Chapter 6

Satellite Attitude Dynamics

Chapter 5 develops the fundamental topics required to study the rotational motion of rigid bodies. These fundamentals are relevant to a wide variety of engineering problems, and especially to the study of the attitude motion of spacecraft. An important step in specializing the rigid body equations of motion for a particular application is to develop a thorough understanding of the torques $\mathbf{g}$ that appear in the equations. In this chapter, we develop the environmental torques that affect spacecraft motion, and investigate some of the standard problems of interest to spacecraft engineers.

6.1 Environmental Torques

Important environmental torques affecting satellite attitude dynamics include gravity gradient, magnetic, aerodynamic, and solar radiation pressure torques. We develop expressions for these torques in this section, and in subsequent sections we study the effects of the torques on attitude motion.

Currently this section only includes the development of the gravity gradient torque.

6.1.1 Gravity Gradient Torque

We assume a rigid spacecraft in orbit about a spherical primary, and every differential mass element of the body is subject to Newton's Universal Gravitational Law:

$$d\vec{\mathbf{f}}_g = -\frac{GM\,dm}{r^2}\hat{\mathbf{e}}_r \tag{6.1}$$

where G is the universal gravitational constant, M is the mass of the spherical primary, dm is the mass of a mass element of the body in orbit, r is the radial distance from the mass center of the primary to the mass element, and $\hat{\mathbf{e}}_r$ is a unit vector from the mass center of the primary to the mass element. Note that $\hat{\mathbf{e}}_r$ for the mass center may be expressed as one of the unit vectors in the orbital frame, $\mathcal{F}_o$, as $\hat{\mathbf{e}}_r = -\hat{\mathbf{o}}_3$.

Furthermore, we find it useful to write the position vector to a differential mass element as the sum of the position vector from the primary to the mass center of the body and the vector from the mass center of the body to the differential mass element:

$$\vec{\mathbf{r}} = {}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}} \tag{6.2}$$

This force acts on every particle in the body, so that we compute the force on the body by integrating this differential force over the body. Thus

$$\vec{\mathbf{f}}_g = -\int_{\mathcal{B}} \frac{GM}{r^2}\hat{\mathbf{e}}_r \, dm \tag{6.3}$$

which can be written as

$$\vec{\mathbf{f}}_g = -\int_{\mathcal{B}} \frac{GM}{r^3}\vec{\mathbf{r}} \, dm \tag{6.4}$$

or, expanding the position vector using Eq. (6.2),

$$\vec{\mathbf{f}}_g = -\int_{\mathcal{B}} \frac{GM}{|{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}|^3}({}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}) \, dm \tag{6.5}$$

In the integrand, the vector ${}_{c}\vec{\mathbf{r}}$ is the only variable that depends on the differential mass element. However, in general, this integral cannot be computed in closed form. The usual approach is to assume that the radius of the orbit is much greater than the size of the body, *i.e.*, $|{}^{c}_{O}\vec{\mathbf{r}}| \gg |{}_{c}\vec{\mathbf{r}}|$, and expanding the integrand as follows:

$$\begin{aligned}
\frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{|{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}|^3} &= \frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{[({}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}})\cdot({}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}})]^{3/2}} && (6.6)\\
&= \frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{[{}^{c}_{O}\vec{\mathbf{r}}\cdot{}^{c}_{O}\vec{\mathbf{r}} + 2\,{}^{c}_{O}\vec{\mathbf{r}}\cdot{}_{c}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}\cdot{}_{c}\vec{\mathbf{r}}]^{3/2}} && (6.7)\\
&= \frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{[{}^{c}_{O}r^2 + 2\,{}^{c}_{O}\vec{\mathbf{r}}\cdot{}_{c}\vec{\mathbf{r}} + {}_{c}r^2]^{3/2}} && (6.8)\\
&= \frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{{}^{c}_{O}r^3\,[1 + 2\,{}^{c}_{O}\vec{\mathbf{r}}\cdot{}_{c}\vec{\mathbf{r}}/{}^{c}_{O}r^2 + {}_{c}r^2/{}^{c}_{O}r^2]^{3/2}} && (6.9)
\end{aligned}$$

Now, since $|{}^{c}_{O}\vec{\mathbf{r}}| \gg |{}_{c}\vec{\mathbf{r}}|$, expansion in a Taylor series leads to

$$\begin{aligned}
\frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{|{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}|^3} &= \frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{{}^{c}_{O}r^3}\left(1 - 3\frac{{}^{c}_{O}\vec{\mathbf{r}}\cdot{}_{c}\vec{\mathbf{r}}}{{}^{c}_{O}r^2} + H.O.T.\right) && (6.10)\\
&\approx \frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{{}^{c}_{O}r^3}\left(1 - 3\frac{{}^{c}_{O}\vec{\mathbf{r}}\cdot{}_{c}\vec{\mathbf{r}}}{{}^{c}_{O}r^2}\right) && (6.11)\\
&\approx \frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{{}^{c}_{O}r^3} && (6.12)
\end{aligned}$$

Substituting the last of these approximations in to the volume integral for $\vec{\mathbf{f}}_g$ leads to

$$\begin{aligned} \vec{\mathbf{f}}_g &= -\int_{\mathcal{B}} \frac{GM\left({}^c_O\vec{\mathbf{r}} + {}_c\vec{\mathbf{r}}\right)}{{}^c_O r^3}\, dm \quad (6.13) \\ &= -\frac{GMm}{{}^c_O r^3}\,{}^c_O\vec{\mathbf{r}} \quad (6.14) \end{aligned}$$

Finally, letting $\vec{\mathbf{r}}$ denote the position vector from the primary to the mass center of the orbiting body, and applying Newton's Second Law, we obtain:

$$\ddot{\vec{\mathbf{r}}} + \frac{GM}{r^3}\vec{\mathbf{r}} = \vec{\mathbf{0}} \quad (6.15)$$

This equation is recognizable as the familiar two-body vector equation of orbital motion, in which the constant GM is usually written as the gravitational parameter μ.

We also need to develop the moment about the mass center due to the gravitational forces. This moment may be expressed as an integral over the body:

$$\vec{\mathbf{g}}_g^c = -\int_{\mathcal{B}} {}_c\vec{\mathbf{r}} \times d\vec{\mathbf{f}}_g \quad (6.16)$$

Applying the same assumptions as those used to compute the approximate force, we obtain an expression for the approximate moment about the mass center:

$$\vec{\mathbf{g}}_g^c = 3\frac{GM}{r^3}\hat{\mathbf{o}}_3 \times \vec{\mathbf{I}}^c \cdot \hat{\mathbf{o}}_3 \quad (6.17)$$

Expressed in a body-fixed reference frame, this gravity-gradient torque is

$$\mathbf{g}_{gb}^c = 3\frac{GM}{r^3}\mathbf{o}_{3b}^{\times}\mathbf{I}_b^c\mathbf{o}_{3b} \quad (6.18)$$

Note that $\mathbf{o}_{3b}$ is the third column of the rotation matrix $\mathbf{R}^{bo}$. Usually the facts that the moment and moment of inertia are about the mass center and that the vectors are expressed in $\mathcal{F}_b$ are understood and we can simplify the notation to

$$\mathbf{g}_g = 3\frac{GM}{r^3}\mathbf{o}_3^{\times}\mathbf{I}\mathbf{o}_3 \quad (6.19)$$

This torque affects the motion of all orbiting bodies, and in the next section we investigate the resulting steady motions and their stability.

6.2 Gravity Gradient Stabilization

For a rigid satellite in a central gravitational field, the equations of motion may be approximated as

$$\begin{aligned} \ddot{\vec{\mathbf{r}}} + \frac{GM}{r^3}\vec{\mathbf{r}} &= \vec{\mathbf{0}} \quad (6.20) \\ \dot{\vec{\mathbf{h}}} &= 3\frac{GM}{r^3}\hat{\mathbf{o}}_3 \times \vec{\mathbf{I}} \cdot \hat{\mathbf{o}}_3 \quad (6.21) \end{aligned}$$

where $\vec{\mathbf{r}}$ is the position vector of the mass center of the body with respect to the center of the gravitational primary, and $\vec{\mathbf{h}}$ is the angular momentum of the body about its mass center. The vector $\hat{\mathbf{o}}_3$ is the nadir vector; *i.e.*, $\hat{\mathbf{o}}_3 = -\vec{\mathbf{r}}/r$.

The first equation is simply the two-body equation of motion for a point mass orbiting a spherical primary, and its solution is well known. The second equation is the vector expression of Euler's equations for the specific case of a gravity gradient torque (the right-hand side). As usual, we want to write this equation in terms of the principal body-frame components of these vectors. Thus, the rotational equations of motion for a rigid body subject only to gravitational forces and moments can be written as

$$\mathbf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + 3\frac{GM}{r^3}\mathbf{o}_3^{\times}\mathbf{I}\mathbf{o}_3 \tag{6.22}$$

Since r appears in this equation, the solution to the orbital (translational) equations of motion must be used in solving the attitude (rotational) equations of motion. Also, suitable kinematics equations must be used with these equations, since the components of $\mathbf{o}_3$ in $\mathcal{F}_b$ are the elements of the third column of $\mathbf{R}^{bo}$, which may be written as $\mathbf{R}^{bi}\mathbf{R}^{io}$.

Now, let us investigate the possibility of equilibrium attitude motion of a satellite in a circular orbit, in which case r is constant, and the term $3GM/r^3$ is $3\omega_c^2$, where ω_c is the orbital angular velocity, or mean motion. Recall that the orbital frame, $\mathcal{F}_o$ is a frame with $\hat{\mathbf{o}}_1$ in the direction of the velocity vector, $\hat{\mathbf{o}}_2$ in the direction of the negative orbit normal, and $\hat{\mathbf{o}}_3$ in the nadir direction (*i.e.*, Earth-pointing). The angular velocity of this frame with respect to the inertial frame may be expressed in the orbital frame as $\boldsymbol{\omega}_o^{oi} = [0, -\omega_c, 0]^T$. The possibility of an Earth-orbiting satellite with the body frame always aligned with the orbital frame has obvious operational advantages, and we investigate this equilibrium in the following section.

If the two frames, $\mathcal{F}_b$ and $\mathcal{F}_o$, are aligned, then $\mathbf{R}^{bo} = \mathbf{1}$, so that $\mathbf{o}_3 = [0, 0, 1]^T$ in $\mathcal{F}_b$. Furthermore, $\boldsymbol{\omega}^{bo} = \mathbf{0}$, so that $\boldsymbol{\omega}^{bi} = \boldsymbol{\omega}^{oi} = [0, -\omega_c, 0]^T$. If we substitute this angular velocity into the right-hand side of Eq. (6.22), we find that $\dot{\boldsymbol{\omega}} = \mathbf{0}$, so this motion is in fact an equilibrium motion. However, just as the rigid pendulum has two equilibrium motions, straight down ($\theta = 0$) and straight up (($\theta = \pi$), with different stability properties, there is a variety of equilibrium motions for the rigid body. Specifically, there are three different principal axes that could be aligned with the $\hat{\mathbf{o}}_1$ axis, and for each of these there are two remaining principal axes that could be aligned with the $\hat{\mathbf{o}}_2$ axis. Thus there are six different types of equilibrium motions possible (actually there are 12 if you count the $-\hat{\mathbf{b}}_1$ axis as being different from the $+\hat{\mathbf{b}}_1$ axis). The way we think about these possibilities is the same for the spin-stability of an asymmetric body: we investigate small motions of $\mathcal{F}_b$ with respect to $\mathcal{F}_o$, and see how the ordering of the principal moments of inertia I_1, I_2, and I_3 affect the stability properties.

To investigate the stability of small motions in this case, we must include the kinematics. We use a 1-2-3 rotation sequence from $\mathcal{F}_o$ to $\mathcal{F}_b$. That is, $\mathbf{R}^{bo} =$

$\mathbf{R}_3(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{R}_1(\theta_1)$. Expanding this expression gives

$$\mathbf{R}^{bo} = \begin{bmatrix} c_2c_3 & s_1s_2c_3 + c_1s_3 & s_1s_3 - c_1s_2c_3 \\ -s_2s_3 & c_1c_3 - s_1s_2s_3 & s_1c_3 + c_1s_2s_3 \\ s_2 & -s_1c_2 & c_1c_2 \end{bmatrix} \tag{6.23}$$

If we make the assumption that all the angles are small, *i.e.*, $\sin\theta_i \approx \theta_i$, $\cos\theta_i \approx 1$, and $\theta_i\theta_j \approx 0$, then this rotation matrix becomes

$$\mathbf{R}^{bo} \approx \mathbf{1} - \boldsymbol{\theta}^\times \tag{6.24}$$

Thus $\mathbf{o}_3 = [-\theta_2, \theta_1, 1]^T$ in the body frame. This approximation may be used immediately to compute the gravity gradient torque (for small angles) as

$$\mathbf{g}_{gg} = 3\omega_c^2\mathbf{o}_3^\times\mathbf{I}\mathbf{o}_3 = 3\omega_c^2 \begin{bmatrix} (I_3 - I_2)\theta_1 \\ (I_3 - I_1)\theta_2 \\ 0 \end{bmatrix} \tag{6.25}$$

Assuming small angles and small angular rates, the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_o$ is simply $\boldsymbol{\omega}^{bo} = \dot{\boldsymbol{\theta}}$. Note that this result requires developing $\boldsymbol{\omega} = \mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ and carrying out the small-angle approximations.

The $\boldsymbol{\omega}$ that appears in the equations of motion (6.22) is the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$, which is

$$\boldsymbol{\omega}^{bi} = \boldsymbol{\omega}^{bo} + \boldsymbol{\omega}^{oi} \tag{6.26}$$

and both vectors must be expressed in $\mathcal{F}_b$. Thus

$$\boldsymbol{\omega}^{bi} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\omega_c \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 - \omega_c\theta_3 \\ \dot{\theta}_2 - \omega_c \\ \dot{\theta}_3 + \omega_c\theta_1 \end{bmatrix} \tag{6.27}$$

In Euler's equations, we also need $\dot{\boldsymbol{\omega}}$, which is easily computed as

$$\dot{\boldsymbol{\omega}} = \begin{bmatrix} \ddot{\theta}_1 - \omega_c\dot{\theta}_3 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 + \omega_c\dot{\theta}_1 \end{bmatrix} \tag{6.28}$$

Now, if we substitute Eqs. (6.25), (6.27), and (6.28) into Eq. (6.22), we obtain the following three coupled linear ordinary differential equations:

$$I_1\ddot{\theta}_1 + (I_2 - I_3 - I_1)\omega_c\dot{\theta}_3 - 4(I_3 - I_2)\omega_c^2\theta_1 = 0 \tag{6.29}$$

$$I_2\ddot{\theta}_2 + 3\omega_c^2(I_1 - I_3)\theta_2 = 0 \tag{6.30}$$

$$I_3\ddot{\theta}_3 + (I_3 + I_1 - I_2)\omega_c\dot{\theta}_1 + (I_2 - I_1)\omega_c^2\theta_3 = 0 \tag{6.31}$$

Note that the pitch equation (θ_2) is decoupled from the roll (θ_1) and yaw (θ_3) equations, and is of the form $\ddot{x} + kx = 0$, so that the pitch motion is stable if $I_1 > I_3$. That is, the Earth-pointing axis, $\hat{\mathbf{b}}_3$, cannot be the major axis. If the other two equations of motion are unstable, then the pitch motion does not necessarily remain small, since the nonlinear terms are no longer negligible.

To investigate the stability of the coupled roll-yaw motion, we write the two equations (6.29,6.31) in the matrix second-order form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{G}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{6.32}$$

where $\mathbf{x} = [\theta_1 \;\; \theta_3]^T$ and the three 2×2 matrices are defined as

$$\mathbf{M} = \begin{bmatrix} I_1 & 0 \\ 0 & I_3 \end{bmatrix}, \tag{6.33}$$

$$\mathbf{G} = (I_1 + I_3 - I_2)\omega_c \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{6.34}$$

$$\mathbf{K} = \omega_c^2 \begin{bmatrix} 4(I_2 - I_3) & 0 \\ 0 & (I_2 - I_1) \end{bmatrix} \tag{6.35}$$

These three matrices are normally called the *mass matrix*, the *gyroscopic damping matrix*, and the *stiffness matrix*, respectively.

Just as we did for the spin stability of a rigid body, we assume an exponential form for the solution to this system of differential equations. Specifically, we seek solutions of the form:

$$\mathbf{x} = e^{\lambda t}\mathbf{c} \tag{6.36}$$

where λ is an eigenvalue, and $\mathbf{c}$ is a 2×1 matrix containing the arbitrary constants that must be determined based on the initial conditions. We need the two derivatives of this form of the solution: $\dot{\mathbf{x}} = \lambda e^{\lambda t}\mathbf{c} = \lambda\mathbf{x}$, and $\ddot{\mathbf{x}} = \lambda^2 e^{\lambda t}\mathbf{c} = \lambda^2\mathbf{x}$. Substituting these into the differential equation and collecting terms, we obtain

$$e^{\lambda t}\left[\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}\right]\mathbf{c}_o = \mathbf{0} \tag{6.37}$$

Since $e^{\lambda t} \neq 0$, we can divide both sides by $e^{\lambda t}$, leading to the matrix equation

$$\left[\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}\right]\mathbf{c}_o = \mathbf{0} \tag{6.38}$$

which is of the form $\mathbf{A}\mathbf{c}_o = \mathbf{0}$. A well-known result of linear algebra states that this equation can only have non-trivial solutions ($\mathbf{c}_o \neq \mathbf{0}$) if the matrix $\mathbf{A}$ is singular. Another well-known result of linear algebra is that a matrix is singular if and only if its determinant is zero. Thus, we can determine the eigenvalues λ by setting the determinant of the matrix equal to zero, and solving for λ:

$$\begin{vmatrix} \lambda^2 I_1 + 4\omega_c^2(I_2 - I_3) & -\lambda\omega_c(I_1 + I_3 - I_2) \\ \lambda\omega_c(I_1 + I_3 - I_2) & \lambda^2 I_3 + \omega_c^2(I_2 - I_1) \end{vmatrix} = 0 \tag{6.39}$$

which expands to give the characteristic polynomial

$$\lambda^4 I_1 I_3 + \lambda^2 \omega_c^2 [I_1(I_2 - I_1) + 4I_3(I_2 - I_3) + (I_1 + I_3 - I_2)^2] + 4\omega_c^4 (I_2 - I_1)(I_2 - I_3) = 0 \quad (6.40)$$

Dividing by $I_1 I_3 \omega_c^4$, and defining the two inertia parameters $k_1 = (I_2 - I_3)/I_1$, $k_3 = (I_2 - I_1)/I_3$, the characteristic equation may be more conveniently written as

$$\left(\frac{\lambda}{\omega_c}\right)^4 + (1 + 3k_1 + k_1 k_3)\left(\frac{\lambda}{\omega_c}\right)^2 + 4k_1 k_3 = 0 \quad (6.41)$$

This equation is a quadratic polynomial in the variable $s = (\lambda/\omega_c)^2$, which can be written as

$$s^2 + b_1 s + b_0 = 0 \quad (6.42)$$

In order for the motion to be stable (oscillatory), the eigenvalues (λ's) must be pure imaginary, so s must be negative. The necessary conditions for s to be real and negative are

$$b_0 > 0 \qquad b_1 > 0 \qquad b_1^2 - 4b_2 > 0 \quad (6.43)$$

Recall that for pitch stability, we need $I_1 > I_3$, which is equivalent to $k_1 > k_3$. The four stability conditions may then be expressed as

$$k_1 > k_3 \quad (6.44)$$
$$k_1 k_3 > 0 \quad (6.45)$$
$$1 + 3k_1 + k_1 k_3 > 0 \quad (6.46)$$
$$(1 + 3k_1 + k_1 k_3)^2 - 16k_1 k_3 > 0 \quad (6.47)$$

We can show that the parameters k_1 and k_2 (called Smelt parameters[1]) are bounded between ± 1. Thus we can construct a stability diagram in the $k_1 k_3$ plane, as shown in Figure 6.1.

In the stability diagram shown in Fig. 6.1, the curves indicate the stability boundaries. The curves labeled I, III, and IV, correspond to the first, third, and fourth of the conditions listed in Eqs. (6.44–6.47). The second condition rules out the second and fourth quadrants of the plane. Thus, the only regions corresponding to stable configurations are the two labeled "Lagrange" and "D^2." The symbol D^2 refers to the "DeBra-Delp" region first reported in Ref. 1. The Lagrange region was established by Lagrange* in his studies of the equilibrium motions of the moon.

*Joseph Louis Lagrange (1736 - 1813) was a major contributor to the field of mechanics, developing the calculus of variations and a new way of writing the equations of motion, now known as Lagrange's equations.

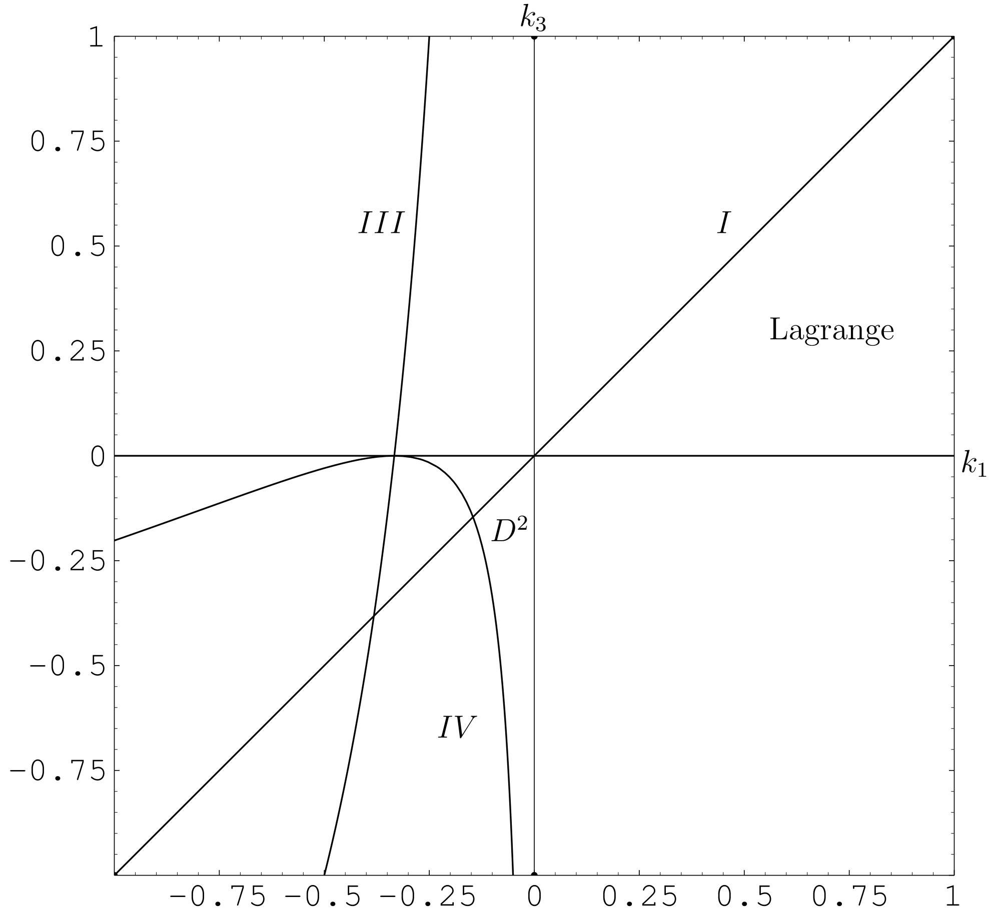

Figure 6.1: The Smelt Parameter Plane

6.3 Spin Stabilization

6.4 Dual-Spin Stabilization

In Chapter 5, we develop the well-known result that a rigid body is stable in spin about either the major or minor axis, but is unstable in spin about the intermediate axis. We also found that in the presence of energy dissipation the major axis spin becomes asymptotically stable whereas the minor axis spin becomes unstable. This latter result is the reason for spin-stabilized spacecraft being shaped more like tuna cans than asparagus cans. In this section, we develop two important results regarding dual-spin stabilization.

Dual-spin stabilization applies to spacecraft with two components that are spinning relative to each other. Typically one body is spinning relatively fast and the other is spinning relatively slow. The first result involves only rigid bodies, and concludes that a spinning wheel can be used to stabilize spin about *any* axis. The second result involves an energy sink analysis and concludes that a minor axis spin is stable if the energy dissipation on the rapidly spinning component is “smaller” than the energy dissipation on the slowly spinning component.

We begin with the equations of motion for a rigid body with an embedded axisymmetric flywheel. Note that the flywheel’s motion does not affect the moment of inertia matrix of the system. That is, if we lock the flywheel relative to the body and compute the moment of inertia matrix and principal axes of the system, these moments of inertia are unchanged by allowing the flywheel to rotate about its symmetry axis. We use a body-fixed, principal reference frame, $\mathcal{F}_b$, which has angular velocity relative to inertial space denoted $\vec{\boldsymbol{\omega}}$, with body frame components $\boldsymbol{\omega}$. The flywheel’s spin axis is assumed (without loss of generality) to be aligned parallel to the spacecraft’s $\hat{\mathbf{b}}_3$ axis. The flywheel has spin axis moment of inertia denoted by I_w. The flywheel’s angular velocity relative to the body frame may be written as $\vec{\boldsymbol{\omega}}^{wb} = \Omega_w \hat{\mathbf{b}}_3$. Expressed in the body frame, this vector has components $\boldsymbol{\omega}^{wb} = \Omega_w[0, 0, 1]^T$. Note that this angular velocity is the relative velocity as would be measured by a tachometer. The inertial angular velocity of the flywheel is $\vec{\boldsymbol{\omega}}^{wi} = \vec{\boldsymbol{\omega}} + \vec{\boldsymbol{\omega}}^{wb}$. In the following development we assume that Ω_w is constant.

The total angular momentum of the system is simply the angular momentum due to the rigid body motion, plus the axial angular momentum contributed by the flywheel. In the body frame, the angular momentum has components:

$$\mathbf{h} = \mathbf{I}\boldsymbol{\omega} + I_w \boldsymbol{\omega}^{wb} \tag{6.48}$$

Now we can apply Euler’s law of moment of momentum, $\dot{\vec{\mathbf{h}}} = \vec{\mathbf{g}}$.Since $\vec{\mathbf{h}}$ is expressed in a rotating reference frame, the resulting differential equation is

$$\dot{\mathbf{h}} + \boldsymbol{\omega}^{\times}\mathbf{h} = \mathbf{g} \tag{6.49}$$

Now substituting the expression for $\mathbf{h}$ into this equation and collecting terms leads to the following coupled system of equations:

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3 - \frac{I_w}{I_1}\Omega_w\omega_2 + \frac{g_1}{I_1} \tag{6.50}$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_1\omega_3 + \frac{I_w}{I_2}\Omega_w\omega_1\frac{g_2}{I_2} \tag{6.51}$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2\frac{g_3}{I_3} \tag{6.52}$$

We restrict attention here to the case of torque-free motion, $\mathbf{g} = \mathbf{0}$. If the body frame has angular velocity $\boldsymbol{\omega} = [0, 0, \Omega]^T$, then $\dot{\boldsymbol{\omega}} = \mathbf{0}$. That is, steady spin about the $\hat{\mathbf{b}}_3$ axis is an equilibrium motion. If the wheel’s relative angular velocity is zero, *i.e.*, $\Omega_w = 0$, then the rigid body result holds: the spin is stable if the $\hat{\mathbf{b}}_3$ axis is the major or minor axis, unstable if it is the intermediate axis. We now want to determine how the wheel affects this stability result.

To determine stability of this steady motion, we linearize the equations of motion, by letting $\boldsymbol{\omega} = \boldsymbol{\omega}_e + \delta\boldsymbol{\omega}$, where $\boldsymbol{\omega}_e$ is the equilibrium motion of interest. On substituting this expression into the equations of motion, and noting that $\dot{\boldsymbol{\omega}}_e = \mathbf{0}$, we obtain

$$\delta\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\Omega\delta\omega_2 - \frac{I_w}{I_1}\Omega_w\delta\omega_2 \tag{6.53}$$

$$\delta\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\Omega\delta\omega_1 + \frac{I_w}{I_2}\Omega_w\delta\omega_1 \tag{6.54}$$

$$\delta\dot{\omega}_3 = 0 \tag{6.55}$$

Notice that there are no $\delta\omega_3$'s in these equations, and that $\delta\dot{\omega}_3 = 0$, so it is constant.

The $\delta\dot{\omega}_1$ and $\delta\dot{\omega}_2$ equations can be rewritten as

$$\delta\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\Omega - \frac{I_w}{I_1}\Omega_w\right)\delta\omega_2 \tag{6.56}$$

$$\delta\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\Omega + \frac{I_w}{I_2}\Omega_w\right)\delta\omega_1 \tag{6.57}$$

If we differentiate the $\delta\dot{\omega}_1$ equation and substitute the $\delta\dot{\omega}_2$ equation into the result, we obtain the following second-order, constant-coefficient, linear ordinary differential equation:

$$\delta\ddot{\omega}_1 + \left(\frac{I_3 - I_2}{I_1}\Omega + \frac{I_w}{I_1}\Omega_w\right)\left(\frac{I_3 - I_1}{I_2}\Omega + \frac{I_w}{I_2}\Omega_w\right)\delta\omega_1 \tag{6.58}$$

This equation is of the form $\ddot{x} + kx = 0$, and as we know, solutions are stable and oscillatory (sines and cosines of $\sqrt{k}t$) if $k > 0$, and are unstable (exponentials of $\pm\sqrt{-k}t$) if $k < 0$. Therefore, to determine stability, we just need to know whether the term

$$k = \left(\frac{I_3 - I_2}{I_1}\Omega - \frac{I_w}{I_1}\Omega_w\right)\left(\frac{I_3 - I_1}{I_2}\Omega + \frac{I_w}{I_2}\Omega_w\right) \tag{6.59}$$

is positive or negative. To analyze this term, we factor out the positive quantities, $1/I_1$, $1/I_2$, and Ω, so that

$$k = \frac{\Omega^2}{I_1 I_2}\left(I_3 - I_2 + I_w\hat{\Omega}_w\right)\left(I_3 - I_1 + I_w\hat{\Omega}_w\right) \tag{6.60}$$

where $\hat{\Omega}_w = \Omega_w/\Omega$. For the case where $\hat{\Omega}_w = 0$, the condition reduces to the familiar asymmetric rigid body conditions for stability: $I_3 > I_1$ and $I_3 > I_2$, or $I_3 < I_1$ and $I_3 < I_2$. However, for the case where the wheel is spinning relative to the body, we get the conditions

$$I_3 > I_2 - I_w\hat{\Omega}_w \quad \text{and} \quad I_3 > I_1 - I_w\hat{\Omega}_w \tag{6.61}$$

or

$$I_3 < I_2 - I_w\hat{\Omega}_w \quad \text{and} \quad I_3 < I_1 - I_w\hat{\Omega}_w \tag{6.62}$$

Thus, the $\hat{\mathbf{b}}_3$ axis does not have to be the major axis. For example, if $\hat{\Omega}_w$ is "large enough," then the expressions $I_1 - I_w\hat{\Omega}_w$ and $I_2 - I_w\hat{\Omega}_w$ can both be made negative, in which case the first set of conditions is satisfied. Likewise, if the wheel is spinning in the opposite direction as the body, then $\hat{\Omega}_w < 0$, and for sufficiently high spin rate, the two expressions $I_1 - I_w\hat{\Omega}_w$ and $I_2 - I_w\hat{\Omega}_w$ are both positive and can be made greater than I_3, so that the second set of conditions are satisfied.

Example 6.1 *Consider a rigid body with flywheel, with $I_1 = 300$, $I_2 = 400$, $I_3 = 350$, $I_w = 10$ (all in kg m^2), which we would like to have spinning about $\hat{\mathbf{b}}_3$ at 2π rad/s (60 RPM). Clearly $\hat{\mathbf{b}}_3$ is the intermediate axis, so this motion is unstable with the wheel locked. How fast does the wheel have to be spinning in order to stabilize this motion?*

In the first of the two sets of stability conditions, since $I_2 > I_1$, if the first condition is satisfied, the second condition is satisfied as well. Thus we need to have

$$I_3 > I_2 - I_w\hat{\Omega}_w \tag{6.63}$$

which we can solve for Ω_w to obtain

$$\Omega_w > \frac{I_2 - I_3}{I_w}\Omega \tag{6.64}$$

Substituting the given numbers into this expression gives

$$\Omega_w = 10\pi \text{ rad/s} = 300 \text{ RPM} \tag{6.65}$$

Thus if we spin the wheel faster than 300 RPM, the intermediate axis spin is stable.

One thing to be careful of here is that for $\mathbf{g} = \mathbf{0}$, the total angular momentum magnitude is constant. Say we decide to spin the wheel at 400 RPM. Then the total angular momentum of the operating point is $I_3\Omega + I_w\Omega_w = 2618$ kg m^2/ s. Thus before we spin up the wheel, the initial total angular momentum must have this same magnitude.

6.5 Summary

This chapter presents the basics of satellite attitude dynamics and stability analysis.

6.6 References and further reading

Thomson,[2] Kaplan,[3] Hughes,[4] Rimrott,[5] Chobotov,[6] Wiesel,[7] Sidi[8]

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6.7 Exercises

6.8 Problems