Goddard Problem

- **Posed by R.H. Goddard, "A Method of Reaching Extreme Altitudes", Smithsonian Inst. Misc. Coll. 71, 1919, reprinted by Am. Rocket Soc., 1946.**

- **Three state variables**

  r : **distance from Earth's center**

  v : **radial velocity**

  m : **rocket's mass**

- **One control variable β : mass flow-rate**

➤ The dynamical model

$$
\begin{aligned}
\dot{r} &= v \\
\dot{v} &= \frac{\beta\, c}{m} - \frac{D(r,v)}{m} - \frac{\mu}{r^2} \\
\dot{m} &= -\beta
\end{aligned}
$$

with $\beta \in [0, \beta_{\max}]$.

- **The initial conditions are:**

$$\begin{aligned} r(0) &= R_e \text{ Earth's radius} \\ v(0) &= 0 \text{ start from rest} \\ m(0) &= M_0 \text{ initial mass} \end{aligned}$$

- **The (only) specified end-condition is**

$$\theta_1(\vec{x}(t_f)) = m(t_f) - M_f$$

where $M_f < M_0$ is the mass with all fuel expended

- **The cost functional is**

$$g(r(t_f), v(t_f), m(t_f)) = -r(t_f)\,,$$

that is, we maximize the final altitude.

Nondimensional Form

- **It's convenient to nondimenionalize. We select**

$$M : M_0 \text{ unit mass}$$

$$L : R_e \text{ unit length}$$

$$T : \sqrt{R_e^3/\mu} \text{ unit time}$$

Note that this leads to

$$\sqrt{\mu / R_e} \; : \; \text{unit speed}$$

$$M_0 \mu / R_e^2 \; : \; \text{unit force}$$

➤ **The scaled dynamical model**

$$\begin{aligned} \dot{\tilde{r}} &= \tilde{v} \\ \dot{\tilde{v}} &= \frac{\tilde{\beta}\,\tilde{c}}{\tilde{m}} - \frac{\tilde{D}(\tilde{r}, \tilde{v})}{\tilde{m}} - \frac{1}{\tilde{r}^2} \\ \dot{\tilde{m}} &= -\tilde{\beta} \end{aligned}$$

with $\tilde{\beta} \in [0, \tilde{\beta}_{\max}]$, where, for example,

$$\tilde{r} = r/R_e$$

- **In the following we drop the ˜ and note that all quantities have been non-dimensionalized.**

Applying the M.P.

- **We form the variational**

Hamiltonian

$$\boldsymbol{H^a = \lambda_r\, v + \lambda_v \left[\frac{\beta\, c}{m} - \frac{D(r, v)}{m} - \frac{1}{r^2}\right] - \lambda_m\, \beta}$$

➤ **The adjoint differential**

equations are

$$\dot{\lambda}_r = \lambda_v \left[\frac{1}{m} \frac{\partial D}{\partial r} - \frac{2}{r^3} \right]$$

$$\dot{\lambda}_v = -\lambda_r + \frac{\lambda_v}{m} \frac{\partial D}{\partial v}$$

$$\dot{\lambda}_m = \frac{\lambda_v}{m^2} \left[\beta c - D(r, v) \right].$$

➤ **The terminal transversality**

conditions imply

$$
\begin{aligned}
H(t_f) &= 0 \\
\lambda_r(t_f) &= \lambda_0 \\
\lambda_v(t_f) &= 0 \\
\lambda_m(t_f) &= \nu_1
\end{aligned}
$$

Note that H is constant along an extremal path.

Applying the M. P.
min H

- We are to minimize the variational Hamiltonian H subject to the bounds.
- Observe that H can be written

as

$$\boldsymbol{H} = \left[\frac{\boldsymbol{\lambda}_v\, \boldsymbol{c}}{\boldsymbol{m}} - \boldsymbol{\lambda}_m\right] \boldsymbol{\beta}$$

$$+ \text{ terms independent of } \boldsymbol{\beta}$$

Use the symbol $\boldsymbol{S}$ for the terms in square brackets

➤ **For the mass flow-rate $\boldsymbol{\beta}$ we find**

three possibilities

$$\beta^* = \begin{cases} 0 & \text{if } S > 0 \\ \beta_{\max} & S < 0 \\ \text{singular} & S = 0 \end{cases}$$

The *singular* case arises only if $S(\cdot)$ vanishes on an arc of finite width.

- **Since m is positive we can multiply S by m without changing the conclusions. Hence we re-define**

$$S = [\lambda_v\, c \;-\; \lambda_m\, m]$$

Singular Control

- **For S to stay constant at zero we require $\dot{S} = 0$ and this leads to**

$$\dot{S} = \dot{\lambda}_v c - \dot{\lambda}_m m - \lambda_m \dot{m}$$

- **Substituting the appropriate**

state/adjoint differential equations this simplifies to

$$\dot{S} = \frac{\lambda_v}{m}\left[D + c\frac{\partial\, D}{\partial\, v}\right] - \lambda_r c$$

Note that the β terms have cancelled out.

- **We could take a second time-derivative and insist that**

$\ddot{S} = 0$. **Equivalently, observe that the three conditions**

$$
\begin{aligned}
H &= 0 \\
S &= 0 \\
\dot{S} &= 0
\end{aligned}
$$

are three linear homogeneous equations in the adjoint

variables λ_r, λ_v, λ_m. Since the adjoints can not all vanish simultaneously, this implies that the determinant must be zero. That is,

$$v\left[D + c\frac{\partial\, D}{\partial\, v}\right] - c\left[D + \frac{m}{r^2}\right] = 0$$

- **Since this involves only state variables it's somewhat simpler than $\dot{S}$. Setting the time-derivative of this expression to zero will lead to an expression for the *singular* control β. It is still somewhat messy.**

- **In general, the control will appear first in an even time-derivative of the switching function. If the control appears first in the $2q$-th time-derivative of S, we say the singular arc is of order q. The Goddard problem has a first order singular arc.**