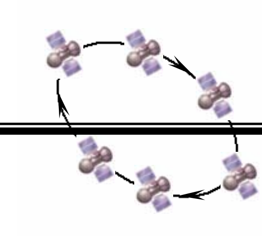
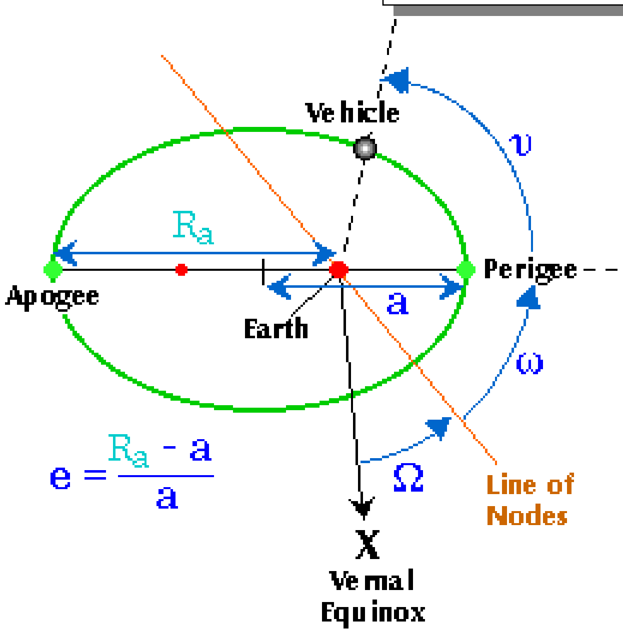
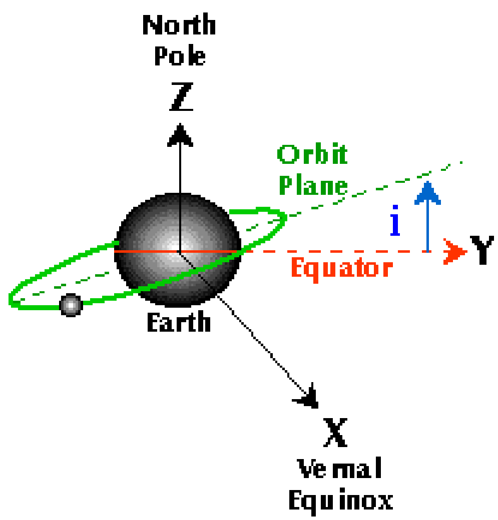


Astrodynamics

- Brief History of Orbital Mechanics
- Basic Facts about Orbits
- Orbital Elements
- Perturbations
- Satellite Coverage





References

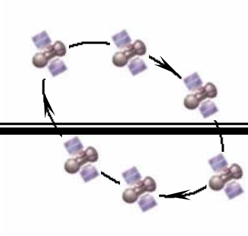
P&M Ch 3

SMAD Chs 6 & 7

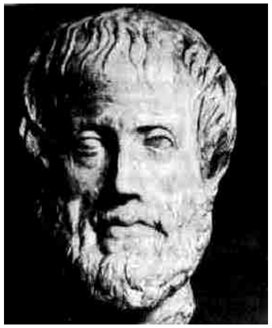
G&F Ch 4

BMW

Vallado



A Brief History of Orbital Mechanics



Aristotle (384-322 BC)

Ptolemy (87-150 AD)

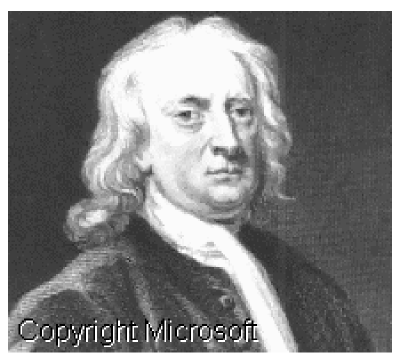
Nicolaus Copernicus (1473-1543)

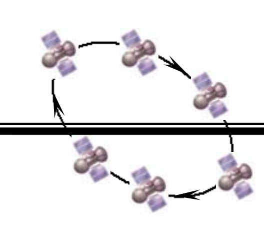
Tycho Brahe (1546-1601)

Johannes Kepler (1571-1630)

Galileo Galilei (1564-1642)

Sir Isaac Newton (1643-1727)





Kepler's Laws

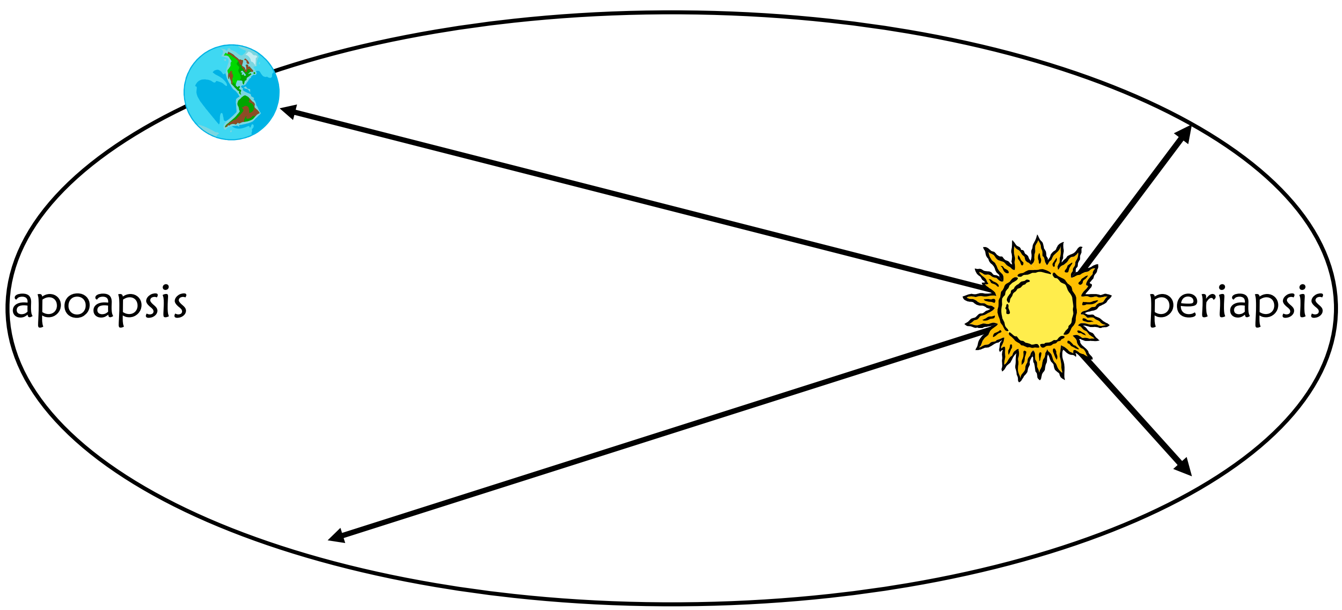
I. The orbit of each planet is an ellipse with the Sun at one focus.

II. The line joining the planet to the Sun sweeps out equal areas in equal times.

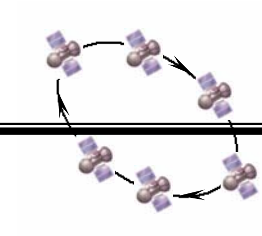
III. The square of the period of a planet is proportional to the cube of its mean distance to the sun.

Kepler's First Two Laws

I. The orbit of each planet is an ellipse with the Sun at one focus.



II. The line joining the planet to the Sun sweeps out equal areas in equal times.



Kepler's Third Law

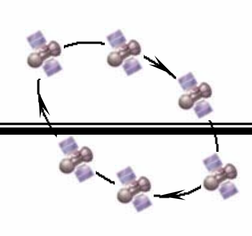
III. The square of the period of a planet is proportional to the cube of its mean distance to the sun.

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Here T is the period, a is the semimajor axis of the ellipse, and m is the gravitational parameter (depends on mass of central body)

$$\mu_{\oplus} = GM_{\oplus} = 3.98601\times10^{5}\ \text{km}^3\text{s}^{-2}$$

$$\mu_{\text{sun}} = GM_{\text{sun}} = 1.32715\times10^{11}\ \text{km}^3\text{s}^{-2}$$



Mean Motion

- The Mean Motion is defined as

$$n = \sqrt{\frac{\mu}{a^3}}$$

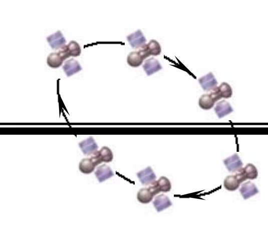
- The period can be written in terms of the mean motion as

$$T = 2\pi / n$$

- The Mean Anomaly is defined as

$$M = n(t - t_p)$$

where t_p is the time of periapsis passage



Earth Satellite Orbit Periods

Orbit	Altitude (km)	Period (min)
LEO	300	90.52
LEO	400	92.56
MEO	3000	150.64
GPS	20232	720
GEO	35786	1436.07

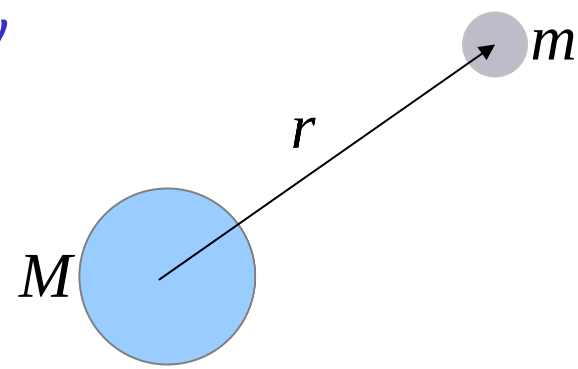
Newton's Laws

- Kepler's Laws were based on observation data: "curve fits"
- Newton established the theory
  - Universal Gravitational Law

$$F_g = -\frac{GMm}{r^2}$$

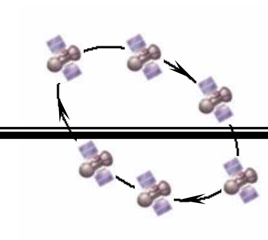
  - Second Law

$$\vec{\mathbf{F}} = m\ddot{\vec{\mathbf{r}}}$$



Universal Gravitational Constant

$$G = 6.672 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$$



Elliptical Orbits

- Planets, comets, and asteroids orbit the Sun in ellipses
- Moons orbit the planets in ellipses
- Artificial satellites orbit the Earth in ellipses
- To understand orbits, you need to understand ellipses (and other conic sections)
- But first, let’s study circular orbits:

  A circle is a special case of an ellipse

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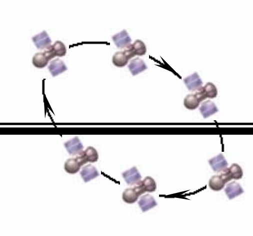
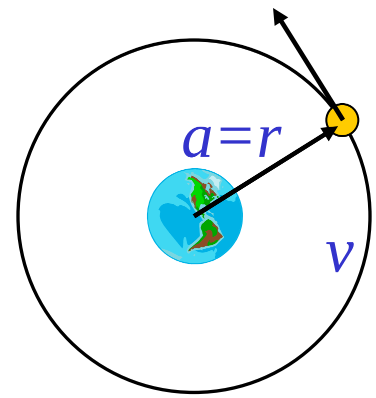


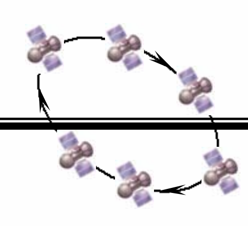
Circular Orbits

- Speed of satellite in circular orbit depends on radius

$$v_c = \sqrt{\frac{\mu}{r}}$$



- If an orbiting object at a particular radius has a speed $< v_c$, then it is in an elliptical orbit with lower energy
- If an orbiting object at radius r has a speed $> v_c$, then it is in a higher-energy orbit: elliptical, parabolic, or hyperbolic



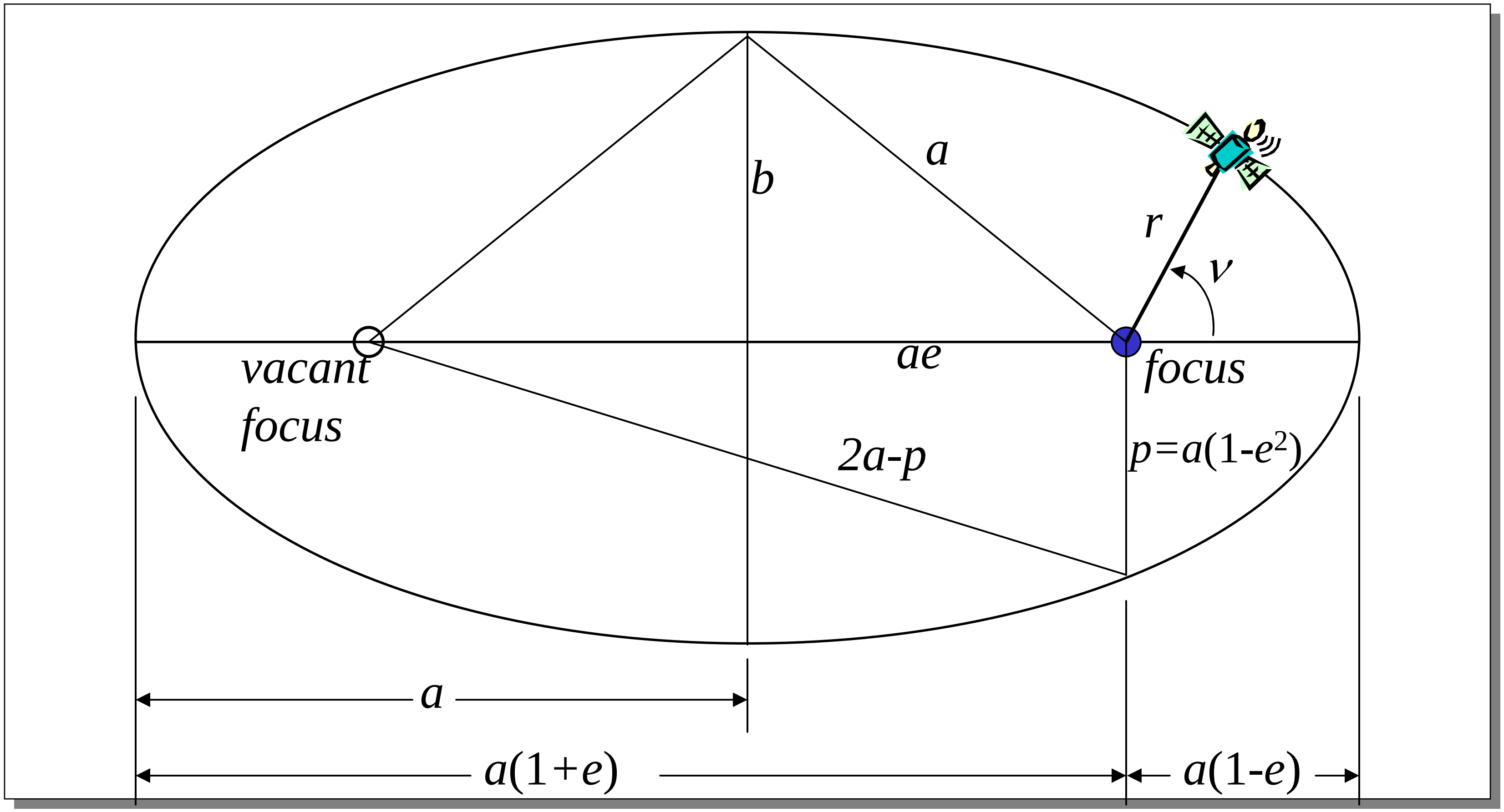
The Energy of an Orbit

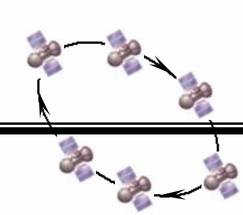
- Orbital energy is the sum of the kinetic energy, $mv^2/2$, and the potential energy, $-\mu m/r$
- Customarily, we use the specific mechanical energy, E (*i.e.*, the energy per unit mass of satellite)

$$E = \frac{v^2}{2} - \frac{\mu}{r} \quad \Leftrightarrow \quad E = \frac{-\mu}{2a}$$

- From this definition of energy, we can develop the following facts
  - $E<0 \Leftrightarrow$ orbit is elliptical or circular
  - $E=0 \Leftrightarrow$ orbit is parabolic
  - $E>0 \Leftrightarrow$ orbit is hyperbolic

Properties of Ellipses

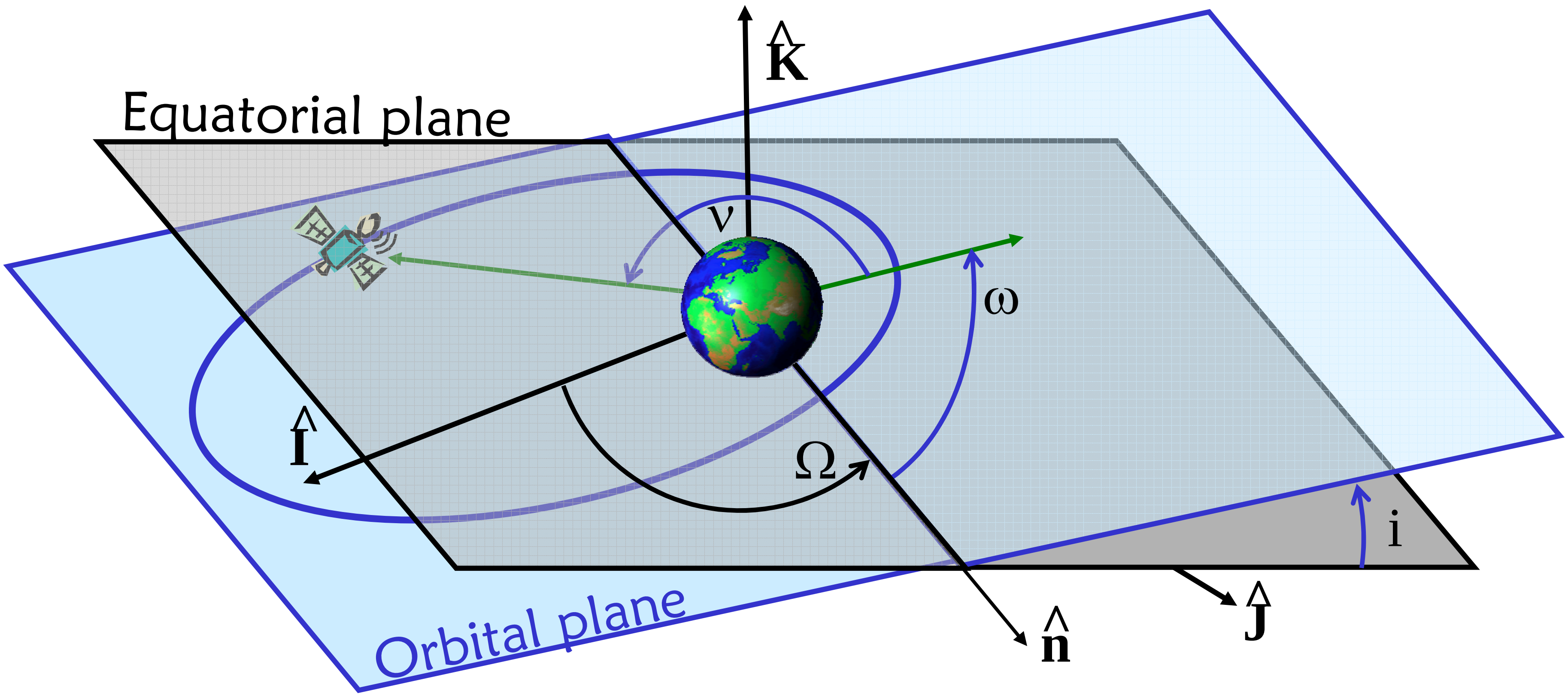




Facts About Elliptical Orbits

- Periapsis is the closest point of the orbit to the central body
  - $r_p = a(1-e)$
- Apoapsis is the farthest point of the orbit from the central body
  - $r_a = a(1+e)$
- Velocity at any point is
  - $v = (2E+2\mu/r)^{1/2}$
- Escape velocity at any point is
  - $v = (2\mu/r)^{1/2}$

Orbital Elements

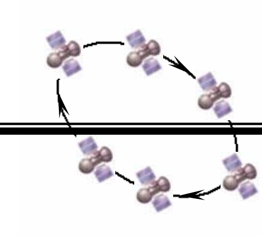


Orbit is defined by 6 *orbital elements* (**oe**'s):

semimajor axis, *a*; eccentricity, *e*;

inclination, *i*; right ascension of ascending node, Ω;

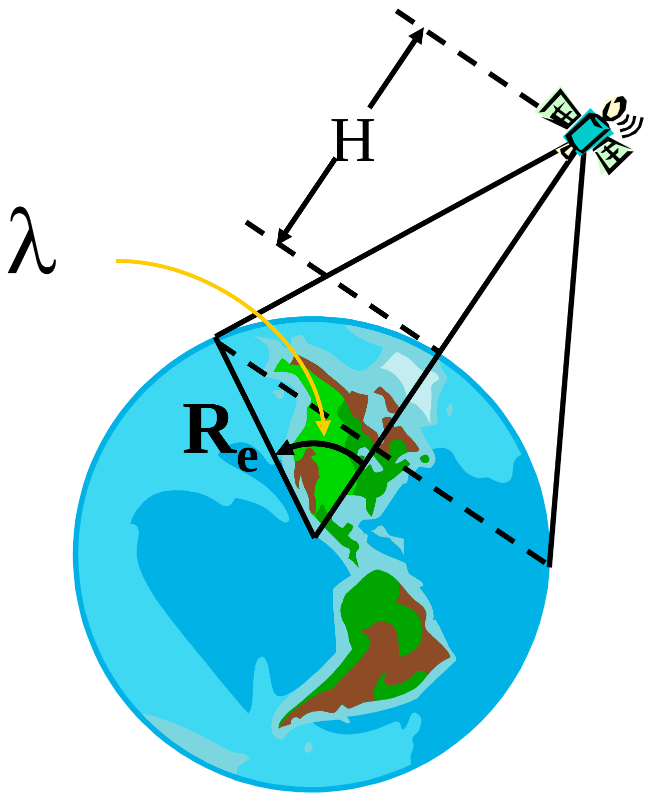
argument of periapsis, ω; and true anomaly, ν



Orbital Elements(continued)

- Semimajor axis a determines the size of the ellipse
- Eccentricity e determines the shape of the ellipse
- Two-body problem
  - a, e, i, Ω, and ω are constant
  - 6^{th} orbital element is the angular measure of satellite motion in the orbit – 2 angles are commonly used:
    - True anomaly, ν
    - Mean anomaly, M
- In reality, these elements are subject to various perturbations
  - Earth oblateness (J_2)
  - atmospheric drag
  - solar radiation pressure
  - gravitational attraction of other bodies

Instantaneous Access Area



IAA

$$IAA = K_A(1-\cos\lambda)$$

$$K_A = 2\pi R_e^2$$

$$K_A = 2.55604187\times10^8\ \text{km}^2$$

$$\cos\lambda = \frac{R_e}{R_e + H}$$

Example: Space shuttle

$$R_e = 6378\ \text{km}$$

$$\text{H} = 300\ \text{km}$$

$$\cos\lambda = 0.9551 \Rightarrow \lambda = 17.24^\circ$$

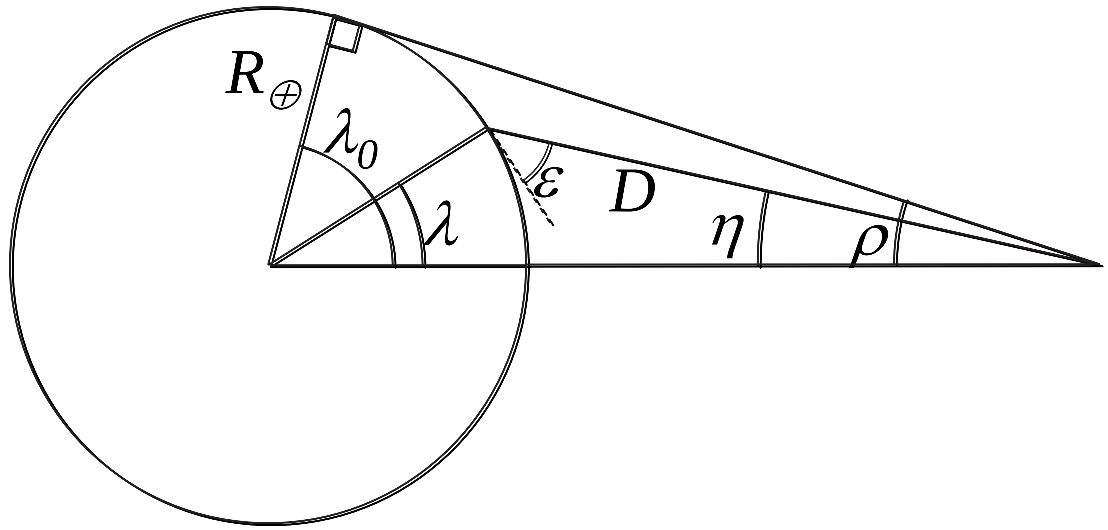
$$IAA = 11{,}476{,}628\ \text{km}^2$$

Elevation Angle

Elevation angle, ε, is measured up from horizon to target

Minimum elevation angle is typically based on the performance of an antenna or sensor

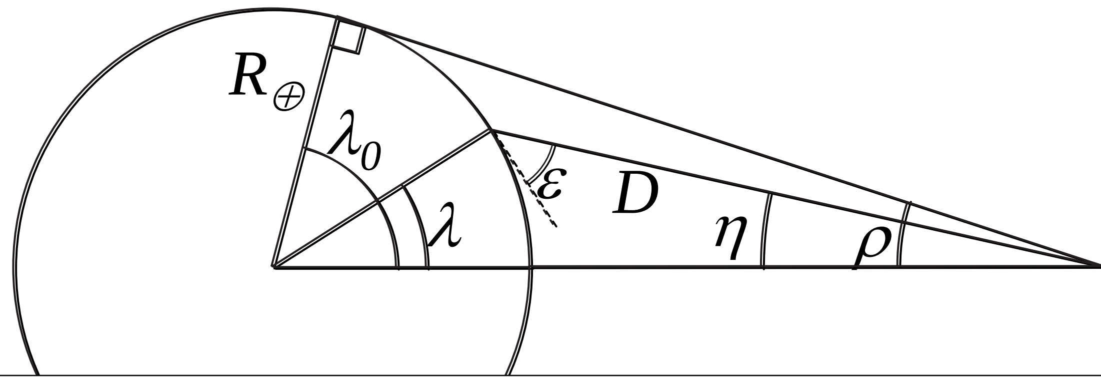
IAA is determined by same formula, but the Earth central angle, λ, is determined from the geometry shown



The angle η is called the nadir angle

The angle ρ is called the apparent Earth radius

Geometry of Earth-Viewing



- Given altitude H, we can state
  $\sin\rho = \cos\lambda_0 = R_\oplus / (R_\oplus + H)$
  $\rho + \lambda_0 = 90°$
- For a target with known position vector, λ is easily computed
  $\cos\lambda = \cos\delta_s \cos\delta_t \cos\Delta L + \sin\delta_s \sin\delta_t$
- Then $\tan\eta = \sin\rho \sin\lambda / (1 - \sin\rho\cos\lambda)$
- And $\eta + \lambda + \varepsilon = 90°$ and $D = R_\oplus \sin\lambda / \sin\eta$

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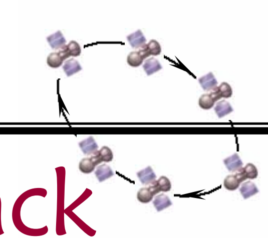


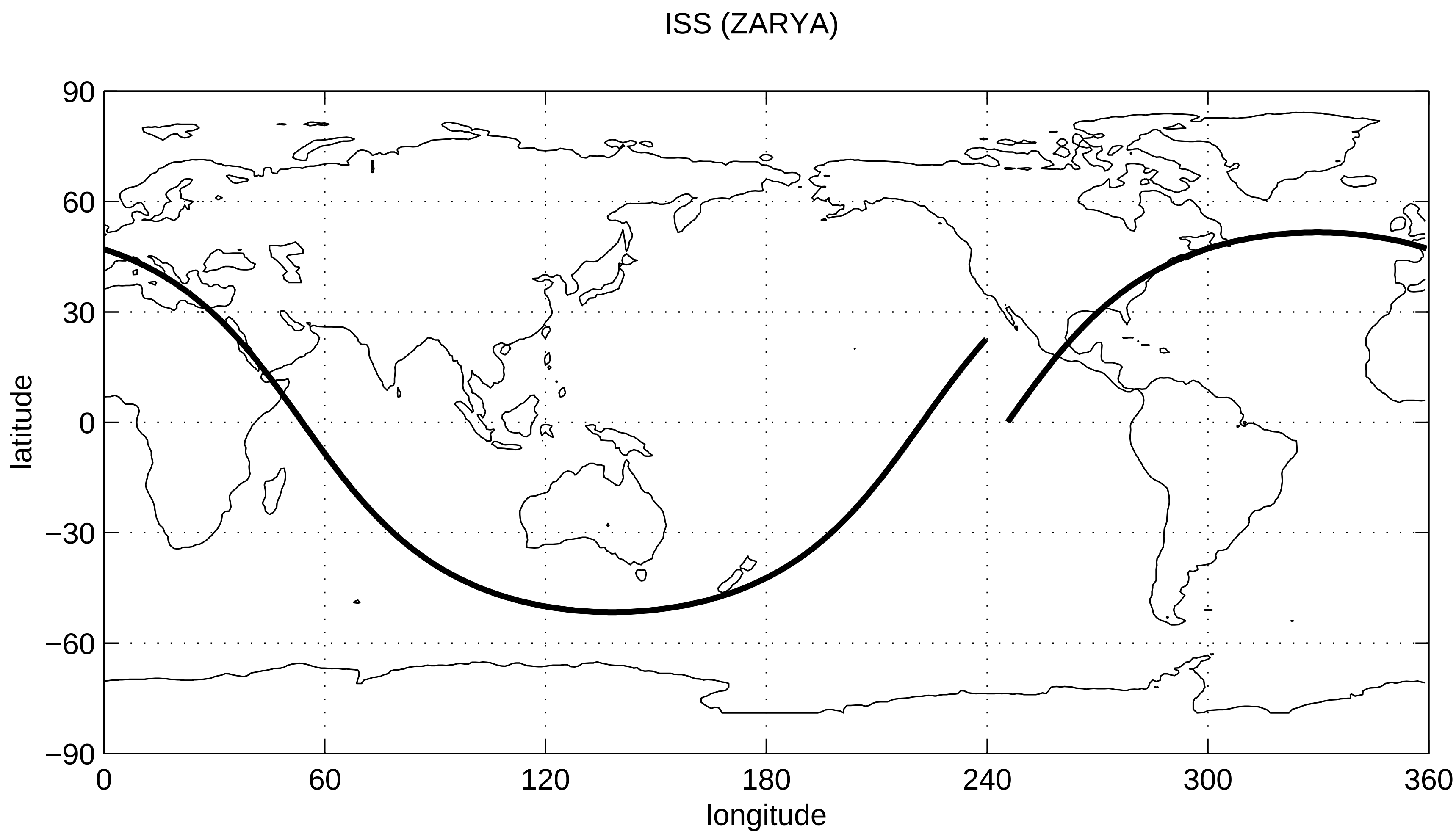
Algorithm for SSP, Ground Track

- Compute position vector in ECI
- Determine Greenwich Sidereal Time θ_g at epoch, θ_{g0}
- Latitude is $\delta_s = \sin^{-1}(r_3/r)$
- Longitude is $L_s = \tan^{-1}(r_2/r_1) - \theta_{g0}$

- Propagate position vector in "the usual way"
- Propagate GST using $\theta_g = \theta_{g0} + \omega_{\oplus}(t-t_0)$
  where $\omega_{\oplus}$ is the angular velocity of the Earth
- Notes:
  http://www.aoe.vt.edu/~chall/courses/aoe4134/sidereal.pdf
  http://aa.usno.navy.mil/data/docs/WebMICA_2.html
  http://tycho.usno.navy.mil/sidereal.html

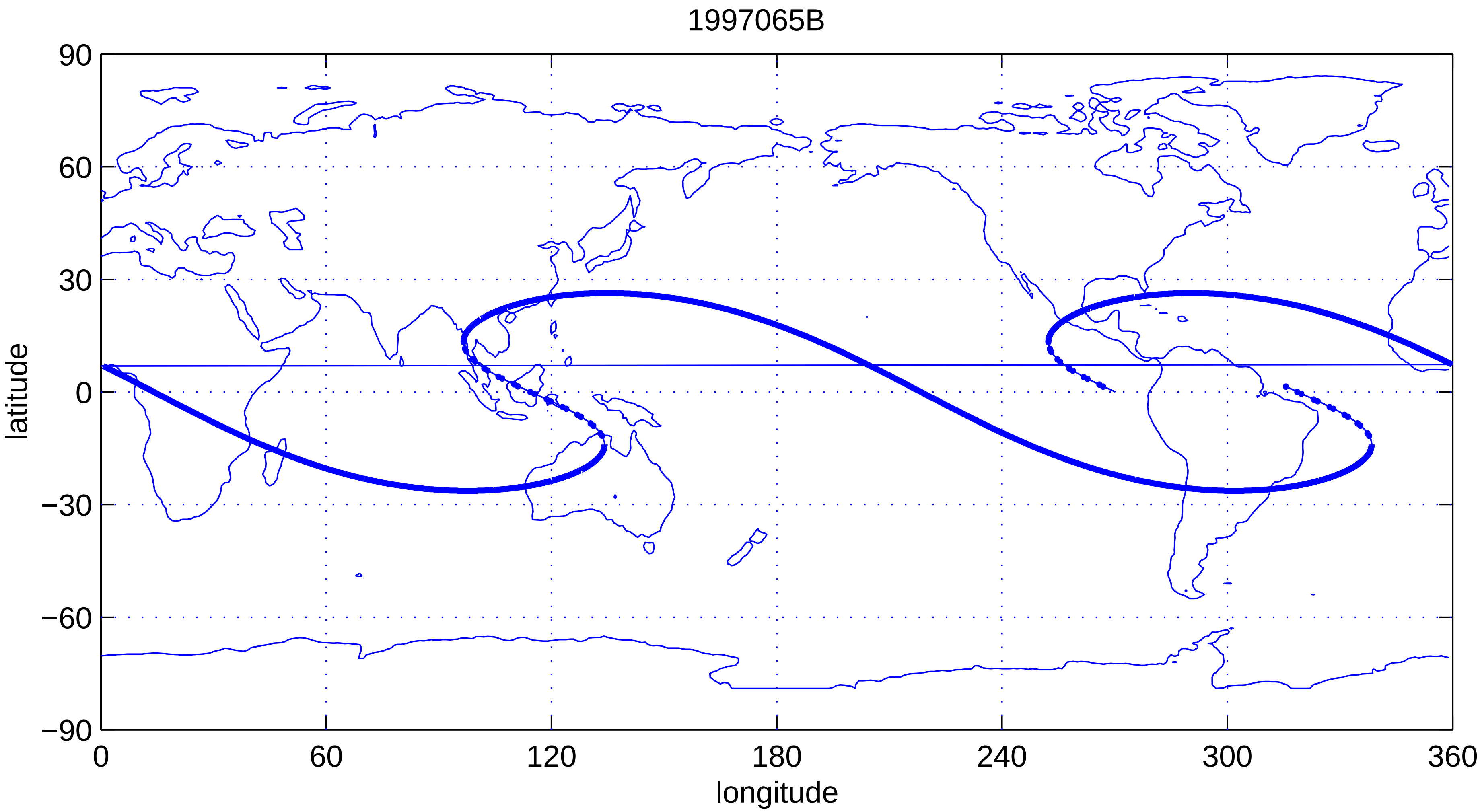
Ground Track

This plot is for a satellite in a nearly circular orbit



Ground Track

This plot is for a satellite in a highly elliptical orbit



Error Sources

Table 2.1: Sources of Pointing and Mapping Errors[2]

Spacecraft Position Errors		
ΔI	In- or along-track	Displacement along the spacecraft's velocity vector
ΔC	Cross-track	Displacement normal to the spacecraft's orbit plane
ΔR_S	Radial	Displacement toward the center of the Earth (nadir)
Sensing Axis Orientation Errors (in polar coordinates about nadir)		
$\Delta\eta$	Elevation	Error in angle from nadir to sensing axis
$\Delta\phi$	Azimuth	Error in rotation of the sensing axis about nadir
Other Errors		
ΔR_T	Target altitude	Uncertainty i the altitude of the observed object
ΔT	Clock error	Uncertainty in the real observation time

Error Budgets

Table 2.2: Pointing and Mapping Error Formulas[2]

Source	Magnitude	Magnitude of Mapping Error (km)	Magnitude of Pointing Error (rad)	Direction of Error
Attitude Errors:				
Azimuth	$\Delta\phi$ (rad)	$\Delta\phi D \sin\eta$	$\Delta\phi \sin\eta$	Azimuthal
Nadir Angle	$\Delta\eta$ (rad)	$\Delta\eta D / \sin\varepsilon$	$\Delta\eta$	Toward nadir
Position Errors:				
In-track	ΔI (km)	$\Delta I (R_T/R_S) \cos H$	$(\Delta I / D) \sin Y_I$	Parallel to ground track
Cross-track	ΔC (km)	$\Delta C (R_T/R_S) \cos G$	$(\Delta C / D) \sin Y_C$	Perpendicular to ground track
Radial	ΔR_S (km)	$\Delta R_S \sin\eta / \sin\varepsilon$	$(\Delta R_S / D) \sin\eta$	Toward nadir
Other Errors:				
Target altitude	ΔR_T (km)	$\Delta R_T / \tan\varepsilon$	—	Toward nadir
S/C Clock	ΔT (s)	$\Delta T V_e \cos\ \text{lat}$	$\Delta T (V_e / D) \cos\ \text{lat} \sin J$	Parallel to Earth's equator

$\sin H = \sin\lambda \sin\phi$

$\sin G = \sin\lambda \cos\phi$

$V_e = 464$ m/s (Earth rotation velocity at equator)

$\cos Y_I = \cos\phi \sin\eta$

$\cos Y_C = \sin\phi \sin\eta$

$\cos J = \cos\phi_E \cos\varepsilon$, where ϕ_E = azimuth relative to East

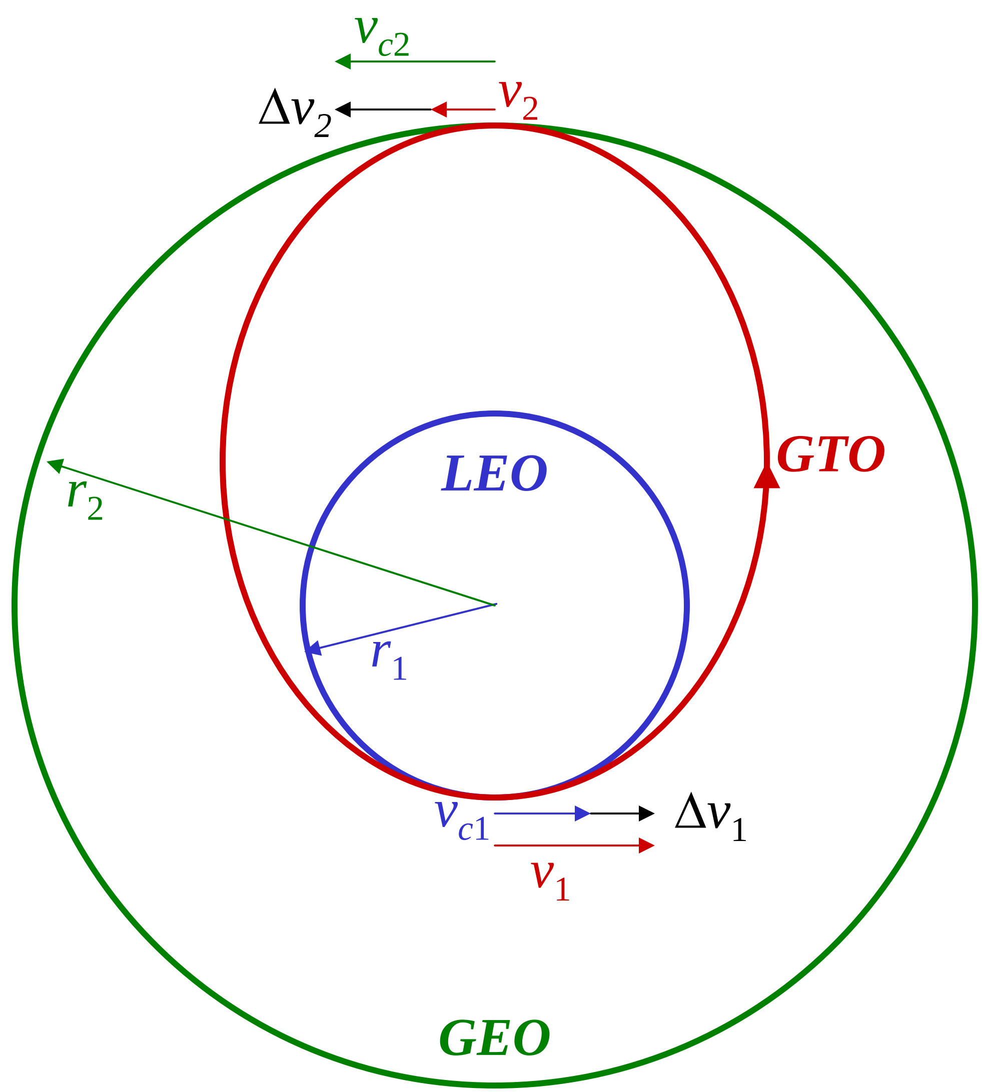
Hohmann Transfer

- Initial LEO orbit has radius r_1, velocity v_{c1}
- Desired GEO orbit has radius r_2, velocity v_{c2}
- Impulsive Δv is applied to get on geostationary transfer orbit (GTO) at perigee

$$\Delta v_1 = \sqrt{\frac{2\mu}{r_1} - \frac{2\mu}{r_1 + r_2}} - \sqrt{\frac{\mu}{r_1}}$$

- Coast to apogee and apply another impulsive Δv

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{2\mu}{r_1 + r_2}}$$

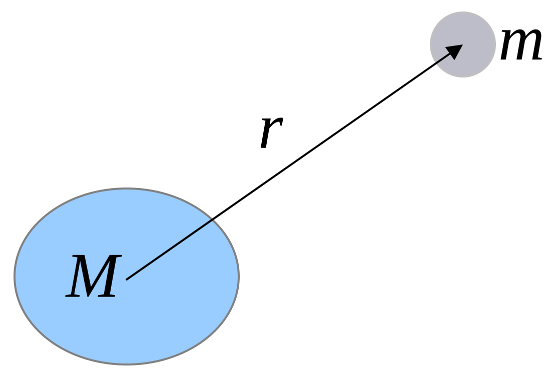


Earth Oblateness Perturbations

- Earth is non-spherical, and to first approximation is an oblate spheroid
- The primary effects are on Ω and ω:

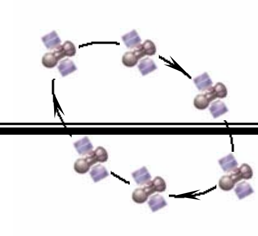
$$\dot{\Omega} = \frac{3J_2 n R_e^2}{2a^2(1-e^2)^2}\cos i$$

$$\dot{\omega} = \frac{-3J_2 n R_e^2}{4a^2(1-e^2)^2}(4-5\sin^2 i)$$



The Oblateness Coefficient

$$J_2 = -1.08263\times10^{-3}$$



Main Applications of J_2 Effects

- Sun-synchronous orbits:
  The rate of change of Ω can be chosen so that the orbital plane maintains the same orientation with respect to the sun throughout the year
- Critical inclination orbits:
  The rate of change of ω can be made zero by selecting $i \approx 63.4°$