euler angle development above. We use a "3-1-3" sequence as follows: Begin with a "3" rotation about the inertial $\hat{\mathbf{i}}_3$ axis through the RAAN, Ω. This rotation is followed by a "1" rotation about the $\hat{\mathbf{i}}_1'$ axis through the inclination, i. The last rotation is another "3" rotation about the $\hat{\mathbf{i}}_3''$ axis through the argument of latitude, u.

We denote the resulting reference frame by $\{\hat{\mathbf{o}}'\}$, since it is not quite the desired orbital reference frame. Recall that the orbital reference frame for a circular orbit has its three vectors aligned as follows: $\{\hat{\mathbf{o}}_1\}$ is in the direction of the orbital velocity vector (the $\vec{\mathbf{v}}$ direction), $\{\hat{\mathbf{o}}_2\}$ is in the direction opposite to the orbit normal (the $-\vec{\mathbf{h}}$ direction), and $\{\hat{\mathbf{o}}_3\}$ is in the nadir direction (or the $-\vec{\mathbf{r}}$ direction). However, the frame resulting from the "3-1-3" rotation developed above has its unit vectors aligned in the $\vec{\mathbf{r}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{h}}$ directions, respectively.

Now, it is possible to go back and choose angles so that the "3-1-3" rotation gives the desired orbital frame; however, it is instructive to see how to use two more rotations to get from the $\{\hat{\mathbf{o}}'\}$ *frame to the* $\{\hat{\mathbf{o}}\}$ *frame. Specifically, if we perform another "3" rotation about* $\{\hat{\mathbf{o}}'_3\}$ *through* $90°$ *and a "1" rotation about* $\{\hat{\mathbf{o}}''_1\}$ *through* $270°$*, we arrive at the desired orbital reference frame. These final two rotations lead to an interesting rotation matrix:*

$$\mathbf{R}^{oo'} = \mathbf{R}_1(270°)\mathbf{R}_3(90°) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \tag{3.40}$$

Careful study of this rotation matrix reveals that its effect is to move the second row to the first row, negate the third row and move it to the second row, and negate the first row and move it to the third row.

So, the rotation matrix that takes vectors from the inertial frame to the orbital frame is

$$\mathbf{R}^{oi} = \mathbf{R}^{oo'}\mathbf{R}_3(u)\mathbf{R}_1(i)\mathbf{R}_3(\Omega) \tag{3.41}$$

which, when expanded, gives

$$\mathbf{R}^{oi} = \begin{bmatrix} -su\,c\Omega - cu\,ci\,s\Omega & -su\,s\Omega + cu\,ci\,c\Omega & cu\,si \\ -si\,s\Omega & si\,c\Omega & -ci \\ -cu\,c\Omega + su\,ci\,s\Omega & -cu\,s\Omega - su\,ci\,c\Omega & -su\,si \end{bmatrix} \tag{3.42}$$

where we have used the letters c *and* s *as abbreviations for* cos *and* sin*, respectively.*

Now, we also need to be able to extract Euler angles from a given rotation matrix. This exercise requires careful consideration of the elements of the rotation matrix and careful application of various inverse trigonometric functions.

Thus, suppose we are given a specific rotation matrix with nine specific numbers. We can extract the three angles associated with $\mathbf{R}^{oi}$ *as developed above as follows:*

$$i = \cos^{-1}(-R_{23}) \tag{3.43}$$

$$u = \tan^{-1}(-R_{33}/R_{13}) \tag{3.44}$$

$$\Omega = \tan^{-1}(-R_{21}/R_{22}) \tag{3.45}$$

3.1.5 Euler's Theorem, Euler Parameters, and Quaternions

The Euler angle sequence approach to describing the relative orientation of two frames is reasonably easy to develop and to visualize, but it is not the most useful approach for spacecraft dynamics. Another of Euler's contributions is the theorem that tells us that only one rotation is necessary to reorient one frame to another. This theorem is known as Euler's Theorem and is formally stated as

> **Euler's Theorem.** The most general motion of a rigid body with a fixed point is a rotation about a fixed axis.

Thus, instead of using three simple rotations (and three angles) to keep track of rotational motion, we only need to use a single rotation (and a single angle) about the "fixed axis" mentioned in the theorem. At first glance, it might appear that we are getting something for nothing, since we are going from three angles to one; however, we also have to know the axis of rotation. This axis, denoted $\hat{\mathbf{a}}$, is called the *Euler axis*, or the *eigenaxis*, and the angle, denoted Φ, is called the *Euler angle*, or the *Euler principal angle*.

For a rotation from $\mathcal{F}_i$ to $\mathcal{F}_b$, about axis $\hat{\mathbf{a}}$ through angle Φ, it is possible to express the rotation matrix $\mathbf{R}^{bi}$, in terms of $\hat{\mathbf{a}}$ and Φ, just as we expressed $\mathbf{R}^{bi}$ in terms of the Euler angles in the previous section. Note that since the rotation is about $\hat{\mathbf{a}}$, the Euler axis vector has the same components in $\mathcal{F}_i$ and $\mathcal{F}_b$; that is,

$$\mathbf{R}^{bi}\mathbf{a} = \mathbf{a} \tag{3.46}$$

and the subscript notation ($\mathbf{a}_i$ or $\mathbf{a}_b$) is not needed. We leave it as an exercise to show that

$$\mathbf{R}^{bi} = \cos\Phi\mathbf{1} + (1-\cos\Phi)\mathbf{a}\mathbf{a}^T - \sin\Phi\mathbf{a}^\times \tag{3.47}$$

where $\mathbf{a}$ is the column matrix of the components of $\hat{\mathbf{a}}$ in either $\mathcal{F}_i$ or $\mathcal{F}_b$. Equation (3.46) provides the justification for the term *eigenaxis* for the Euler axis, since this equation defines $\mathbf{a}$ as the eigenvector of $\mathbf{R}^{bi}$ associated with the eigenvalue 1. A corollary to Euler's Theorem is that every rotation matrix has one eigenvalue that is unity.

Given an Euler axis, $\hat{\mathbf{a}}$, and Euler angle, Φ, we can easily compute the rotation matrix, $\mathbf{R}^{bi}$. We also need to be able to compute the component matrix, $\mathbf{a}$, and the angle Φ, for a given rotation matrix, $\mathbf{R}$. One can show that

$$\Phi = \cos^{-1}\left[\frac{1}{2}\left(\text{trace}\,\mathbf{R} - 1\right)\right] \tag{3.48}$$

$$\mathbf{a}^\times = \frac{1}{2\sin\Phi}\left(\mathbf{R}^T - \mathbf{R}\right) \tag{3.49}$$

So, we can write a rotation matrix in terms of Euler angles, or in terms of the Euler axis/angle set. There are several other approaches, or *parameterizations of the attitude*, and we introduce one of the most important of these: *Euler parameters*, or *quaternions*.

We define four new variables in terms of $\mathbf{a}$ and Φ.

$$\mathbf{q} = \mathbf{a}\sin\frac{\Phi}{2} \tag{3.50}$$

$$q_4 = \cos\frac{\Phi}{2} \tag{3.51}$$

The 3×1 matrix $\mathbf{q}$ forms the *Euler axis component* of the quaternion, also called the vector component. The scalar q_4 is called the *scalar component*. Collectively, these four variables are known as a *quaternion*, or as the *Euler parameters*. We use the notation $\bar{\mathbf{q}}$ to denote the 4×1 matrix containing all four variables; that is,

$$\bar{\mathbf{q}} = \left[\mathbf{q}^T \;\; q_4\right]^T \tag{3.52}$$

A given $\mathbf{a}$ and Φ correspond to a particular relative orientation of two reference frames. Thus a given $\bar{\mathbf{q}}$ also corresponds to a particular orientation. It is relatively easy to show that the rotation matrix can be written as

$$\mathbf{R} = \left(q_4^2 - \mathbf{q}^T\mathbf{q}\right)\mathbf{1} + 2\mathbf{q}\mathbf{q}^T - 2q_4\mathbf{q}^\times \tag{3.53}$$

We also need to express $\bar{\mathbf{q}}$ in terms of the elements of $\mathbf{R}$:

$$q_4 = \pm\frac{1}{2}\sqrt{1 + \text{trace}\,\mathbf{R}} \tag{3.54}$$

$$\mathbf{q} = \frac{1}{4q_4}\begin{bmatrix} R_{23} - R_{32} \\ R_{31} - R_{13} \\ R_{12} - R_{21} \end{bmatrix} \tag{3.55}$$

We now have three basic ways to parameterize a rotation matrix: Euler angles, Euler axis/angle, and Euler parameters. Surprisingly there are many other parameterizations, some of which are not named after Euler. However, these three suffice for the topics in this course. To summarize, a rotation matrix can be written as

$$\mathbf{R} = \mathbf{R}_i(\theta_3)\mathbf{R}_j(\theta_2)\mathbf{R}_k(\theta_1) \tag{3.56}$$

$$\mathbf{R} = \cos\Phi\mathbf{1} + (1 - \cos\Phi)\mathbf{a}\mathbf{a}^T - \sin\Phi\mathbf{a}^\times \tag{3.57}$$

$$\mathbf{R} = \left(q_4^2 - \mathbf{q}^T\mathbf{q}\right)\mathbf{1} + 2\mathbf{q}\mathbf{q}^T - 2q_4\mathbf{q}^\times \tag{3.58}$$

The subscripts i, j, k in the Euler angle formulation indicate that any of the twelve Euler angle sequences may be used. That is, using set notation, $k \in \{1,2,3\}$, $j \in \{1,2,3\}\backslash k$, and $i \in \{1,2,3\}\backslash j$.

Before leaving this topic, we need to establish the following rule:

Rotations do not add like vectors.

The Euler axis/angle description of attitude suggests the possibility of representing a rotation by the vector quantity $\Phi\hat{\mathbf{a}}$. Then, if we had two sequential rotations, say $\Phi_1\hat{\mathbf{a}}_1$ and $\Phi_2\hat{\mathbf{a}}_2$, then we might represent the net rotation by the vector sum of these two: $\Phi_1\hat{\mathbf{a}}_1 + \Phi_2\hat{\mathbf{a}}_2$. This operation is not valid, as the following example illustrates. Suppose that Φ_1 and Φ_2 are both 90°, then the vector sum of the two supposed "rotation vectors" would be $\pi/2(\hat{\mathbf{a}}_1 + \hat{\mathbf{a}}_2)$. Since vector addition is commutative, the resulting "rotation vector" does not depend on the order of performing the two rotations. However, it is easy to see that the actual rotation resulting from the two individual rotations does depend on the order of the rotations. Thus the "rotation vector" description of attitude motion is not valid.

3.2 Attitude Kinematics

In the previous sections, we developed several different ways to describe the attitude, or orientation, of one reference frame with respect to another, in terms of attitude variables. The comparison and contrast of rotational and translational motion is summarized in Table 3.1. The purpose of this section is to develop the kinematics

Table 3.1: Comparison of Rotational and Translational Motion

	Variables	Kinematics D.E.s
Translational Motion	(x, y, z)	$\dot{\vec{\mathbf{r}}} = \vec{\mathbf{p}}/m$
Rotational Motion	$(\theta_1, \theta_2, \theta_3)$	?
	$(\mathbf{a}, \Phi)$	?
	$(\mathbf{q}, q_4)$	?

differential equations (D.E.s) to fill in the "?" in Table 3.1. To complete the table, we first need to develop the concept of angular velocity.

3.2.1 Angular Velocity

The easiest way to think about angular velocity is to first consider the simple rotations developed in Section 3.1.4. The first example developed in that section was for a "3-2-1" Euler angle sequence. Thus we are interested in the rotation of one frame, $\mathcal{F}_{i'}$, with respect to another frame, $\mathcal{F}_i$, where the rotation is about the "3" axis (either $\hat{\mathbf{i}}'_3$ or $\hat{\mathbf{i}}_3$). Then, the *angular velocity* of $\mathcal{F}_{i'}$ with respect to $\mathcal{F}_i$ is

$$\vec{\omega}^{i'i} = \dot{\theta}_1 \hat{\mathbf{i}}'_3 = \dot{\theta}_1 \hat{\mathbf{i}}_3 \tag{3.59}$$

Note the ordering of the superscripts in this expression. Also, note that this vector quantity has the same components in either frame; that is,

$$\omega_{i'}^{i'i} = \omega_{i}^{i'i} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \tag{3.60}$$

This simple expression results because it is a simple rotation. For the "2" rotation from $\mathcal{F}_{i'}$ to $\mathcal{F}_{i''}$, the angular velocity vector is

$$\vec{\omega}^{i''i'} = \dot{\theta}_2 \hat{\mathbf{i}}''_2 = \dot{\theta}_2 \hat{\mathbf{i}}'_2 \tag{3.61}$$

which has components

$$\omega_{i''}^{i''i'} = \omega_{i'}^{i''i'} = \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix} \tag{3.62}$$

Finally, for the "1" rotation from $\mathcal{F}_{i''}$ to $\mathcal{F}_b$, the angular velocity vector is

$$\vec{\omega}^{bi''} = \dot{\theta}_3 \hat{\mathbf{b}}_1 = \dot{\theta}_3 \hat{\mathbf{i}}_1'' \tag{3.63}$$

with components

$$\omega_b^{bi''} = \omega_{i''}^{bi''} = \begin{bmatrix} \dot{\theta}_3 \\ 0 \\ 0 \end{bmatrix} \tag{3.64}$$

Thus, the angular velocities for simple rotations are also simple angular velocities.

Now, angular velocity vectors add in the following way: the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$ is equal to the sum of the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_{i''}$, the angular velocity of $\mathcal{F}_{i''}$ with respect to $\mathcal{F}_{i'}$, and the angular velocity of $\mathcal{F}_{i'}$ with respect to $\mathcal{F}_i$. Mathematically,

$$\vec{\omega}^{bi} = \vec{\omega}^{bi''} + \vec{\omega}^{i''i'} + \vec{\omega}^{i'i} \tag{3.65}$$

However, this expression involves vectors, which are mathematically abstract objects. In order to do computations involving angular velocities, we must choose a reference frame, and express all these vectors in that reference frame and add them together. Notice that in Eqs. (3.60,3.62, and 3.64), the components of these vectors are given in different reference frames. To add them, we must transform them all to the same frame. In most attitude dynamics applications, we use the body frame, so for this example, we develop the expression for $\vec{\omega}^{bi}$ in $\mathcal{F}_b$, denoting it ω_b^{bi}.

The first vector on the right hand side of Eq. (3.65) is already expressed in $\mathcal{F}_b$ in Eq. (3.64), so no further transformation is required. The second vector in Eq. (3.65) is given in $\mathcal{F}_{i''}$ and $\mathcal{F}_{i'}$ in Eq. (3.62). Thus, in order to transform $\omega_{i''}^{i''i'}$ (or $\omega_{i'}^{i''i'}$) into $\mathcal{F}_b$, we need to premultiply the column matrix by either $\mathbf{R}^{bi''}$ or $\mathbf{R}^{bi'}$. Both matrices give the exact same result, which is again due to the fact that we are working with simple rotations. Since $\mathbf{R}^{bi''}$ is simpler [$\mathbf{R}^{bi''} = \mathbf{R}_1(\theta_3)$, whereas $\mathbf{R}^{bi'} = \mathbf{R}_1(\theta_3)\mathbf{R}_2(\theta_2)$], we use $\omega_b^{i''i'} = \mathbf{R}^{bi''}\omega_{i''}^{i''i'}$, or

$$\omega_b^{i''i'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\theta_3 \; \dot{\theta}_2 \\ -\sin\theta_3 \; \dot{\theta}_2 \end{bmatrix} \tag{3.66}$$

Similarly, $\omega_{i'}^{i'i}$ must be premultiplied by $\mathbf{R}^{bi'} = \mathbf{R}_1(\theta_3)\mathbf{R}_2(\theta_2)$ hence

$$\omega_b^{i'i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} -\sin\theta_2 \; \dot{\theta}_1 \\ \cos\theta_2 \sin\theta_3 \; \dot{\theta}_1 \\ \cos\theta_2 \cos\theta_3 \; \dot{\theta}_1 \end{bmatrix} \tag{3.67}$$

Now we have all the angular velocity vectors of Eq. (3.65) expressed in $\mathcal{F}_b$ and can add them together:

$$\begin{aligned}
\omega_b^{bi} &= \omega_b^{bi''} + \omega_b^{i''i'} + \omega_b^{i'i} && (3.68)\\
&= \begin{bmatrix} \dot\theta_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos\theta_3\ \dot\theta_2 \\ -\sin\theta_3\ \dot\theta_2 \end{bmatrix} + \begin{bmatrix} -\sin\theta_2\ \dot\theta_1 \\ \cos\theta_2\ \sin\theta_3\ \dot\theta_1 \\ \cos\theta_2\ \cos\theta_3\ \dot\theta_1 \end{bmatrix} && (3.69)\\
&= \begin{bmatrix} \dot\theta_3 - \sin\theta_2\ \dot\theta_1 \\ \cos\theta_3\ \dot\theta_2 + \cos\theta_2\ \sin\theta_3\ \dot\theta_1 \\ -\sin\theta_3\ \dot\theta_2 + \cos\theta_2\ \cos\theta_3\ \dot\theta_1 \end{bmatrix} && (3.70)\\
&= \begin{bmatrix} -\sin\theta_2 & 0 & 1 \\ \cos\theta_2\ \sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_2\ \cos\theta_3 & -\sin\theta_3 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix} && (3.71)
\end{aligned}$$

The last version of this equation is customarily abbreviated as

$$\omega_b^{bi} = \mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{3.72}$$

where $\boldsymbol{\theta} = [\theta_1\ \theta_2\ \theta_3]^T$, and $\mathbf{S}(\boldsymbol{\theta})$ obviously depends on which Euler angle sequence is used. For a given Euler angle sequence, it is relatively straightforward to develop the appropriate $\mathbf{S}(\boldsymbol{\theta})$. Often it is clear what angular velocity vector and reference frame we are working with, and we drop the sub and superscripts on ω.

Thus if the $\dot\theta$'s are known, then they can be integrated to determine the θ's, and then the components of $\vec{\omega}$ can be determined. This integration is entirely analogous to knowing $\dot x$, $\dot y$, and $\dot z$, and integrating these to determine the position x, y, and z. In this translational case however, one usually knows $\dot x$, *etc.*, from determining the *velocity*; *i.e.*, $\mathbf{v} = [\dot x\ \dot y\ \dot z]^T$. The velocity is determined from the kinetics equations of motion as in Eq. (3.3). Similarly, for rotational motion, the kinetics equations of motion are used to determine the angular velocity, which is in turn used to determine the $\dot\theta$'s, not *vice versa*. In the next section, we develop relationships between the time derivatives of the attitude variables and the angular velocity.

3.2.2 Kinematics Equations

We begin by solving Eq. (3.72) for $\dot{\boldsymbol{\theta}}$, which requires inversion of $\mathbf{S}(\boldsymbol{\theta})$. It is straightforward to obtain:

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} 0 & \sin\theta_3/\cos\theta_2 & \cos\theta_3/\cos\theta_2 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 1 & \sin\theta_3\sin\theta_2/\cos\theta_2 & \cos\theta_3\sin\theta_2/\cos\theta_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \mathbf{S}^{-1}\omega \tag{3.73}$$

Thus, if we know the ω's as functions of time, and have initial conditions for the three Euler angles, then we can integrate these three differential equations to obtain the

θ's as functions of time. Careful examination of $\mathbf{S}^{-1}$ shows that some of the elements of this matrix become large when θ_2 approaches $\pi/2$, and indeed become infinite when $\theta_2 = \pi/2$. This problem is usually called a *kinematic singularity*, and is one of the difficulties associated with using Euler angles as attitude variables. Even though the angular velocity may be small, the Euler angle rates can become quite large. For a different Euler angle sequence, the kinematic singularity occurs at a different point. For example, with symmetric Euler angle sequences, the kinematic singularity always occurs when the middle angle (θ_2) is 0 or π. Because the singularity occurs at a different point for different sequences, one way to deal with the singularity is to switch Euler angle sequences whenever a singularity is approached. Hughes[1] provides a table of $\mathbf{S}^{-1}$ for all 12 Euler angle sequences. Another difficulty is that it is computationally expensive to compute the sines and cosines necessary to integrate Eq. (3.73).

As we indicated in Section 3.1.5, other attitude variables may be used to represent the orientation of two reference frames, with the Euler axis/angle set, and quaternions, being the most common. We now provide the differential equations relating $(\mathbf{a}, \Phi)$ and $\bar{\mathbf{q}}$ to ω. For the Euler axis/angle set of attitude variables, the differential equations are

$$\dot{\Phi} = \mathbf{a}^T\omega \tag{3.74}$$

$$\dot{\mathbf{a}} = \frac{1}{2}\left[\mathbf{a}^\times - \cot\frac{\Phi}{2}\mathbf{a}^\times\mathbf{a}^\times\right]\omega \tag{3.75}$$

The kinematic singularity in these equations is evidently at $\Phi = 0$ or 2π, both of which correspond to $\mathbf{R} = \mathbf{1}$ which means the two reference frames are identical. Thus it is reasonably straightforward to deal with this singularity. There is, however, the one trig function that must be computed as Φ varies.

For Euler parameters or quaternions, the kinematic equations of motion are

$$\dot{\bar{\mathbf{q}}} = \frac{1}{2}\begin{bmatrix} \mathbf{q}^\times + q_4\mathbf{1} \\ -\mathbf{q}^T \end{bmatrix}\omega = \mathbf{Q}(\bar{\mathbf{q}})\omega \tag{3.76}$$

There are no kinematic singularities associated with $\dot{\bar{\mathbf{q}}}$, and there are no trig functions to evaluate. For these reasons, the quaternion, $\bar{\mathbf{q}}$, is the attitude variable of choice for most satellite attitude dynamics applications.

3.3 Summary

This chapter provides the basic background for describing reference frames, their orientations with respect to each other, and the transformation of vectors from one frame to another.

3.4 Summary of Notation

There are several subscripts and superscripts used in this and preceding chapters. This table summarizes the meanings of these symbols.

Symbol	Meaning
$\vec{\mathbf{v}}$	vector, an abstract mathematical object with direction and length
$\left\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\right\}$	the three unit base vectors of a reference frame
$\mathcal{F}_i$	the reference frame with base vectors $\left\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\right\}$ typically $\mathcal{F}_i$ denotes an inertial reference frame whereas $\mathcal{F}_b$ denotes a body-fixed frame, and $\mathcal{F}_o$ denotes an orbital reference frame
$\left\{\hat{\mathbf{i}}\right\}$	a column matrix whose 3 elements are the unit vectors of $\mathcal{F}_i$
$\mathbf{v}_i$	a column matrix whose 3 elements are the components of the vector $\vec{\mathbf{v}}$ expressed in $\mathcal{F}_i$
$\mathbf{v}_b$	a column matrix whose 3 elements are the components of the vector $\vec{\mathbf{v}}$ expressed in $\mathcal{F}_b$
$\mathbf{R}^{bi}$	rotation matrix that transforms vectors from $\mathcal{F}_i$ to $\mathcal{F}_b$
$\boldsymbol{\theta}$	a column matrix whose 3 elements are the Euler angles θ_1, θ_2, θ_3
$\vec{\omega}$	an angular velocity vector
$\vec{\omega}^{bi}$	the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$
ω_b^{bi}	the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$ expressed in $\mathcal{F}_b$ typically used in Euler's equations
ω_i^{bi}	the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$ expressed in $\mathcal{F}_i$ not commonly used
ω_a^{bi}	the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$ expressed in $\mathcal{F}_a$ there are applications for this form

3.5 References and further reading

Most satellite attitude dynamics and control textbooks cover kinematics only as a part of the dynamics presentation. Pisacane and Moore[2] is a notable exception, providing a detailed treatment of kinematics before covering dynamics and control. Shuster[3] provided an excellent survey of the many attitude representation approaches, including many interesting historical comments. Wertz's handbook[4] also covers some of this material. Kuipers[5] provides extensive details about quaternions and their applications, but does not include much on spacecraft attitude applications.

Bibliography

[1] Peter C. Hughes. *Spacecraft Attitude Dynamics*. John Wiley & Sons, New York, 1986.

[2] Vincent L. Pisacane and Robert C. Moore, editors. *Fundamentals of Space Systems*. Oxford University Press, Oxford, 1994.

[3] Malcolm D. Shuster. A survey of attitude representations. *Journal of the Astronautical Sciences*, 41(4):439–517, 1993.

[4] J. R. Wertz, editor. *Spacecraft Attitude Determination and Control*. D. Reidel, Dordrecht, Holland, 1978.

[5] Jack B. Kuipers. *Quaternions and Rotation Sequences*. Princeton University Press, Princeton, 1999.

3.6 Exercises

1. Verify the validity of Eq. (3.8) by direct calculation.

2. Develop the rotation matrix for a "3-1-2" rotation from $\mathcal{F}_a$ to $\mathcal{F}_b$. Your result should be a single matrix in terms of θ_1, θ_2, and θ_3 [similar to Eq. (3.42)].

3. Using the relationship between the elements of the quaternion and the Euler angle and axis, verify that Eq. (3.53) and Eq. (3.47) are equivalent.

4. Select two unit vectors and convince yourself that the statement "Rotations do not add like vectors" described on p. 3-12 is true.

5. Develop $\mathbf{S}(\boldsymbol{\theta})$ for a "2-3-1" rotation from $\mathcal{F}_i$ to $\mathcal{F}_b$ so that $\omega_b^{bi} = \mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$. Where is $\mathbf{S}(\boldsymbol{\theta})$ singular?

6. A satellite at altitude 700 km is pointing at a target that is 7° away from nadir (*i.e.*, $\eta = 7°$). What is the range D to the target, and what is the spacecraft elevation angle ε.

7. A satellite in a circular orbit with radius 7000 km is intended to target any point within its instantaneous access area (IAA) for which the elevation angle ε is greater than 5°. What must the range of operation be for the attitude control system?

8. The Tropical Rainfall Measuring Mission is in a circular orbit with altitude 350 km, and inclination $i = 35°$. If it must be able to target any point that is within its IAA and within the tropics, then what must the ACS range of

operations be, and at what minimum elevation angle must the sensor be able to operate?

9. A satellite is in an elliptical orbit with $a = 7000$, $e = 0.1$. Make plots of the IAA as a function of time for one period, using ε_{min} of 0°, 5°, and 10°.

3.7 Problems

Consider the following rotation matrix $\mathbf{R}^{ab}$ that transforms vectors from $\mathcal{F}_b$ to $\mathcal{F}_a$:

$$\mathbf{R}^{ab} = \begin{bmatrix} 0.45457972 & 0.43387382 & -0.77788868 \\ -0.34766601 & 0.89049359 & 0.29351236 \\ 0.82005221 & 0.13702069 & 0.55564350 \end{bmatrix}$$

1. Use at least two different properties of rotation matrices to convince yourself that $\mathbf{R}^{ab}$ is indeed a rotation matrix. Remark on any discrepancies you notice.

2. Determine the Euler axis $\mathbf{a}$ and Euler principal angle Φ, directly from $\mathbf{R}^{ab}$. Verify your results using the formula for $\mathbf{R}(\mathbf{a}, \Phi)$. Verify that $\mathbf{Ra} = \mathbf{a}$.

3. Determine the components of the quaternion $\bar{\mathbf{q}}$, directly from $\mathbf{R}^{ab}$. Verify your results using the formula for $\mathbf{R}(\bar{\mathbf{q}})$. Verify your results using the relationship between $\bar{\mathbf{q}}$ and $(\mathbf{a}, \Phi)$.

4. Derive the formula for a 2-3-1 rotation. Determine the Euler angles for a 2-3-1 rotation, directly from $\mathbf{R}^{ab}$. Verify your results using the formula you derived.

5. Write a short paragraph discussing the relative merits of the three different representations of $\mathbf{R}$.

6. This problem requires numerical integration of the kinematics equations of motion. You should have a look at the MatLab Appendix.

   Suppose that $\mathcal{F}_b$ and $\mathcal{F}_a$ are initially aligned, so that $\mathbf{R}^{ba}(0) = \mathbf{1}$. At $t = 0$, $\mathcal{F}_b$ begins to rotate with angular velocity $\omega_b = e^{-4t}[\sin t \;\; \sin 2t \;\; \sin 3t]^T$ with respect to $\mathcal{F}_a$.

   Use a 1-2-3 Euler angle sequence and make a plot of the three Euler angles *vs.* time for $t = 0$ to 10 s. Use a sufficiently small step size so that the resulting plot is "smooth."

   Use the quaternion representation and make a plot of the four components of the quaternion *vs.* time time for $t = 0$ to 10 s. Use a sufficiently small step size so that the resulting plot is "smooth."

   Use the results of the two integrations to determine the rotation matrix at $t = 10$ s. Do this using the expression for $\mathbf{R}(\boldsymbol{\theta})$ and for $\mathbf{R}(\bar{\mathbf{q}})$. Compare the results.