# Analytical Hierarchy Process

The Analytical Hierarchy Process (AHP) is a systematic method for comparing a list of objectives or alternatives. When used in the systems engineering process, AHP can be a powerful tool for comparing alternative design concepts.

We begin by assuming that a set of objectives has been established (Value System Design, Objective Hierarchy), and that we are trying to establish a normalized set of weights to be used when comparing alternatives using these objectives. For simplicity, we assume that there are 4 objectives: $O_1$, $O_2$, $O_3$, and $O_4$.

We form a pairwise comparison matrix $A$, where the number in the $i_{\text{th}}$ row and $j_{\text{th}}$ column gives the relative importance of $O_i$ as compared with $O_j$. We use a 1–9 scale, with $a_{ij} = 1$ if the two objectives are equal in importance, $a_{ij} = 3$ if $O_i$ is weakly more important than $O_j$, $a_{ij} = 5$ if $O_i$ is strongly more important than $O_j$, $a_{ij} = 7$ if $O_i$ is very strongly more important than $O_j$, and $a_{ij} = 9$ if $O_i$ is absolutely more important than $O_j$.

Thus we might arrive at the following matrix:

$$A = \begin{bmatrix} 1 & 1/5 & 1/3 & 1/7 \\ 5 & 1 & 3 & 5 \\ 3 & 1/3 & 1 & 3 \\ 7 & 1/5 & 1/3 & 1 \end{bmatrix} = \begin{bmatrix} 1.000 & 0.200 & 0.333 & 0.143 \\ 5.000 & 1.000 & 3.000 & 5.000 \\ 3.000 & 0.333 & 1.000 & 3.000 \\ 7.000 & 0.200 & 0.333 & 1.000 \end{bmatrix}$$

To normalize the weights, we compute the sum of each column and then divide each column by the corresponding sum. Thus, using an overbar to denote normalization, we get:

$$\bar{A} = \begin{bmatrix} 0.063 & 0.115 & 0.071 & 0.016 \\ 0.313 & 0.577 & 0.643 & 0.547 \\ 0.188 & 0.192 & 0.214 & 0.328 \\ 0.438 & 0.115 & 0.071 & 0.109 \end{bmatrix}$$

Notice that the numbers in the second row are generally larger than the rest of the numbers, except for the case of column 1. This indicates some inconsistency in the comparisons used in the original matrix. Ideally, the 4 normalized columns would all be identical if the pairwise comparisons were consistent. In practice, one can compute a consistency measure using the eigenvalues of the normalized comparison matrix.

The next step is to compute the average values of each row and use these as the weights in the Objective Hierarchy. Thus, for this example, the weights would be:

$$w = [0.066 \quad 0.520 \quad 0.231 \quad 0.183]^T$$

Note that by construction, $\sum_{i=1}^{4} w_i = 1$. These weights would be used in summing the measures as required in the evaluation of the Objective Hierarchy.

**Reference**: Ernest H. Forman, *Decision by Objectives*, http://mdm.gwu.edu/Forman/DBO.pdf